Tamsui Oxford Journal of Mathematical Sciences **26(4)** (2010) 397-409
Aletheia University

Some New Ostrowski-Grüss Type Inequalities with Two Mappings and Applications*

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Received April 2, 2009, Accepted August 10, 2010.

Abstract

In this paper we establish some new Ostrowski-Grüss type inequalities with two mappings. Some applications for two dependent cumulative distribution functions are also given.

Keywords and Phrases: *Ostrowski-Grüss type inequality, Lipschizian type function, Grüss type inequality, Cumulative distribution function.*

1. Introduction

In 1938, Ostrowski proved that the following interesting and useful theorem (see [1,p.468]).

Theorem 1.1. *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in $\mathring{A}$, the interior of A, and let $a, b \in \mathring{A}$ with $a < b$. If $\| f' \|_\infty = \sup_{x\in(a,b)} |f'(x)| < \infty$, then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right](b-a) \| f' \|_\infty, \quad \forall x \in [a,b]. \quad (1)$$

*2000 *Mathematics Subject Classification.* Primary 26D10, 26D15.
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Grüss [2] obtained the following well-known theorem (Grüss type inequality) which establishes a connection between the integral of the product of two functions and the product of the integrals of the two functions.

Theorem 1.2. *Let $f, g : [a, b] \to \mathbb{R}$ be two integralable functions such that $\phi \le f(x) \le \Phi$ and $\gamma \le g(x) \le \Gamma$ for all $x \in [a, b]$, $\phi, \Phi, \gamma, \Gamma$ are constants. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx\right| \le \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma). \quad (2)$$

Dragomir and Wang [3] first derived a new inequality of Ostrowski type using Grüss type inequality (2) as follows.

Theorem 1.3. *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in $\mathring{A}$ and let $a, b \in \mathring{A}$ with $a < b$. If $f' \in L_1[a, b]$ and $\gamma \le f'(x) \le \Gamma$, for all $x \in [a, b]$, γ, Γ are constants. Then we have the following inequality:*

$$\left|f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(y)dy\right| \le \frac{1}{4}(b-a)(\Gamma-\gamma),$$
$$\forall x \in [a, b]. \quad (3)$$

Cheng [4] gave a sharp version of the Ostrowski-Grüss type integral inequality, i.e., $\frac{1}{4}$ is replaced by $\frac{1}{8}$ in inequality (3). Ujević [5] provided new estimations of the left part of (1) as follows.

Theorem 1.4. *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping in $\mathring{A}$ and let $a, b \in \mathring{A}$ with $a < b$. If $f' \in L_1[a, b]$ and $\gamma \le f'(x) \le \Gamma$, for all $x \in [a, b]$, γ, Γ are constants. Then for all $x \in [a, b]$, we have the following inequalities:*

$$\left|f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(y)dy\right| \le \frac{1}{2}(b-a)(S-\gamma), \quad (4)$$

$$\left|f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(y)dy\right| \le \frac{1}{2}(b-a)(\Gamma-S), \quad (5)$$

where $S = (f(b) - f(a))/(b - a)$.

In order to narrate conveniently, we cites the definition of $L-$Lipschitzian and $(l, L)-$Lipschitzian.

Definition 1.5.[6] The function $f : [a, b] \to \mathbb{R}$ is said to be $L-$Lipschitzian on $[a, b]$ if

$$|f(x) - f(y)| \leq L|x - y|, \forall x, y \in [a, b],$$

where $L > 0$ is given.

Definition 1.6.[6] The function $f : [a, b] \to \mathbb{R}$ is said to be $(l, L)-$Lipschitzian on $[a, b]$ if

$$l(x - y) \leq f(x) - f(y) \leq L(x - y), \forall x, y \in [a, b],$$

where $l, L \in \mathbb{R}$ with $l < L$.

By means of the $L-$Lipschitzian and $(l, L)-$Lipschitzian properties of the functions, Z. Liu [6] gave the following results.

