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On a Certain Fredholm Type Sum-difference Equation*

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Abstract

In this paper we study some basic qualitative properties of solutions of a certain Fredholm type sum-difference equation. A finite difference inequality with explicit estimate is used to establish the results.

Keywords and Phrases: *Fredholm type, Sum-difference equation, Finite difference inequality, Estimates on the solutions, Uniqueness, Continuous dependence.*

1. Introduction

The theory of finite difference equations and their wide applications has drawn much attention in the past few decades. A simple way of solving Fredholm integral equation is to write down the equation for a set of equidistant points and to approximate the integral terms by appropriate quadrature formulas. This procedure in general leads to the study of Fredholm type sum-difference equation

$$y(n) = h(n) + \sum_{s=\alpha}^{\beta} k(n, s, y(s)),$$

on the fixed region of summation (see [1,3,5]).

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In this paper we consider the following more general Fredholm type sum-difference equation

$$x\left(n\right)=f\left(n\right)+\sum_{s=\alpha}^{\beta}g\left(n,s,x\left(s\right),\Delta x\left(s\right)\right), \tag{1.1}$$

where f and g are given functions and x is the unknown function to be found. The origin of equation (1.1) can be traced back to the recent study of its integral analogue by Bica, Căuş and Mureşan in [2] (see also [4,10]). In fact the study of qualitative properties of solutions of equation (1.1) is a challanging task, because of the occurance of the extra factor $\Delta x\left(s\right)$ in the sum on the right hand side in (1.1), about which almost nothing seems to be known. The problem of existence of solutions for equations like (1.1) can be dealt with the method employed in [9,10] (see also [2,5,6]). The aim of the present paper is to study some basic qualitative properties of solutions of equation (1.1) under some suitable conditions on the functions involved therein. The main tool employed in the analysis is based on the application of a certain finite difference inequality with explicit estimate given in [8, Theorem 4.5.1, part (a_2), p. 224] (see also [7] for similar results).

2. Estimates on the Solutions

Let R^m denote the real m-dimensional Euclidean space with appropriate norm denoted by $|.|$. Let $R_+ = [0, \infty)$, $N_0 = \{0, 1, 2, ...\}$, $N = \{1, 2, ...\}$, $N_{\alpha,\beta} = \{\alpha, \alpha + 1, ..., \alpha + n = \beta\}$ $(\alpha \in N_0, n \in N)$ be the given subsets of R, the set of real numbers and $D(A, B)$ the class of discrete functions from the set A to the set B. For the functions $w(n), z(n, ., .)$ $(n \in N_0)$, we define the operators Δ and Δ_1 by $\Delta w\left(n\right) = w\left(n + 1\right) - w\left(n\right)$, $\Delta_1 z\left(n, ., .\right) = z\left(n + 1, ., .\right) - z\left(n, ., .\right)$. Throughout, we assume that $f \in D\left(N_{\alpha,\beta}, R^m\right), g \in D\left(N^2_{\alpha,\beta} \times R^m \times R^m, R^m\right)$ and all the functions involved in our discussion satisfy the condition $w\left(n\right) = 0$ for $n \notin N_{\alpha,\beta}$.

By a solution of equation (1.1) we mean a function $x\left(n\right) : N_{\alpha,\beta} \to R^m$ for which $\Delta x\left(n\right)$ exists and satisfies the equation (1.1). It is easy to observe that the solution $x(n)$ of equation (1.1) satisfies the following sum-difference equation

$$\Delta x(n) = \Delta f(n) + \sum_{s=\alpha}^{\beta} \Delta_1 g(n, s, x(s), \Delta x(s)), \tag{2.1}$$

for $n \in N_{\alpha,\beta}$.

We need the following special version of the finite difference inequality given in [8, Theorem 4.5.1, part (a_2), p. 224]. We shall state it in the following lemma for completeness.

