Tamsui Oxford Journal of Mathematical Sciences **26(4)** (2010) 383-395
Aletheia University

A Theorem Connecting the H-Transform and Fractional Integral Operators Involving the Multivariable H-Function*

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Received March 31, 2009, Accepted June 19, 2009.

Abstract

In the present paper, we first establish a general theorem that gives the image of a modified H-transform in the fractional integral operator involving the multivariable H-function. Next, we deduce two important corollaries involving Wright generalized Bessel function, Mittag-Leffler function, Appell function F_1 and the product of Whittaker functions which are also quite general in nature and of interest by themselves. Several other new and known results can also be obtained from our main theorem. We record here exact reference to one such known result.

Keywords and Phrases: *Modified H-function transform, Multivariable H-function, Appell function F_1.*

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*2000 *Mathematics Subject Classification.* Primary 26A33, 33C60.

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1. Introduction

The new modified H-transform that is a generalization of transform studied by Saigo *et al.* [6] will be defined and represented in the following manner:

$$\mathbf{h}(s_1,...,s_n) = h^{0,0\ :M_1',N_1';...;M_n',N_n'}_{P',Q'\,:P_1',Q_1'\ ;...;P_n',Q_n'}\left[F_1(x_1,...,x_n)\,;\,\rho_1,...,\rho_n\,;\,s_1,...,s_n\right]$$

$$= \int_{d_1}^{\infty}...\int_{d_n}^{\infty}\prod_{i=1}^{n}(s_ix_i)^{\rho_i-1}\,H^{0,0\ :M_1',N_1';...;M_n',N_n'}_{P',Q'\,:P_1',Q_1';...;P_n',Q_n'}\left[\begin{array}{c}(s_1x_1)^{k_1}\\ \vdots\\ (s_nx_n)^{k_n}\end{array}\middle|\begin{array}{l}(a_{2j};\alpha_{2j}^{(1)},...,\alpha_{2j}^{(n)})_{1,P'}:\\ (b_{2j};\beta_{2j}^{(1)},...,\beta_{2j}^{(n)})_{1,Q'}:\end{array}\right.$$

$$\left.\begin{array}{l}(c_{2j}^{(1)},\gamma_{2j}^{(1)})_{1,P_1'};...;(c_{2j}^{(n)},\gamma_{2j}^{(n)})_{1,P_n'}\\ (d_{2j}^{(1)},\eta_{2j}^{(1)})_{1,Q_1'};...;(d_{2j}^{(n)},\eta_{2j}^{(n)})_{1,Q_n'}\end{array}\right]F\,(x_1,...,x_n)\,dx_1...\,dx_n \qquad (1)$$

for $k_1,\ldots,k_n>0$, where

$$F\,(x_1,...,x_n) = f\left(a_1\sqrt{x_1^2-d_1^2}U\,(x_1-d_1)\,,...,\,a_n\sqrt{x_n^2-d_n^2}U(x_n-d_n)\right)$$

$$x_1 > d_1 > 0\,,\cdots,\,x_n > d_n > 0 \qquad (2)$$

Here $U(x_i--d_i)$ $(i=1,\ldots,n)$ is the well-known Heaviside's unit function. Further, we assume that $\mathbf{h}(s_1,\ldots,s_n)$ belongs to V1, where V1 is the class of functions $f(x_1,\ldots,x_n)$ on $R_+^{(n)}$ which are infinitely differentiable with partial derivatives of any order such that

$$f\,(x_1,...,x_n) = \left[\begin{array}{ll}0\,(\,|\,x_i\,|^{w_i}) & (\,\max\,\{\,|\,x_1|\,,...,\,|\,x_n|\,\}\ \to\ 0)\\ 0\,(\,|\,x_i\,|^{-\tau_i}) & (\,\min\,\{\,|\,x_1\,|\,,...,\,|\,x_n|\,\}\ \to\ \infty)\end{array}\right] \qquad (3)$$

The transform defined by (1) exists provided the following (sufficient) conditions are satisfied.

(i) $|\arg\ s_i| < \frac{1}{2}\pi\ \ \Omega_i/k_i$

where

$$\Omega_i = -\sum_{j=1}^{P'}\alpha_{2j}^{(i)}-\sum_{j=1}^{Q'}\beta_{2j}^{(i)}+\sum_{j=1}^{N_i'}\gamma_{2j}^{(i)}-\sum_{j=N_i'+1}^{P_i'}\gamma_{2j}^{(i)}+\sum_{j=1}^{M_i'}\eta_{2j}^{(i)}-\sum_{j=M_i'+1}^{Q_i'}\eta_{2j}^{(i)} > 0$$

$$\forall i \in \{1, ..., n\}.$$

(ii) $\Re(w_i) + 1 > 0$.

