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t-Path Sigraphs*

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Abstract

Given a sigraph S and a positive integer t, the *t-path sigraph* $(S)_t$ of S is formed by taking a copy of the vertex set $V(S)$ of S, joining two vertices u and v in the copy by a single edge $e = uv$ whenever there is a $u - v$ path of length t in S and then by defining its sign to be $-$ whenever in every $u - v$ path of length t in S all the edges are negative. In this paper, we introduce a variation of the concept of t-path sigraphs studied above. The motivation stems naturally from one's mathematically inquisitiveness as to ask why not define the sign of an edge $e = uv$ in $(S)_t$ as the product of the signs of the vertices u and v in S. It is shown that for any sigraph S, its t-path sigraph $(S)_t$ is balanced. We then give structural characterization of t-path sigraphs. Further, in this paper we characterize sigraphs which are switching equivalent to their 2(3)-path sigraphs.

Keywords and Phrases: *Sigraphs, Balance, Switching, t-Path sigraphs, Negation of a sigraph.*

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1. Introduction

For standard terminology and notion in graph theory we refer the reader to West [14]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A *sigraph* is an ordered pair $S = (G, \sigma)$, where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \to \{+, -\}$ is a function. A sigraph $S = (G, \sigma)$ is *balanced* if every cycle in S has an even number of negative edges (See [7]). Equivalently, a sigraph is balanced if product of signs of the edges on every cycle of S is positive.

A *marking* of S is a function $\mu : V(G) \to \{+, -\}$; a sigraph S together with a marking μ is denoted by S_μ. Given a sigraph S one can easily define a marking μ of S as follows: For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{u \in N(v)} \sigma(uv),$$

the marking μ of S is called *canonical* marking of S.

The following characterization of balanced sigraphs is well known.

Proposition 1. (**E. Sampathkumar [12]**) *A sigraph $S = (G, \sigma)$ is balanced if, and only if, there exist a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

The idea of switching a sigraph was introduced by Abelson and Rosenberg [2] in connection with structural analysis of marking μ of a sigraph S. Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched sigraph* or just *switched sigraph*. Two sigraphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further, a sigraph $S_1 = (G, \sigma)$ *switches* to a sigraph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective sigraphs.

Two sigraphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *cycle isomorphic* (see [15]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result will be useful in our further investigation (See [15]):

Proposition 2. (**T. Zaslavasky** [15]) *Two sigraphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

One can extend the study of sigraph equations w.r.t. isomorphism to sigraph equations w.r.t. switching equivalence. The purpose of this paper is to determine the solutions of one such extended sigraph equations.

2. t-Path Sigraphs

Given a graph G and a positive integer t, the *t-path graph* $(G)_t$ of G is formed by taking a copy of the vertex set $V(G)$ of G, joining two vertices u and v in the copy by a single edge $e = uv$ whenever there is a $u - v$ path of length t in G. The notion of *t-path graphs* was introduced by Escalante et al. [4] as a generalization of the notion of *open neighborhood graphs* introduced by Acharya [1] (also see Escalante & Montejano [3, 5], Harary et al. [9], Simic [13], for further studies as also Kovchegov [10] for application of the notion to analyze evolution of social networks triggered by local interactions). A graph G for which

$$(G)_t \cong G \tag{1}$$

has been termed as *t-path invariant graph* by Esclante et. al [4], Escalante & Montejano [5] where the explicit solution to (1) has been determined for $t = 2, 3$. The structure of t-path invariant graphs are still remains uninvestigated in literature for all $t \geq 4$. The following result characterize the structure of 2-path invariant graphs.

Proposition 3. (**Escalante et al.** [4])
A graph G order p is a 2-path invariant graph if, and only if, $G \cong \overline{K_p}$ or K_P with $p \geq 3$ or the odd p-cycle C_p, $p = 2m+1,\ m \geq 2$.

The structure of 3-path invariant graph is given by the following result:

Proposition 4. (**Escalante & Montejano** [5])
A graph G order p is a 3-path invariant graph if, and only if, it is isomorphic to any of the following graphs:

[a] C_m, $4 \leq m \neq 3k$, k is a positive integer

[b] K_n, $n \geq 4$

[c] $K_{m,n}$, $2 \leq m \leq n$

[d] The double star $K_{t,2}^{m,n}$

[e] $E_1 = K_4 - x$

[f] $E_2 =$ the subdivision of E_1

[g] E_3 (See [5])

[h] E_4 (See [5])

The notion of t-path graph of a given graph was extended to the class of sigraphs by Mishra [11] as follows: Given a sigraph S and a positive integer t, the *t-path sigraph* $(S)_t$ of S is formed by taking a copy of the vertex set $V(S)$ of S, joining two vertices u and v in the copy by a single edge $e = uv$ whenever there is a $u-v$ path of length t in S and then by defining its sign to be $-$ whenever in every $u-v$ path of length t in S all the edges are negative.

In this paper, we shall now introduce a variation of the concept of t-path sigraphs studied above. The motivation stems naturally from one's mathematically inquisitiveness as to ask why not define the sign of an edge $e = uv$ in $(S)_t$ as the product of the signs of the vertices u and v in S. The *t-path sigraph* $(S)_t = ((G)_t, \sigma')$ of a sigraph $S = (G, \sigma)$ is a sigraph whose underlying graph is $(G)_t$ called *t-path graph* and sign of any edge $e = uv$ in $(S)_t$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S. Further, a sigraph $S = (G, \sigma)$ is called

t-path sigraph, if $S \cong (S')_t$, for some sigraph S'.

The following result indicates the limitations of the notion of t-path sigraphs as introduced above, since the entire class of unbalanced sigraphs is forbidden to be t-path sigraphs.