Lemma. *Let $u, a, c, g \in D(N_{\alpha,\beta}, R_+)$. Suppose that*

$$u(n) \leq a(n) + c(n) \sum_{s=\alpha}^{\beta} g(s) u(s), \tag{2.2}$$

for $n \in N_{\alpha,\beta}$. If

$$d = \sum_{s=\alpha}^{\beta} g(s) c(s) < 1, \tag{2.3}$$

then

$$u(n) \leq a(n) + c(n) \left\{ \frac{1}{1-d} \sum_{s=\alpha}^{\beta} g(s) a(s) \right\}, \tag{2.4}$$

for $n \in N_{\alpha,\beta}$.

First, we shall give the following theorem which deals with the estimate on the solution of equation (1.1).

Theorem 1. *Suppose that the functions $g, \Delta_1 g$ satisfy the conditions*

$$|g(n, s, u, v)| \leq p(n) q(s) [|u| + |v|], \tag{2.5}$$

$$|\Delta_1 g(n, s, u, v)| \leq p(n) q_1(s) [|u| + |v|], \tag{2.6}$$

where $p, q, q_1 \in D(N_{\alpha,\beta}, R_+)$ and

$$d_1 = \sum_{s=\alpha}^{\beta} [q(s) + q_1(s)] p(s) < 1, \tag{2.7}$$

holds. Then for every solution $x \in D(N_{\alpha,\beta}, R^m)$ of equation (1.1), we have the estimate

$$|x(n)| + |\Delta x(n)| \leq [|f(n)| + |\Delta f(n)|]$$
$$+p(n)\left\{\frac{1}{1-d_1}\sum_{s=\alpha}^{\beta}[q(s)+q_1(s)][|f(s)|+|\Delta f(s)|]\right\}, \tag{2.8}$$

for $n \in N_{\alpha,\beta}$.

Proof. Let $x \in D(N_{\alpha,\beta}, R^m)$ be a solution of equation (1.1). Then from the hypotheses, we have

$$\begin{aligned}
&|x(n)| + |\Delta x(n)| \\
&\leq |f(n)| + \sum_{s=\alpha}^{\beta}|g(n,s,x(s),\Delta x(s))| + |\Delta_1 f(n)| + \sum_{s=\alpha}^{\beta}|\Delta_1 g(n,s,x(s),\Delta x(s))| \\
&\leq |f(n)| + |\Delta f(n)| + \sum_{s=\alpha}^{\beta} p(n)\, q(s)[|x(s)| + |\Delta x(s)|] \\
&\qquad + \sum_{s=\alpha}^{\beta} p(n)\, q_1(s)[|x(s)| + |\Delta x(s)|] \\
&= |f(n)| + |\Delta f(n)| + p(n)\sum_{s=\alpha}^{\beta}[q(s)+q_1(s)][|x(s)| + |\Delta x(s)|]. \qquad (2.9)
\end{aligned}$$

Now, an application of Lemma to (2.9) gives the desired estimate in (2.8).

Remark 1. We note that the estimate obtained in (2.8) yields not only the bound on the solution $x(n)$ of equation (1.1), but also the bound on $\Delta x(n)$. If the bound on the right hand side in (2.8) is bounded, then the solution $x(n)$ of equation (1.1) and $\Delta x(n)$ are bounded.

Next, we shall obtain the estimate on the solution of equation (1.1), assuming that the functions $g, \Delta_1 g$ satisfy Lipschitz type conditions.

Theorem 2. *Suppose that the functions $g, \Delta_1 g$ satisfy the conditions*

$$|g(n,s,u,v) - g(n,s,\bar{u},\bar{v})| \leq r(n)\, w(s)[|u-\bar{u}| + |v-\bar{v}|], \tag{2.10}$$

$$|\Delta_1 g(n,s,u,v) - \Delta_1 g(n,s,\bar{u},\bar{v})| \le r(n)\, w_1(s)\,[|u-\bar{u}| + |v-\bar{v}|], \tag{2.11}$$