(iii)

$$\Re\,(\rho_i - \tau_i) \;+\; k_i \max_{1 \leqq j \leqq N_n'} \left[\Re\, \frac{(c_{2j}^{(i)} - 1)}{\gamma_{2j}^{(i)}} \right] \;<\; 0$$

$$\forall i \in \{1, ..., n\}.$$

The multivariable H-function has been studied extensively by Srivastava and Panda in their two basic papers on the subject (see [10, pp.119-137] and [11, pp.169-197]). In the present paper, we shall define and represent it in the following manner [9, p. 251, Eq. (C.1)]:

$$H\;[z_1, ..., z_r] \;=\; H^{0,N:\,M_1,N_1;...;\,M_r,N_r}_{P,Q\,:\,P_1,Q_1\;;...;\,P_r,Q_r} \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \left| \begin{matrix} (a_j;\; \alpha_j^1,...,\alpha_j^{(r)})_{1,P} : (c_j^1,\gamma_j^1)_{1,P_1} \;;...;\; (c_j^{(r)},\gamma_j^{(r)})_{1,P_r} \\ (b_j:\; \beta_j^1,...,\beta_j^{(r)})_{1,Q} \;:\; (d_j^1,\eta_j^1)_{1,Q_1};...;\; (d_j^{(r)},\eta_j^{(r)})_{1,Q_r} \end{matrix} \right. \right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(\xi_1, ..., \xi_r) \prod_{i=1}^{r} \{\phi_i(\xi_i)\, z_i^{\xi_i}\}\; d\xi_1 ... d\xi_r \tag{4}$$

where $\omega \;=\; \sqrt{-1}$,

$$\phi_i(\xi_i) \;=\; \frac{\prod_{j=1}^{M_i} \Gamma\,(d_j^{(i)} - \eta_j^{(i)}\xi_i)\, \prod_{j=1}^{N_i} \Gamma\,(1 - c_j^{(i)} + \gamma_j^{(i)}\xi_i)}{\prod_{j=M_i+1}^{Q_i} \Gamma(1 - d_j^{(i)} + \eta_j^{(i)}\xi_i)\, \prod_{j=N_i+1}^{P_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)}\xi_i)},\;\; \forall\; i \;\in\; \{1, ..., r\} \tag{5}$$

$$\psi\,(\xi_1, ..., \xi_r) \;=\; \frac{\prod_{j=1}^{N} \Gamma\,(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)}\xi_i)}{\prod_{j=N+1}^{P} \Gamma\,(a_j - \sum_{j=1}^{r} \alpha_j^{(i)}\xi_i)\; \prod_{j=1}^{Q} \Gamma\,(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)}\xi_i)} \tag{6}$$

$$\forall\, i \;\in\; \{1, ..., r\}.$$

For the convergence, existence conditions and other details of the multivariable H-function, we refer to the book mentioned earlier [9, pp. 251-253, Eqs.(C.2) to (C.8)]

An excellent survey of the work done on operators of fractional integration and their applications has been made in the highly useful work of Srivastava and Saxena [12]. Further, theory and applications of fractional differential equations have been recently studied in a text book by Kilbas *et al.* [4]. Generalized multivariable fractional integral operators have been studied from time to time by several authors including Srivastava *et al.* [13], Gupta *et al.* [2] and Saxena *et al.* [7].

In the present paper, we shall study the fractional integral operator involving multivariable H-function, which is a generalization of an operator studied by Saxena and Kumbhat [8] and defined as follows:

$$R^{\eta_1,\ldots,\eta_n,\alpha}_{x_1,\ldots,x_n,r_1,\ldots,r_n}[f(x_1,\ldots,x_n)] = \prod_{i=1}^{n}(r_i\, x_i^{-\eta_i-r_i\alpha-1}) \int_{t_1=0}^{x_1}\cdots\int_{t_n=0}^{x_n} \prod_{i=1}^{n}\{t_i^{\eta_i}(x_i^{r_i}-t_i^{r_i})^{\alpha}\}$$

$$\cdot H^{0,N\,:\,M_1,N_1;\ldots;M_n,N_n}_{P,Q\,:\,P_1,Q_1\;;\ldots;P_n,Q_n}\left[\begin{array}{c} K_1\left(\frac{t_1^{r_1}}{x_1^{r_1}}\right)^{m_l}\left(1-\frac{t_1^{r_1}}{x_1^{r_1}}\right)^{n_1} \\ \vdots \\ K_n\left(\frac{t_n^{r_n}}{x_n^{r_n}}\right)^{m_n}\left(1-\frac{t_n^{r_n}}{x_n^{r_n}}\right)^{n_n} \end{array}\right|$$