Proposition 5. *For any sigraph $S = (G, \sigma)$, its t-path sigraph $(S)_t$ is balanced.*

Proof. Since sign of any edge $e = uv$ is $(S)_t$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S, by Proposition 1, $(S)_t$ is balanced.

Remark. For any two signed graphs S and S' with same underlying graph, their path signed graphs are switching equivalent.

Corollary 6. *For any sigraph $S = (G, \sigma)$, its* 2 (3)*-path sigraph* $(S)_2$ $((S)_3)$ *is balanced.*

The following result characterize sigraphs which are t-path sigraphs.

Proposition 7. *A sigraph $S = (G, \sigma)$ is a t-path sigraph if, and only if, S is balanced sigraph and its underlying graph G is a t-path graph.*

Proof. Suppose that S is balanced and G is a t-path graph. Then there exists a graph H such that $(H)_t \cong G$. Since S is balanced, by Proposition 1, there exists a marking μ of G such that each edge $e = uv$ in S satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the sigraph $S' = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $(S')_t \cong S$. Hence S is a t-path sigraph.

Conversely, suppose that $S = (G, \sigma)$ is a t-path sigraph. Then there exists a signed graph $S' = (H, \sigma')$ such that $(S')_t \cong S$. Hence G is the t-path graph of H and by Proposition 5, S is balanced.

3. Switching Invariant Two-Path Sigraphs

Let $\psi(G)$ denotes the set of all sigraphs whose underlying graph is G. In view of Proposition 3, we see that if S is a solution to

$$(S)_2 \sim S, \tag{2}$$

then S is either a totally disconnected sigraph, or $S \in \psi(K_p)$ for some integer $p \geq 3$, or $S \in \psi(C_{2m+1})$ for $m \geq 2$. Therefore, in order to completely determine of the structure of switching invariant 2-path sigraphs, it is enough to search for solutions of (2) in the sets $\psi(K_p),\ p \geq 3$ and $\psi(C_{2m+1}),\ m \geq 2$.

Proposition 8. *For any sigraph $S = (G, \sigma)$, $S \sim (S)_2$ if, and only if, G is isomorphic to either $K_p,\ p \geq 3$ or $C_{2m+1},\ m \geq 2$ and S is balanced.*

Proof. Suppose $S \sim (S)_2$. This implies, $G \cong (G)_2$ and hence by Proposition 3, we see that the G must be isomorphic to either K_p or C_{2m+1}. Now, if S is any sigraph on any of these graphs, Corollary 6 implies that $(S)_2$ is balanced and hence if S is unbalanced its 2-path sigraph $(S)_2$ being balanced cannot be switching equivalent to S in accordance with Proposition 2. Therefore, S must be balanced.

Conversely, suppose that S is a balanced sigraph on K_p or C_{2m+1}. Then, since $(S)_2$ is balanced as per Corollary 6 and since $G \cong (G)_2$ in each of these cases, the result follows from Proposition 2.

Remark. In [6], the authors has proved the above result using notion t-path sigraph defined by Mishra [11]. The result and proof given here is very simple and straightforward from that given in [6].

The notion of *negation* $\eta(S)$ of a given sigraph S defined in [8] as follows: $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a sigraph $S = (G, \sigma)$, the $(S)_t$ is balanced (Proposition 5). We now examine, the condition under which negation of $(S)_t$ (i.e., $\eta((S)_t)$) is balanced.

Proposition 9. *Let $S = (G, \sigma)$ be a sigraph. If $(G)_t$ is bipartite then $\eta((S)_t)$ is balanced.*

Proof. Since, by Proposition 5, $(S)_t$ is balanced, then every cycle in $(S)_t$ contains even number of negative edges. Also, since $(G)_t$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C

in $(S)_t$ are also even. This implies that the same thing is true in negation of $(S)_t$. Hence $\eta((S)_t)$ is balanced. Proposition 8 provides easy solutions to three other sigraph switching equivalence relations, which are given in the following results.

Corollary 10. *For any sigraph $S = (G, \sigma)$, $\eta(S) \sim (S)_2$ if, and only if, S is unbalanced sigraph on K_p, $p \geq 3$ or C_{2m+1}, $m \geq 2$.*

Corollary 11. *For any sigraph $S = (G, \sigma)$, $(\eta(S))_2 \sim (S)_2$.*

Corollary 12. *For any sigraph $S = (G, \sigma)$, $\eta(S) \sim (S)_2$ if, and only if, G is isomorphic to either K_p, $p \geq 3$ or C_{2m+1}, $m \geq 2$ and $\eta(S)$ is balanced.*

4. Switching Invariant Three-Path Sigraphs

In view of Proposition 4, we see that if S is a solution to

$$(S)_3 \sim S, \tag{3}$$

then G is isomorphic to any of the graphs C_m, $4 \leq m \neq 3k$, k is a positive integer, K_n, $n \geq 4$, $K_{m,n}$, $2 \leq m \leq n$, The double star $K_{t,2}^{m,n}$, $E_1 = K_4 - x$, $E_2 =$ the subdivision of E_1, E_3, E_4 and S must be balanced. The proof of technique of the following result is similar to the Proposition 8.

Proposition 13. *For any sigraph $S = (G, \sigma)$, $S \sim (S)_3$ if, and only if, G isomorphic to any of the graphs: C_m, $4 \leq m \neq 3k$, k is a positive integer, K_n, $n \geq 4$, $K_{m,n}$, $2 \leq m \leq n$, The double star $K_{t,2}^{m,n}$, $E_1 = K_4 - x$, $E_2 =$ the subdivision of E_1, E_3, E_4 and S is balanced.*

Remark. The other results are in Section 3 one can easily apply to three-path sigraphs.

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