Theorem 1.7. *Let $f : [a, b] \to \mathbb{R}$ be $(l, L)-$Lipschitzian on $[a, b]$. Then we have*

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) S - \frac{1}{b-a} \int_a^b f(y)dy \right| \leq \frac{1}{8}(b-a)(L-l), \quad (6)$$

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) S - \frac{1}{b-a} \int_a^b f(y)dy \right| \leq \frac{1}{2}(b-a)(S-l), \quad (7)$$

and

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) S - \frac{1}{b-a} \int_a^b f(y)dy \right| \leq \frac{1}{2}(b-a)(L-S), \quad (8)$$

for all $x \in [a, b]$, where $S = (f(b) - f(a))/(b - a)$.

Theorem 1.8. *Let $f : [a, b] \to \mathbb{R}$ be $L-$Lipschitzian on $[a, b]$. Then we have*

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) S - \frac{1}{b-a} \int_a^b f(y)dy \right| \leq \frac{1}{4}(b-a)L, \quad (9)$$

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) S - \frac{1}{b-a} \int_a^b f(y)dy \right| \leq \frac{1}{2}(b-a)(\Gamma - |S|), \quad (10)$$

for all $x \in [a, b]$, where $S = (f(b) - f(a))/(b - a)$.

In [7], Pachpatte established the inequality involving two functions similar to (1) as follows.

Theorem 1.9. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions on $[a, b]$ and differentiable on (a, b), whose derivatives $f', g' : (a, b) \to \mathbb{R}$ are bounded on (A, B), i.e., $\|f'\|_\infty = \sup_{x\in(a,b)} |f'(x)| < \infty$, $\|g'\|_\infty = \sup_{x\in(a,b)} |g'(x)| < \infty$. Then*

$$\mathcal{F}(f,g) \leq \frac{1}{2}\left[|g(x)|\|f'\|_\infty + |f(x)|\|g'\|_\infty\right]\left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a), \forall x \in [a,b]. \quad (11)$$

where $\mathcal{F}(f,g) = \left|f(x)g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(y)dy + f(x)\int_a^b g(y)dy\right]\right|$.

In this paper, we establish some new estimations of inequality (11) similar to Theorem 1.3, 1.4, 1.7, 1.8. Some applications for two dependent cumulative distribution functions are also given.

2. Main Results

Theorem 2.1. *Let $f, g : A \subseteq \mathbb{R} \to \mathbb{R}$ be two differentiable mappings in $\mathring{A}$ and let $a, b \in \mathring{A}$ with $a < b$. If $f', g' \in L_1[a, b]$, $\gamma = \min_{x\in[a,b]} f'(x)$, $\Gamma = \max_{x\in[a,b]} f'(x)$, $\phi = \min_{x\in[a,b]} g'(x)$, and $\Phi = \max_{x\in[a,b]} f'(x)$. Then we have the following inequality:*

$$\mathcal{G}(f,g) \leq \frac{1}{8}(b-a)[(\Gamma-\gamma)|g(x)| + (\Phi-\phi)|f(x)|], \quad \forall x \in [a,b]. \quad (12)$$

where

$$\mathcal{G}(f,g) = \Bigg|f(x)g(x) - \frac{1}{2(b-a)}\Bigg[g(x)\left(\int_a^b f(y)dy + (f(b)-f(a))\left(x-\frac{a+b}{2}\right)\right) + f(x)\left(\int_a^b g(y)dy + (g(b)-g(a))\left(x-\frac{a+b}{2}\right)\right)\Bigg]\Bigg|.$$

Proof. By Theorem 1.3, we have

$$\left|f(x) - \frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(y)dy\right| \leq \frac{1}{4}(b-a)(\Gamma-\gamma), \quad (13)$$

$$\left|g(x) - \frac{g(b)-g(a)}{b-a}\left(x-\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b g(y)dy\right| \leq \frac{1}{4}(b-a)(\Phi-\phi). \quad (14)$$