$$\left.\begin{array}{l} (a_{1j};\alpha_{1j}^{(1)},\ldots,\alpha_{1j}^{(n)})_{1,P} : (c_{1j}^{(1)},\gamma_{1j}^{(1)})_{1,P_1};\ldots;(c_{1j}^{(n)},\gamma_{1j}^{(n)})_{1,P_n} \\ (b_{1j};\beta_{1j}^{(1)},\ldots,\beta_{1j}^{(n)})_{1,Q} : (d_{1j}^{(1)},\eta_{1j}^{(1)})_{1,Q_1};\ldots;(d_{1j}^{(n)},\eta_{1j}^{(n)})_{1,Q_n} \end{array}\right] f\,(t_1,\ldots,t_n)\,dt_1\ldots dt_n \qquad (7)$$

where $N, P, Q, M_i, N_i, P_i, Q_i$ are non-negative integers such that

$0 = N = P, Q = 0, 1 = M_i = Q_i$ and $0 = N_i = P_i$

and

$$|\arg\ K_i| < \frac{1}{2}\ \Omega_i'\pi\ (\Omega_i' > 0)$$

$$\Omega_i' = -\sum_{j=N+1}^{P}\alpha_{1j}^{(i)} - \sum_{j=1}^{Q}\beta_{1j}^{(i)} + \sum_{j=1}^{N_i}\gamma_{1j}^{(i)} - \sum_{j=N_i+1}^{P_i}\gamma_{1j}^{(i)} + \sum_{j=1}^{M_i}\eta_{1j}^{(i)} - \sum_{j=M_i+1}^{Q_i}\eta_{1j}^{(i)} > 0$$

$$\forall i \in \{1,\ldots,n\}.$$

Here r_i, m_i, n_i are non-negative integers. The (sufficient) conditions of validity of this operator are given below

(i) $\Re(\eta_i) + r_i m_i \min_{1 \leqq j \leqq M_i} \left\{ \Re \left(\frac{d_{1j}^{(i)}}{\eta_{1j}^{(i)}} \right) \right\} + 1 > 0$ (i = 1,. . . ,n)

(ii) $\Re(\alpha) + n_i \min_{1 \leqq j \leqq M_i} \left\{ \Re \left(\frac{d_{1j}^{(i)}}{\eta_{1j}^{(i)}} \right) \right\} + 1 > 0$ (i = 1,. . . ,n)

2. Main Theorem

In this section, we first prove our main result as detailed below.

Theorem. *If*

$$\mathbf{h}(s_1, ..., s_n) = \int_{d_1}^{\infty} ... \int_{d_n}^{\infty} \prod_{i=1}^{n} (s_i x_i)^{\rho_i - 1} H^{0,0\ :\ M_1', N_1';...;\ M_n', N_n'}_{P',Q'\ :\ P_1', Q_1'\ ;...;\ P_n', Q_n'}$$

$$\left[\begin{matrix} (s_1 x_1)^{k_1} \\ \vdots \\ (s_n x_n)^{k_n} \end{matrix} \middle| \begin{matrix} (a_{2j}; \alpha_{2j}^{(1)}, ..., \alpha_{2j}^{(n)})_{1,P'} : & (c_{2j}^{(1)}, \gamma_{2j}^{(1)})_{1,P_1'} ;...; (c_{2j}^{(n)}, \gamma_{2j}^{(n)})_{1,P_n'} \\ (b_{2j}; \beta_{2j}^{(1)}, ..., \beta_{2j}^{(n)})_{1,Q'} : & (d_{2j}^{(1)}, \eta_{2j}^{(1)})_{1,Q_1'} ;...; (d_{2j}^{(n)}, \eta_{2j}^{(n)})_{1,Q_n'} \end{matrix} \right] F(x_1, ..., x_n)\, dx_1 ... dx_n$$

and

$$R^{\eta_1,...,\eta_n,\alpha}_{x_1,...,x_n,r_1,...,r_n}[f(x_1, ..., x_n)] = \prod_{i=1}^{n} (r_i\, x_i^{-\eta_i - r_i \alpha - 1}) \int_{t_1=0}^{x_1} ... \int_{t_n=0}^{x_n} \prod_{i=1}^{n} \{ t_i^{\eta_i} (x_i^{r_i} - t_i^{r_i})^{\alpha} \}$$