where $r, w, w_1 \in D(N_{\alpha,\beta}, R_+)$ *and*

$$d_2 = \sum_{s=\alpha}^{\beta} [w(s) + w_1(s)]\, r(s) < 1. \tag{2.12}$$

If $x \in D(N_{\alpha,\beta}, R^m)$ *is any solution of equation (1.1), then*

$$|x(n) - f(n)| + |\Delta x(n) - \Delta f(n)|$$
$$\le Q(n) + r(n)\left\{\frac{1}{1-d_2}\sum_{s=\alpha}^{\beta}[w(s) + w_1(s)]\, Q(s)\right\}, \tag{2.13}$$

for $n \in N_{\alpha,\beta}$, *where*

$$Q(n) = \sum_{\tau=\alpha}^{\beta} [|g(n,\tau,f(\tau),\Delta f(\tau))| + |\Delta_1 g(n,\tau,f(\tau),\Delta f(\tau))|]. \tag{2.14}$$

Proof. Since $x(n)$ is a solution of equation (1.1), by using the hypotheses, we have

$$|x(n) - f(n)| + |\Delta x(n) - \Delta f(n)|$$
$$\le \sum_{s=\alpha}^{\beta} |g(n,s,x(s),\Delta x(s)) - g(n,s,f(s),\Delta f(s))| + \sum_{s=\alpha}^{\beta} |g(n,s,f(s),\Delta f(s))|$$
$$+ \sum_{s=\alpha}^{\beta} |\Delta_1 g(n,s,x(s),\Delta x(s)) - \Delta_1 g(n,s,f(s),\Delta f(s))| + \sum_{s=\alpha}^{\beta} |\Delta_1 g(n,s,f(s),\Delta f(s))|$$
$$\le Q(n) + \sum_{s=\alpha}^{\beta} r(n)\, w(s)\,[|x(s) - f(s)| + |\Delta x(s) - \Delta f(s)|]$$
$$+ \sum_{s=\alpha}^{\beta} r(n)\, w_1(s)\,[|x(s) - f(s)| + |\Delta x(s) - \Delta f(s)|]$$
$$= Q(n) + r(n)\sum_{s=\alpha}^{\beta}[w(s) + w_1(s)]\,[|x(s) - f(s)| + |x(s) - f(s)| + |\Delta x(s) - \Delta f(s)|]. \tag{2.15}$$

Now, an application of Lemma to (2.15) yields (2.13).

3. Uniqueness and Continuous Dependence

In this section, we study the uniqueness of solutions of equation (1.1) and also the dependency of solutions of equations of the form (1.1) on patrameters.

The following theorem deals with the uniqueness of solutions of equation (1.1).

Theorem 3. *Suppose that the functions $g, \Delta_1 g$ satisfy the conditions (2.10), (2.11) and the condition (2.12) holds. Then the equation (1.1) has at most one solution on $N_{\alpha,\beta}$.*

Proof. Let $x_1(n)$ and $x_2(n)$ be two solutions of equation (1.1) on $N_{\alpha,\beta}$. Using these facts and the hypotheses, we have

$$\begin{aligned}
&|x_1(n) - x_2(n)| + |\Delta x_1(n) - \Delta x_2(n)| \\
&\leq \sum_{s=\alpha}^{\beta} |g(n, s, x_1(s), \Delta x_1(s)) - g(n, s, x_2(s), \Delta x_2(s))| \\
&+ \sum_{s=\alpha}^{\beta} |\Delta_1 g(n, s, x_1(s), \Delta x_1(s)) - \Delta_1 g(n, s, x_2(s), \Delta x_2(s))| \\
&\leq \sum_{s=\alpha}^{\beta} r(n) w(s) [|x_1(s) - x_2(s)| + |\Delta x_1(s) - \Delta x_2(s)|] \\
&+ \sum_{s=\alpha}^{\beta} r(n) w_1(s) [|x_1(s) - x_2(s)| + |\Delta x_1(s) - \Delta x_2(s)|] \\
&= r(n) \sum_{s=\alpha}^{\beta} [w(s) + w_1(s)] [|x_1(s) - x_2(s)| + |\Delta x_1(s) - \Delta x_2(s)|]. \qquad (3.1)
\end{aligned}$$

Now, a suitable application of Lemma to (3.1) yields $|x_1(n) - x_2(n)| + |\Delta x_1(n) - \Delta x_2(n)| \leq 0$, which implies $x_1(n) = x_2(n)$. Thus, there is at most one solution to equation (1.1).