In combination with (13) and (14), we obtain

$$\begin{aligned}
\mathcal{G}(f,g) &= \left|\frac{1}{2}\left\{g(x)\left[f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(y)dy\right]\right.\right.\\
&\quad\left.\left.+f(x)\left[g(x)-\frac{g(b)-g(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(y)dy\right]\right\}\right|\\
&\leq \frac{1}{2}\left[|g(x)|\left|f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(y)dy\right|\right.\\
&\quad\left.+|f(x)|\left|g(x)-\frac{g(b)-g(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(y)dy\right|\right]\\
&\leq \frac{1}{8}(b-a)[(\Gamma-\gamma)|g(x)|+(\Phi-\phi)|f(x)|].
\end{aligned}$$

Remark 2.2. By means of the method of Cheng[4], we give also a sharp version of inequality (12), i.e., $\frac{1}{8}$ is replaced by $\frac{1}{16}$.

Theorem 2.3. *Let the assumptions of Theorem 2.1 hold. Then we have the following inequalities:*

$$\mathcal{G}(f,g) \leq \frac{1}{4}(b-a)[(S-\gamma)|g(x)|+(T-\phi)|f(x)|], \tag{15}$$

$$\mathcal{G}(f,g) \leq \frac{1}{4}(b-a)[(\Gamma-S)|g(x)|+(\Phi-T)|f(x)|], \tag{16}$$

for all $x \in [a,b]$, *where* $S = (f(b)-f(a))/(b-a)$, $T = (g(b)-g(a))/(b-a)$.

Proof. The proof of inequality (15) is only given, since the proof of inequality (16) is the similar to that of inequality (15). By Theorem 1.4, we have

$$\left|f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(y)dy\right| \leq \frac{1}{2}(b-a)(S-\gamma), \tag{17}$$

$$\left|g(x)-\frac{g(b)-g(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(y)dy\right| \leq \frac{1}{2}(b-a)(T-\phi). \tag{18}$$

In combination with (17) and (18), we obtain

$$\begin{aligned}\mathcal{G}(f,g) &\le \frac{1}{2}\Bigg[|g(x)|\left|f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(y)dy\right| \\ &\quad +|f(x)|\left|g(x)-\frac{g(b)-g(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(y)dy\right|\Bigg] \\ &\le \frac{1}{4}(b-a)[(S-\gamma)|g(x)|+(T-\phi)|f(x)|].\end{aligned}$$

Theorem 2.4. *Let $f,g:[a,b]\to\mathbb{R}$ is $(l,L),(m,M)-$Lipschitzian on $[a,b]$, respectively. Then we have*

$$\mathcal{G}(f,g)\le\frac{1}{16}(b-a)[(L-l)|g(x)|+(M-m)|f(x)|], \tag{19}$$

$$\mathcal{G}(f,g)\le\frac{1}{4}(b-a)[(S-l)|g(x)|+(T-m)|f(x)|], \tag{20}$$

and

$$\mathcal{G}(f,g)\le\frac{1}{4}(b-a)[(L-S)|g(x)|+(M-T)|f(x)|], \tag{21}$$

for all $x\in[a,b]$, where $S=(f(b)-f(a))/(b-a)$, $T=(g(b)-g(a))/(b-a)$.

Proof. The proof of inequality (19) is only given, since the proofs of inequalities (20) and (21) are the similar to that of inequality (19). By inequality (6) in Theorem 1.7, we have

$$\left|f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(y)dy\right|\le\frac{1}{8}(b-a)(L-l), \tag{22}$$

$$\left|g(x)-\frac{g(b)-g(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(y)dy\right|\le\frac{1}{8}(b-a)(M-m). \tag{23}$$

In combination with (22) and (23), we obtain

$$\begin{aligned}\mathcal{G}(f,g) &\le \frac{1}{2}\Bigg[|g(x)|\left|f(x)-\frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(y)dy\right| \\ &\quad +|f(x)|\left|g(x)-\frac{g(b)-g(a)}{b-a}\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(y)dy\right|\Bigg] \\ &\le \frac{1}{4}(b-a)[(L-l)|g(x)|+(M-m)|f(x)|].\end{aligned}$$