$$\cdot H^{0,N\ :\ M_1, N_1;...;\ M_n, N_n}_{P,Q\ :\ P_1, Q_1\ ;...;\ P_n, Q_n} \left[\begin{matrix} K_1 \left(\frac{t_1^{r_1}}{x_1^{r_1}} \right)^{m_l} \left(1 - \frac{t_1^{r_1}}{x_1^{r_1}} \right)^{n_1} \\ \vdots \\ K_n \left(\frac{t_n^{r_n}}{x_n^{r_n}} \right)^{m_n} \left(1 - \frac{t_n^{r_n}}{x_n^{r_n}} \right)^{n_n} \end{matrix} \right|$$

$$\left. \begin{matrix} (a_{1j}; \alpha_{1j}^{(1)}, ..., \alpha_{1j}^{(n)})_{1,P} : (c_{1j}^{(1)}, \gamma_{1j}^{(1)})_{1,P_1} ;...; (c_{1j}^{(n)}, \gamma_{1j}^{(n)})_{1,P_n} \\ (b_{1j}; \beta_{1j}^{(1)}, ..., \beta_{1j}^{(n)})_{1,Q} : (d_{1j}^{(1)}, \eta_{1j}^{(1)})_{1,Q_1} ;...; (d_{1j}^{(n)}, \eta_{1j}^{(n)})_{1,Q_n} \end{matrix} \right] f(t_1, ..., t_n)\, dt_1 ... dt_n \qquad (8)$$

then

$$R^{\eta_1,...,\eta_n,\alpha}_{s_1,...,s_n,r_1,...,r_n} [\mathbf{h}(s_1, ..., s_n)] = \int_{d_1}^{\infty} ... \int_{d_n}^{\infty} \prod_{i=1}^{n} (s_i x_i)^{\rho_i - 1}$$

$$H^{0,n+N\,:\,M_1,1+N_1\,;...;\,M_n,1+N_n;\,M_1',N_1'\,;...;\,M_n',N_n'}_{n+P+P',n+Q+Q'\,:\,P_1+1,Q_1\,;...;\,P_n+1,Q_n\,;\,P_1',Q_1'\,;...;\,P_n',Q_n'}\left[\begin{array}{l|c} K_1 & A^* : C^* \\ \vdots & \\ K_n & \\ (s_1x_1)^{k_1} & \\ \vdots & B^* : D^* \\ (s_nx_n)^{k_n} & \end{array}\right]$$

$$\cdot F\,(x_1,...,x_n)\,dx_1...\,dx_n \tag{9}$$

where

$$A^* = \left(\begin{array}{l} \left(1-\left(\frac{\eta_1+\rho_1}{r_1}\right);\, m_1,\, \underbrace{0,...,0}_{n-1},\, \frac{k_1}{r_1},\, \underbrace{0,...,0}_{n-1}\right),\, \left(1-\left(\frac{\eta_2+\rho_2}{r_2}\right);\, 0,\, m_2,\, \underbrace{0,...,0}_{n-2},\, 0,\, \frac{k_2}{r_2},\underbrace{0,...,0}_{n-2}\right) \\ ,...,\, \left(1-\left(\frac{\eta_n+\rho_n}{r_n}\right),\, \underbrace{0,...,0}_{n-1},\, m_n,\, \underbrace{0,...,0}_{n-1},\, \frac{k_n}{r_n}\right) \\ \left(a_{1j};\, \alpha_{1j}^{(1)},...,\alpha_{1j}^{(n)},\, \underbrace{0,...,0}_{n}\right)_{1,P},\, \left(a_{2j};\, \underbrace{0,...,0}_{n},\, \alpha_{2j}^{(1)},...,\, \alpha_{2j}^{(n)}\right)_{1,P'} \end{array}\right)$$

$$B^* = \left(\begin{array}{l} \left(-\alpha-\left(\frac{\eta_1+\rho_1}{r_1}\right);\, (m_1+n_1),\, \underbrace{0,...,0}_{n-1},\, \frac{k_1}{r_1},\, \underbrace{0,...,0}_{n-1}\right), \\ \left(-\alpha-\left(\frac{\eta_2+\rho_2}{r_2}\right);\, 0,\, (m_2+n_2),\, \underbrace{0,...,0}_{n-2},\, 0,\, \frac{k_2}{r_2},\underbrace{0,...,0}_{n-2}\right),..., \\ \left(-\alpha-\left(\frac{\eta_n+\rho_n}{r_n}\right),\, \underbrace{0,...,0}_{n-1},\, (m_n+n_n),\, \underbrace{0,...,0}_{n-1},\, \frac{k_n}{r_n}\right) \\ \left(b_{1j};\, \beta_{1j}^{(1)},...,\beta_{1j}^{(n)},\, \underbrace{0,...,0}_{n}\right)_{1,Q},\, \left(b_{2j};\, \underbrace{0,...,0}_{n},\, \beta_{2j}^{(1)},...,\, \beta_{2j}^{(n)}\right)_{1,Q'} \end{array}\right)$$