We next consider the following sum-difference equations

$$z(n) = f(n) + \sum_{s=\alpha}^{\beta} g(n, s, z(s), \Delta z(s), \mu), \tag{3.2}$$

$$z(n) = f(n) + \sum_{s=\alpha}^{\beta} g(n, s, z(s), \Delta z(s), \mu_0), \tag{3.3}$$

for $n \in N_{\alpha,\beta}$, where $f \in D(N_{\alpha,\beta}, R^m), g \in D\left(N_{\alpha,\beta}^2 \times R^m \times R^m \times R, R^m\right)$ and μ, μ_0 are real parameters.

Finally, we give the following theorem which shows the dependency of solutions of equations (3.2), (3.3) on parameters.

Theorem 4. *Suppose that the functions $g, \Delta_1 g$ satisfy the conditions*

$$|g(n, s, u, v, \mu) - g(n, s, \bar{u}, \bar{v}, \mu)| \le \bar{r}(n)\, \bar{w}(s)\, [|u - \bar{u}| + |v - \bar{v}|], \tag{3.4}$$

$$|g(n, s, u, v, \mu) - g(n, s, u, v, \mu_0)| \le \gamma(n, s)\, |\mu - \mu_0|, \tag{3.5}$$

$$|\Delta_1 g(n, s, u, v, \mu) - \Delta_1 g(n, s, \bar{u}, \bar{v}, \mu)| \le \bar{r}(n)\, \bar{w}_1(s)\, [|u - \bar{u}| + |v - \bar{v}|], \tag{3.6}$$

$$|\Delta_1 g(n, s, u, v, \mu) - \Delta_1 g(n, s, u, v, \mu_0)| \le \gamma_1(n, s)\, |\mu - \mu_0|, \tag{3.7}$$

where $\bar{r}, \bar{w}, \bar{w}_1 \in D(N_{\alpha,\beta}, R_+)$, $\gamma, \gamma_1 \in D\left(N_{\alpha,\beta}^2, R_+\right)$. Let

$$\bar{\gamma}(n) = |\mu - \mu_0| \sum_{\tau=\alpha}^{\beta} [\gamma(n, \tau) + \gamma_1(n, \tau)], \tag{3.8}$$

and suppose that

$$d_3 = \sum_{s=\alpha}^{\beta} [\bar{w}(s) + \bar{w}_1(s)]\, \bar{r}(s) < 1. \tag{3.9}$$

Let $z_1(n)$ and $z_2(n)$ be the solutions of equations (3.2) and (3.3) respectively. Then

$$|z_1(n) - z_2(n)| + |\Delta z_1(n) - \Delta z_2(n)|$$
$$\le \bar{\gamma}(n) + \bar{r}(n) \left\{ \frac{1}{1 - d_3} \sum_{s=\alpha}^{\beta} [\bar{w}(s) + \bar{w}_1(s)]\, \bar{\gamma}(s) \right\}, \tag{3.10}$$

for $n \in N_{\alpha,\beta}$.

Proof. Let $w(n) = z_1(n) - z_2(n)$. Using the facts that $z_1(n)$ and $z_2(n)$ are the solutions of equations (3.2) and (3.3), we have