Remark 2.5. We also obtain the following inequalities:

$$\begin{aligned}
\mathcal{G}(f,g) &\leq \frac{1}{4}(b-a)\left[1/4(S-l)|g(x)|+(T-m)|f(x)|\right], \\
\mathcal{G}(f,g) &\leq \frac{1}{4}(b-a)\left[1/4(S-l)|g(x)|+(M-T)|f(x)|\right], \\
\mathcal{G}(f,g) &\leq \frac{1}{4}(b-a)\left[(S-l)|g(x)|+1/4(M-m)|f(x)|\right], \\
\mathcal{G}(f,g) &\leq \frac{1}{4}(b-a)\left[(L-S)|g(x)|+1/4(M-m)|f(x)|\right], \\
\mathcal{G}(f,g) &\leq \frac{1}{4}(b-a)[(S-l)|g(x)|+(M-T)|f(x)|],
\end{aligned}$$

and

$$\mathcal{G}(f,g) \leq \frac{1}{4}(b-a)[(L-S)|g(x)|+(T-m)|f(x)|],$$

for all $x \in [a,b]$, where $S=(f(b)-f(a))/(b-a)$, $T=(g(b)-g(a))/(b-a)$.

Corollary 2.6. *Under the assumptions of Theorem 2.4, we have*

$$\begin{aligned}
\mathcal{M}(f,g) &\leq \frac{1}{16}(b-a)\left[(L-l)\left|g\left(\frac{a+b}{2}\right)\right|+(M-m)\left|f\left(\frac{a+b}{2}\right)\right|\right], \\
\mathcal{M}(f,g) &\leq \frac{1}{4}(b-a)\left[(S-l)\left|g\left(\frac{a+b}{2}\right)\right|+(T-m)\left|f\left(\frac{a+b}{2}\right)\right|\right],
\end{aligned}$$

and

$$\mathcal{M}(f,g) \leq \frac{1}{4}(b-a)\left[(L-S)\left|g\left(\frac{a+b}{2}\right)\right|+(M-T)\left|f\left(\frac{a+b}{2}\right)\right|\right],$$

where $S=(f(b)-f(a))/(b-a)$, $T=(g(b)-g(a))/(b-a)$, *and*

$$\begin{aligned}
&\mathcal{M}(f,g) \\
&= \left|f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{2(b-a)}\left[g\left(\frac{a+b}{2}\right)\int_a^b f(y)dy+f\left(\frac{a+b}{2}\right)\int_a^b g(y)dy\right]\right|.
\end{aligned}$$

Proof. We set $x=\frac{a+b}{2}$ in (19)-(21) to get Corollary 2.6.

Corollary 2.7. *Under the assumptions of Theorem 2.4, we have*

$$\mathcal{N}(f,g) \le \frac{1}{8}(b-a)[(L-l)(|g(a)|+|g(b)|)+(M-m)(|f(a)|+|f(b)|)],$$
$$\mathcal{N}(f,g) \le \frac{1}{2}(b-a)[(S-l)(|g(a)|+|g(b)|)+(T-m)(|f(a)|+|f(b)|)],$$

and

$$\mathcal{N}(f,g) \le \frac{1}{2}(b-a)[(L-S)(|g(a)|+|g(b)|)+(M-T)(|f(a)|+|f(b)|)],$$

where $S=(f(b)-f(a))/(b-a)$, $T=(g(b)-g(a))/(b-a)$,

$$\begin{aligned}&\mathcal{N}(f,g)\\ =&\left|(f(b)g(b)-f(a)g(a))-\frac{1}{b-a}\left[(g(b)-g(a))\int_a^b f(y)dy+(f(b)-f(a))\int_a^b g(y)dy\right]\right|.\end{aligned}$$