$$C^* = \begin{pmatrix} (-\alpha, n_1),\, (c_{1j}^{(1)},\, \gamma_{1j}^{(1)})_{1,P_1};\, (-\alpha, n_2),\, (c_{1j}^{(2)}, \gamma_{1j}^{(2)})_{1,P_2};\, ...;\, (-\alpha, n_n),\, (c_{1j}^{(n)}, \gamma_{1j}^{(n)})_{1,P_n}; \\ (c_{2j}^{(1)}, \gamma_{2j}^{(1)})_{1,P_1'};\, (c_{2j}^{(2)}, \gamma_{2j}^{(2)})_{1,P_2'};\, ...;\, (c_{2j}^{(n)}, \gamma_{2j}^{(n)})_{1,P_n'} \end{pmatrix}$$

$$D^* = \begin{pmatrix} (d_{1j}^{(1)},\, \eta_{1j}^{(1)})_{1,Q_1};\; (d_{1j}^{(2)}, \eta_{1j}^{(2)})_{1,Q_2};\, ...;\; (d_{1j}^{(n)}, \eta_{1j}^{(n)})_{1,Q_n}; \\ (d_{2j}^{(1)}, \eta_{2j}^{(1)})_{1,Q_1'};\, (d_{2j}^{(2)}, \eta_{2j}^{(2)})_{1,Q_2'};\, ...;\, (d_{2j}^{(n)}, \eta_{2j}^{(n)})_{1,Q_n'} \end{pmatrix}$$

provided that

(i) $|\arg\ s_i| \ < \ \frac{1}{2}\pi\,\Omega_i/k_i\ (\Omega_i \ > \ 0)$

(ii) $|\arg\ K_i| \ < \ \frac{1}{2}\pi\,\Omega_i'\ (\Omega_i' \ > \ 0)$

(iii) r_i, m_i, n_i *are non-negative integers.*

(iv) $\Re\,(w_i) \ + \ 1 \ > \ 0$

and $\Re\,(\rho_i - \tau_i) \ + \ k_i \max\limits_{1 \leqq j \ \leqq N_i'} \left\{\Re\left(\frac{c_{2j}^{(i)}-1}{\gamma_{2j}^{(i)}}\right)\right\} \ < \ 0$

(v) $\Re\,(\eta_i) \ + \ r_i m_i \min\limits_{1 \leqq j \ \leqq M_i} \left\{\Re\left(\frac{d_{1j}^{(i)}}{\eta_{1j}^{(i)}}\right)\right\} \ + \ 1 \ > \ 0$

and $\Re\,(\alpha) \ + \ n_i \min\limits_{1 \leqq j \ \leqq M_i} \left\{\Re\left(\frac{d_{1j}^{(i)}}{\eta_{1j}^{(i)}}\right)\right\} \ + \ 1 \ > \ 0$

(vi) $\Re\left(\frac{\eta_i+\rho_i}{r_i}\right) \ + \ m_i \min\limits_{1 \leqq j \leqq M_i} \left\{\Re\left(\frac{d_{1j}^{(i)}}{\eta_{1j}^{(i)}}\right)\right\} \ + \ \frac{k_i}{r_i} \min\limits_{1 \leqq j \ \leqq M_i'} \left\{\Re\left(\frac{d_{2j}^{(i)}}{\eta_{2j}^{(i)}}\right)\right\} \ >$ $0 \quad (\forall i \in \{1, \ldots, n\})$.