$$\begin{aligned}
&|w(n)| + |\Delta w(n)| \\
&\leq \sum_{s=\alpha}^{\beta} |g(n,s,z_1(s),\Delta z_1(s),\mu) - g(n,s,z_2(s),\Delta z_2(s),\mu_0)| \\
&+ \sum_{s=\alpha}^{\beta} |\Delta_1 g(n,s,z_1(s),\Delta z_1(s),\mu) - \Delta_1 g(n,s,z_2(s),\Delta z_2(s),\mu_0)| \\
&\leq \sum_{s=\alpha}^{\beta} |g(n,s,z_1(s),\Delta z_1(s),\mu) - g(n,s,z_2(s),\Delta z_2(s),\mu)| \\
&+ \sum_{s=\alpha}^{\beta} |g(n,s,z_2(s),\Delta z_2(s),\mu) - g(n,s,z_2(s),\Delta z_2(s),\mu_0)| \\
&+ \sum_{s=\alpha}^{\beta} |\Delta_1 g(n,s,z_1(s),\Delta z_1(s),\mu) - \Delta_1 g(n,s,z_2(s),\Delta z_2(s),\mu)| \\
&+ \sum_{s=\alpha}^{\beta} |\Delta_1 g(n,s,z_2(s),\Delta z_2(s),\mu) - \Delta_1 g(n,s,z_2(s),\Delta z_2(s),\mu_0)| \\
&\leq \sum_{s=\alpha}^{\beta} \bar{r}(n)\bar{w}(s)[|z_1(s) - z_2(s)| + |\Delta z_1(s) - \Delta z_2(s)|] \\
&+ \sum_{s=\alpha}^{\beta} \gamma(n,s)|\mu - \mu_0| \\
&+ \sum_{s=\alpha}^{\beta} \bar{r}(n)\bar{w}_1(s)[|z_1(s) - z_2(s)| + |\Delta z_1(s) - \Delta z_2(s)|] \\
&+ \sum_{s=\alpha}^{\beta} \gamma_1(n,s)|\mu - \mu_0| \\
&= \bar{\gamma}(n) + \bar{r}(n)\sum_{s=\alpha}^{\beta}[\bar{w}(s) + \bar{w}_1(s)][|w(s)| + |\Delta w(s)|]. \qquad (3.11)
\end{aligned}$$

Now an application of Lemma to (3.11) yields (3.10), which shows the dependency of solutions of equations (3.2) and (3.3) on parameters.

Remark 2. We note that the results obtained in this paper can be very easily extended to the study of discrete analogue of the more general Fredholm type integrodifferential equation recently studied in [9]. Moreover, our approach here can be used to study the qualitative behavior of solutions of equations of the form (1.1) in more than one dimension. Naturally, these considerations will make the analysis more complicated and we leave them for future investigations.

References

[1] K. E. Atkinson, The numerical solution of Fredholm integral equations of the second kind, *SIAM J. Numer. Anal.,* **4**(1967), 337-348.

[2] A. M. Bica, V.A. Căuş, and S. Mureşan, Application of a trapezoid inequality to neutral Fredholm Integro-differential equations in Banach spaces, *J. Inequal. Pure and Appl. Math.,* **7** no. 5, Art.173, 2006.

[3] H. Brunner and P. J. Van der Houwen, *The numerical solution of Volterra equations*, CWI Monographs, **3**, North-Holland, Amsterdam, 1986.

[4] B. Cahlon and D. Westreich, Complete continuity of integro-differential operators with discontinuous kernels and collectively compact approximations, *J. Math. Anal. Appl.,* **71**(1979), 313-332.

[5] B. Cahlon, L. J. Nachman, and D. Schmidt, Numerical solution of Volterra integral equations with delay arguments, *J. Integral Equations*, **7**(1984), 191-208.

[6] M. Kwapisz, Some existence and uniqueness results for boundary value problems for difference equations, *Applicable Analysis*, **37**(1990), 169-182.

[7] B. G. Pachpatte, *Inequalities for Finite Difference Equations*, Marcel Dekker, Inc., New York, 2002.

[8] B. G. Pachpatte, *Integral and Finite Difference Inequalities and Applications*, North-Holland Mathematics Studies, Vol. 205, Elsevier Science B.V., Amsterdam 2006.

[9] B. G. Pachpatte, On Fredholm type integrodifferential equation, *Tamkang J. Math.,* **39**(2008), 85-94.

[10] B. G. Pachpatte, On Volterra and Fredholm type integrodifferential equations, *Tamusi Oxford J. Math. Sci.,* **24**(2008), 289-310.