Proof. We set $x=a$ and $x=b$ in $\mathcal{G}(f,g)$ defined in Theorem 2.1, we have

$$\begin{aligned}&\mathcal{G}(f(a),g(a))\\ &=\left|\frac{f(a)g(a)}{2}+\frac{f(a)g(b)+f(b)g(a)}{4}-\frac{1}{2(b-a)}\left[g\,(a)\int_a^b f(y)dy+f\,(a)\int_a^b g(y)dy\right]\right|,\\ &\mathcal{G}(f(b),g(b))\\ &=\left|\frac{f(b)g(b)}{2}+\frac{f(a)g(b)+f(b)g(a)}{4}-\frac{1}{2(b-a)}\left[g\,(b)\int_a^b f(y)dy+f\,(b)\int_a^b g(y)dy\right]\right|.\end{aligned}$$

Since

$$\begin{aligned}&\mathcal{N}(f,g)\\ =\;&2\left|\frac{(f(b)g(b)-f(a)g(a))}{2}-\frac{1}{2(b-a)}\left[(g(b)-g(a))\int_a^b f(y)dy+(f(b)-f(a))\int_a^b g(y)dy\right]\right|\\ =\;&2\left|\left\{\frac{f(a)g(a)}{2}+\frac{f(a)g(b)+f(b)g(a)}{4}-\frac{1}{2(b-a)}\left[g\,(a)\int_a^b f(y)dy+f\,(a)\int_a^b g(y)dy\right]\right\}\right.\\ &\left.-\left\{\frac{f(b)g(b)}{2}+\frac{f(a)g(b)+f(b)g(a)}{4}-\frac{1}{2(b-a)}\left[g\,(b)\int_a^b f(y)dy+f\,(b)\int_a^b g(y)dy\right]\right\}\right|\\ \le\;&2(\mathcal{G}(f(a),g(a))+\mathcal{G}(f(b),g(b))).\end{aligned}$$

In combination with the right of (19)-(21), we get Corollary 2.7.

Theorem 2.8. *Let $f, g : [a, b] \to \mathbb{R}$ is $L, M-$Lipschitzian on $[a, b]$, respectively. Then we have*

$$\mathcal{G}(f, g) \leq \frac{1}{8}(b-a)[M|g(x)| + L|f(x)|], \tag{24}$$

$$\mathcal{G}(f, g) \leq \frac{1}{4}(b-a)[(L-|S|)|g(x)| + (M-|T|)|f(x)|], \tag{25}$$

for all $x \in [a, b]$, where $S = (f(b) - f(a))/(b-a)$, $T = (g(b) - g(a))/(b-a)$.

Proof. It is easy to build the proof of Theorem 2.6, since the the proofs of Theorem 2.6 and Theorem 1.8 are similar. So it is omitted.

Remark 2.9. Let $g(x) = 1$, then $\phi = \Phi = 0$. We recapture Theorem 1.3, 1.4, 1.7, 1.8 from Theorem 2.1, 2.3, 2.4, 2.6, respectively.

3. Some Applications

Let X (Y) be a random variable having the probability density function f $(g) : [a, b] \to \mathbb{R}+$ and the cumulative distribution function $F(x) = Pr(X \leq x)$ $(G(y) = Pr(Y \leq y))$, i.e.,

$$F(x) = \int_a^x f(t)dt \quad \left(G(y) = \int_a^y g(t)dt\right), \quad \forall x \in [a, b].$$

Joint distribution function of X and Y is defined by $F(x, y) = Pr(X \leq x, Y \leq y)$. Let X and Y be two dependent random variables, then $Pr(X \leq x, Y \leq y) = Pr(X \leq x) \cdot Pr(Y \leq y)$. $E(X)$ is the expectation of X. Then we have the following inequalities.