Proof. On substituting the value of $\mathbf{h}(s_1, \ldots, s_n)$ from (2) in the left-hand side of (8), we get

$R_{s_1,\ldots,s_n;\, r_1,\ldots,r_n}^{\eta_1,\ldots,\eta_n\,;\,\alpha}\ [\mathbf{h}(s_1, ..., s_n)]$

$$= \prod_{i=1}^{n}\{r_i\, s_i^{-\eta_i - r_i\alpha - 1}\} \int_{t_1=0}^{s_1} ... \int_{t_n=0}^{s_n} \prod_{i=1}^{n}\{t_i^{\eta_i}(s_i^{r_i} - \ t_i^{r_i})^{\alpha}\}$$

$$\cdot H_{P,Q\,:\,P_1,Q_1\ ;...;\,P_n,Q_n}^{0,N\,:\,M_1,N_1\,;...;\,M_n,N_n} \left[K_1\left(\frac{t_1^{r_1}}{s_1^{r_1}}\right)^{m_1}\left(1 - \ \frac{t_1^{r_1}}{s_1^{r_1}}\right)^{n_1}, ..., \ K_n\left(\frac{t_n^{r_n}}{s_n^{r_n}}\right)^{m_n}\left(1 - \ \frac{t_n^{r_n}}{s_n^{r_n}}\right)^{n_n}\right]$$

$$\cdot \int_{d_1}^{\infty} ... \int_{d_n}^{\infty} \prod_{i=1}^{n}(t_i x_i)^{\rho_i - 1} \ \cdot H_{P',Q'\,:\,P_1',Q_1'\ ;...;\,P_n',Q_n'}^{0,0\ \ :\,M_1',N_1'\,;...;\,M_n',N_n'}\ \ [(t_1 x_1)^{k_1}, ..., (t_n x_n)^{k_n}\,]$$

$$\cdot F(x_1,...,x_n)\ dx_1...dx_n\}\ \ dt_1...\,dt_n. \tag{10}$$

Now, interchanging the order of x_i and t_i integrals which is permissible under the given conditions, we obtain

$$R^{\eta_1,...,\eta_n\,;\,\alpha}_{s_1,...,s_n;\,r_1,...,r_n}\,[\mathbf{h}(s_1,...,s_n)] = \prod_{i=1}^{n}\{r_i\, s_i^{-\eta_i - r_i\alpha - 1}\}\int_{d_1}^{\infty}...\int_{d_n}^{\infty}\prod_{i=1}^{n}(x_i)^{\rho_i - 1}\cdot F(x_1,...,x_n)$$

$$\cdot\left\{\int_{t_1=0}^{s_1}...\int_{t_n=0}^{s_n}\prod_{i=1}^{n}\{\,t_i^{\eta_i+\rho_i-1}(s_i^{r_i} - t_i^{r_i})^{\alpha}\}\right.$$

$$\cdot H^{0,N:\,M_1,N_1\,;...;\,M_n,N_n}_{P,Q\,:\,P_1,Q_1;...;\,P_n,Q_n}\left[K_1\left(\frac{t_1^{r_1}}{s_1^{r_1}}\right)^{m_1}\left(1-\ \frac{t_1^{r_1}}{s_1^{r_1}}\right)^{n_1},...,\ K_n\left(\frac{t_n^{r_n}}{s_n^{r_n}}\right)^{m_n}\left(1-\ \frac{t_n^{r_n}}{s_n^{r_n}}\right)^{n_n}\right]$$

$$\left.\cdot H^{0,0\,:\,M_1',N_1'\,;...;\,M_n',N_n'}_{P',Q'\,:\,P_1',Q_1'\,;...;\,P_n',Q_n'}\,[(t_1x_1)^{k_1},...,\ (t_nx_n)^{k_n}]\,dt_1....dt_n\right\}\ dx_1...dx_n. \tag{11}$$

Further, on expressing both the multivariable H-functions in terms of their corresponding Mellin-Barnes contour integrals with the help of (4) and changing the order of contour integrals and t_i-integrals, we arrive at the following result:

$$R^{\eta_1,...,\eta_n\,;\,\alpha}_{s_1,...,s_n;\,r_1,...,r_n}\ [\mathbf{h}\ (s_1,...,s_n)] =\ \prod_i = 1^n\left(\tfrac{r_i}{s_i}\right)\int_{d_1}^{\infty}...\int_{d_n}^{\infty}\prod_{i=1}^{n}(s_ix_i)^{\rho_i - 1}\,F\ (x_1,...,x_n)$$

$$\left[\frac{1}{(2\pi\omega)^{2n}}\int_{L_1}...\int_{L_n}\int_{L_1'}...\int_{L_n'}\psi(\xi_1,...,\xi_n)\,\psi'(\xi_1',...,\xi_n')\ \prod_{i=1}^{n}\{\phi_i(\xi_i)\,K_i^{\xi_i}\}\right.$$

$$\cdot\prod_i = 1^n\{\phi_i'(\xi_i')\,(s_ix_i)^{k_i\xi_i'}\}$$

$$\left\{\int_{t_1=0}^{s_1}...\int_{t_n=0}^{s_n}\prod_{i=1}^{n}\left(\frac{t_i^{r_i}}{s_i^{r_i}}\right)^{\frac{\eta_i+\rho_i}{r_i}+m_{i\xi_i}+\frac{k_i\xi_i'-1}{r_i}}\left(1-\frac{t_i^{r_i}}{s_i^{r_i}}\right)^{\alpha+n_{i\xi_i}}\ dt_1...\ dt_n\right\}$$

$$\left.\cdot d\xi_1...d\xi_n d\xi_1'...d\xi_n'\right]dx_1...\,dx_n \tag{12}$$

Next, we transform the t_i-integrals to well known multiple Beta integrals by the following transformation:

$$1 - \frac{t_i^{r\,i}}{s_i^{r\,i}} \;=\; y_i \;\; or \;\; t_i \;=\; s_i(1-y_i)^{1/r\,i}. \tag{13}$$

Further, we evaluate the multiple Beta integrals thus obtained and finally on reinterpreting the resulting expression thus obtained in terms H-function of $2n$ variables, we easily arrive at the right-hand side of the main theorem after a little simplification.

3. Special Cases

On reducing the multivariable H-function involved in (2) to the product of Wright generalized Bessel function [9, p.19] and the Mittag-Leffler function [3, p.65] and the multivariable H-function occurring in (8) to Appell function F_1 [9, p.89],we arrive at the following corollary after a little simplification.

Corollary 1. *If*

$$\mathbf{h}_1(s_1,s_2) \;=\; \int_{d_1}^{\infty}\int_{d_2}^{\infty}\prod_{i=1}^{2}(s_i x_i)^{\rho_i-1}\; J_{\lambda}^{\upsilon}\;(s_1x_1)\;\; E_{\gamma\,,\,\mu}\,(-s_2x_2)\;\; F(x_1,x_2)\,dx_1\,dx_2$$

and

$$R_{x_1,x_2;\,1,1}^{\eta_1,\eta_2,\alpha}[f(x_1,x_2)] \;=\; \prod_{i=1}^{2} x_i^{-\eta_i-\alpha-1}\int_{t_1=0}^{x_1}\int_{t_2=0}^{x_2}\prod_{i=1}^{2}\{t_i^{\eta_i}(x_i-t_i)^{\alpha}\}$$

$$\cdot F_1\left(a,\;c,\;e\;;b\;;\frac{t_1}{x_1}\,,\;\frac{t_2}{x_2}\right)\;f\;(t_1,t_2)\,dt_1\,dt_2 \tag{14}$$

then

$$R_{s_1,s_2;\,1,1}^{\eta_1,\eta_2,\alpha}[\mathbf{h}_1(s_1,s_2)] \;=\; \frac{[\Gamma(1+\alpha)]^2\Gamma\,(\eta_1+\rho_1)\;\Gamma\,(\eta_2+\rho_2)}{\Gamma\,(1+\alpha+\eta_1+\rho_1)\;\Gamma\,(1+\alpha+\eta_2+\rho_2)\;\;\Gamma\,(1+\lambda)\;\Gamma\,(\mu)}$$

$$\cdot \int_{d_1}^{\infty} \int_{d_2}^{\infty} \prod_{i=1}^{2} (s_i x_i)^{\rho_i - 1} F^{3:1;1;0;1}_{3:0;0;1;1} \left[\begin{array}{ccc} A_1^* & : & C_1^* \\ B_1^* & : & D_1^* \end{array} ;\ 1\,,\ 1\,,\ -s_1 x_1\,,\ -s_2 x_2 \right] F\,(x_1, x_2)\, dx_1 dx_2 \tag{15}$$

where

$$A_1^* = \ ((\eta_1 + \rho_1;\ 1,\ 0,\ 1,\ 0),\ \ (\eta_2 + \rho_2;\ 0, 1, 0, 1),\ \ (a,\ 1, 1,\ 0\ ,\ 0))$$

$$B_1^* = \ ((1 + \alpha + \eta_1 + \rho_1; 1,\ 0, 1,\ 0),\ \ (1 + \alpha + \eta_2 + \rho_2; 0, 1,\ 0, 1,\ ),\ \ (b;\ , 1,\ 1\ , 0, 0))$$

$$C_1^* \ = \ ((c\,,\ 1)\ ;\ (e\,,\ 1)\ \ ;\ -\ \ ;\ \ (1\,,\ 1))$$

$$D_1^* \ = \ (-\ ;\ -\ ;\ (1 + \lambda, \nu)\,;\ (\mu\ ,\ \gamma\,))$$

The conditions of validity of Corollary 1 can be easily derived from the existence conditions of the main theorem.
Again, if we reduce H-function of several variables involved in (2) to the product of the Whittaker functions [5, p.(ix), Eq.(A.16)] and multivariable H-function involved in (8) to a generalized hypergeometric function of several variables [5, p. ix, Eq.(A.15)], we easily obtain the following corollary after a little simplification.