Theorem 3.1. *Let X and Y be two dependent random variables and F and G be defined by the above. If there exist constants L, l, M, m such that $0 \leq l \leq f(t) \leq L$ and $0 \leq m \leq g(t) \leq M$ for all $t \in [a, b]$, then we have the inequalities*

$$\mathcal{P}(X, Y) \leq \frac{1}{16}(b-a)[(L-l)G(x) + (M-m)F(x)], \tag{26}$$

$$\mathcal{P}(X, Y) \leq \frac{1}{4}(b-a)[(S-l)G(x) + (T-m)F(x)], \tag{27}$$

and

$$\mathcal{P}(X, Y) \leq \frac{1}{4}(b-a)[(L-S)G(x) + (M-T)F(x)], \tag{28}$$

for all $x \in [a,b]$, *where* $S = T = (b-a)^{-1}$, *and*

$$\mathcal{P}(X,Y) = \Bigg| Pr(X \le x, Y \le x) - \frac{1}{2(b-a)} \Bigg[G(x) \left((b - E(X)) + \left(x - \frac{a+b}{2} \right) \right) + F(x) \left((b - E(Y)) + \left(x - \frac{a+b}{2} \right) \right) \Bigg] \Bigg|.$$

Proof. It is see to observe that $F(x) = \int_a^x f(t)dt$, $G(x) = \int_a^x g(t)dt$ is (l,L), (m,M)-Lipschitzian on $[a,b]$, respectively. So, by Theorem 2.4, we get

$$\mathcal{G}(F,G) \le \frac{1}{16}(b-a)[(L-l)|G(x)| + (M-m)|F(x)|],$$
$$\mathcal{G}(F,G) \le \frac{1}{4}(b-a)[(S-l)|G(x)| + (T-m)|F(x)|],$$

and

$$\mathcal{G}(F,G) \le \frac{1}{4}(b-a)[(L-S)|G(x)| + (M-T)|F(x)|],$$

where $\mathcal{G}(*,\circ)$ is defined in Theorem 2.1. As $F(a) = G(a) = 0$, $F(b) = G(b) = 1$,

$$\int_a^b F(t)dt = b - E(X) \quad \text{and} \quad \int_a^b G(t)dt = b - E(Y),$$

then $\mathcal{G}(F,G) = \mathcal{P}(X,Y)$, this completes the proof Theorem 3.1.

Remark 3.2. We also obtain the following inequalities:

$$\mathcal{P}(X,Y) \le \frac{1}{4}(b-a)\left[1/4(S-l)G(x) + (T-m)F(x)\right],$$
$$\mathcal{P}(X,Y) \le \frac{1}{4}(b-a)\left[1/4(S-l)G(x) + (M-T)F(x)\right],$$
$$\mathcal{P}(X,Y) \le \frac{1}{4}(b-a)\left[(S-l)G(x) + 1/4(M-m)F(x)\right],$$
$$\mathcal{P}(X,Y) \le \frac{1}{4}(b-a)\left[(L-S)G(x) + 1/4(M-m)F(x)\right],$$
$$\mathcal{P}(X,Y) \le \frac{1}{4}(b-a)[(S-l)G(x) + (M-T)F(x)],$$

and

$$\mathcal{P}(X,Y) \le \frac{1}{4}(b-a)[(L-S)G(x) + (T-m)F(x)],$$

for all $x \in [a,b]$, where $S = T = (b-a)^{-1}$.

Corollary 3.3. *Under the assumptions of Theorem 3.1, we have*

$$|(E(X)+E(Y))-(a+b)| \le \frac{1}{8}(b-a)^2[(L-l)+(M-m)], \quad (29)$$

$$|(E(X)+E(Y))-(a+b)| \le \frac{1}{2}(b-a)^2[(S-l)+(T-m)], \quad (30)$$

and

$$|(E(X)+E(Y))-(a+b)| \le \frac{1}{2}(b-a)^2[(L-S)+(M-T)], \quad (31)$$

where $S = T = (b-a)^{-1}$.

Proof. We set $x = b$ in (26)-(28) to get (29)-(31).