Corollary 2. *If*

$$\mathbf{h}_2(s_1, ..., s_n) \ = \ \int_{d_1}^{\infty} ... \int_{d_n}^{\infty} \prod_{i=1}^{n} (s_i x_i)^{\rho_i - 1} \prod_{i=1}^{n} e^{-\frac{s_i x_i}{2}} W_{\lambda_i, \mu_i}(s_i x_i) F(x_1, ..., x_n) dx_1 ... dx_n$$

and

$$R_{2^{\eta_1, \ldots, \eta_n, \alpha}_{s_1, \ldots, s_n, 1, \ldots, 1}}[f(x_1, ..., x_n)] \ = \ \prod_{i=1}^{n} x_i^{-\eta\,_i - \alpha - 1} \int_{t_1 = 0}^{x_1} ... \int_{t_n = 0}^{x_n} \prod_{i=1}^{n} t_i^{\eta\,_i} (x_i - t_i)^{\alpha}$$

$$ {}_PF_Q\left[\begin{matrix}(a_{1j})_P\,; \\ (b_{1j})_Q\,;\end{matrix}\ \prod_{i=1}^{n}\left(1-\ \frac{t_i}{x_i}\right)\right]\ f\ (t_1,...,t_n)\, dt_1...dt_n \tag{16} $$

then

$$ R_{2\,\eta_1,...,\eta_n,\alpha \atop s_1,...,s_n;\underbrace{1,...,1}_{n}}[\mathbf{h2}(s_1,...,s_n)] \ =\ \frac{\prod_{j=1}^{Q}\Gamma(b_{1j})}{\prod_{j=1}^{P}\Gamma(a_{1j})}\int_{d_1}^{\infty}...\int_{d_n}^{\infty}\prod_{i=1}^{n}(s_ix_i)^{\rho_i-1} $$

$$ H^{0,P\ :\,1,1;...;1,1;\,2,1;...;2,1}_{P,n+Q\,:\,\underbrace{1,1;...;1,1}_{n-times};\underbrace{2,2;...;2,2}_{n-times}}\left[\begin{matrix}-1\\ \vdots\\ -1\\ s_1x_1\\ \vdots\\ s_nx_n\end{matrix}\ \middle|\ \begin{matrix}A_2^*\ :\ C_2^*\\ \\ B_2^*\ :\ D_2^*\end{matrix}\right]F(x_1,...,x_n)\ dx_1...dx_n \tag{17} $$

where $A_2^* = \ (1-\ a_{1j};\ \underbrace{1,...,1}_{n}\,;\ \underbrace{0,...,0}_{n})_{1,P}$

$$ B_2^* \ =\ \Big[(-\alpha-\eta_1-\rho_1;\,1,\,\underbrace{0,...,0}_{n-1},\,1\,,\,\underbrace{0,...,0}_{n-1})\,,\,(-\alpha-\eta_2-\rho_2;0,1,\,\underbrace{0,...,0}_{n-2},0,1,\,\underbrace{0,...,0}_{n-2}),..., $$

$$ (-\alpha-\eta_n-\rho_n;\,\underbrace{0,...,0}_{n-1},1,\,\underbrace{0,...,0}_{n-1},1),\ (1-b_{1j};\,\underbrace{1,...,1}_{n},\,\underbrace{0,...,0}_{n})_{1,Q}\Big] $$

$$ C_2^* = \ [\,\underbrace{(-\alpha,1)\,;...;\,(-\alpha,1)}_{ntimes};\,(1-\eta_1-\rho_1,1),\,(1-\lambda_1,1);...;\,(1-\eta_n-\rho_n,1),\,(1-\lambda_n,1)\,] $$

$$ D_2^* = \ [\,\underbrace{(0,1)\,;...;\,(0,1)}_{ntimes};\,(\frac{1}{2}\ \pm\ \mu_1,1)\,;...;\,(\frac{1}{2}\ \pm\ \mu_n,1)]. $$

The conditions of validity of Corollary 2 follow easily from the conditions of the main theorem.

Finally, if we reduce both multivariable H-functions involved in the main theorem to the H-functions, we get a known result of Gupta [1].

Acknowledgements. The authors are highly thankful to Professor H. M. Srivastava for his valuable suggestions and kind help. They are also thankful to Professor K. C. Gupta for his encouragement.

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