Corollary 3.4. *Under the assumptions of Theorem 3.1, we have*

$$\mathcal{Q}(X,Y) \le \frac{1}{16}(b-a)\left[(L-l)G\left(\frac{a+b}{2}\right)+(M-m)F\left(\frac{a+b}{2}\right)\right], (32)$$

$$\mathcal{Q}(X,Y) \le \frac{1}{4}(b-a)\left[(S-l)G\left(\frac{a+b}{2}\right)+(T-m)F\left(\frac{a+b}{2}\right)\right], \quad (33)$$

and

$$\mathcal{Q}(X,Y) \le \frac{1}{4}(b-a)\left[(L-S)G\left(\frac{a+b}{2}\right)+(M-T)F\left(\frac{a+b}{2}\right)\right], \quad (34)$$

where $S = T = (b-a)^{-1}$, *and*

$$\mathcal{Q}(X,Y) = \left|Pr\left(X \le \frac{a+b}{2}, Y \le \frac{a+b}{2}\right) - \frac{1}{2(b-a)}\left[G\left(\frac{a+b}{2}\right)(b-E(X))+F\left(\frac{a+b}{2}\right)(b-E(Y))\right]\right|.$$

Proof. We set $x = b$ in (26)-(28) to get (32)-(34).

Corollary 3.5. *Under the assumptions of Theorem 3.1, we have*

$$\mathcal{R}(X,Y) \le \frac{1}{8}(b-a)\left[(L-l)G\left(\frac{a+b}{2}\right)+(M-m)F\left(\frac{a+b}{2}\right)\right], (35)$$

$$\mathcal{R}(X,Y) \le \frac{1}{2}(b-a)\left[(S-l)G\left(\frac{a+b}{2}\right)+(T-m)F\left(\frac{a+b}{2}\right)\right], \quad (36)$$

and

$$\mathcal{R}(X,Y) \le \frac{1}{2}(b-a)\left[(L-S)G\left(\frac{a+b}{2}\right) + (M-T)F\left(\frac{a+b}{2}\right)\right], \qquad (37)$$

where $S = T = (b-a)^{-1}$, *and*

$$\mathcal{R}(X,Y) = \left|Pr\left(X \le \frac{a+b}{2}, Y \le \frac{a+b}{2}\right) - \frac{1}{4}\left[G\left(\frac{a+b}{2}\right) + F\left(\frac{a+b}{2}\right)\right]\right|.$$

Proof. Using the triangle inequality, we get

$$\begin{aligned}
\mathcal{R}(X,Y) &= \left|Pr\left(X \le \frac{a+b}{2}, Y \le \frac{a+b}{2}\right) - \frac{1}{4}\left[G\left(\frac{a+b}{2}\right) + F\left(\frac{a+b}{2}\right)\right]\right. \\
&\quad + \frac{1}{2}\left[\frac{G\left(\frac{a+b}{2}\right)}{b-a}\left(E(X) - \frac{a+b}{2}\right) + \frac{F\left(\frac{a+b}{2}\right)}{b-a}\left(E(Y) - \frac{a+b}{2}\right)\right] \\
&\quad \left. - \frac{1}{2}\left[\frac{G\left(\frac{a+b}{2}\right)}{b-a}\left(E(X) - \frac{a+b}{2}\right) + \frac{F\left(\frac{a+b}{2}\right)}{b-a}\left(E(Y) - \frac{a+b}{2}\right)\right]\right| \\
&\le \left|Pr\left(X \le \frac{a+b}{2}, Y \le \frac{a+b}{2}\right)\right. \\
&\quad \left. - \frac{1}{2(b-a)}\left[G\left(\frac{a+b}{2}\right)(b - E(X)) + F\left(\frac{a+b}{2}\right)(b - E(Y))\right]\right| \\
&\quad + \frac{G\left(\frac{a+b}{2}\right)}{2(b-a)}\left|E(X) - \frac{a+b}{2}\right| + \frac{F\left(\frac{a+b}{2}\right)}{2(b-a)}\left|E(Y) - \frac{a+b}{2}\right|.
\end{aligned}$$

In combination with (32)-(34) and Corollary 3 in Liu [6], we get our results: (35)-(37).

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