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Growth of Entire Harmonic Functions in R^n, $n \geq 2$ *

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Abstract

Let h be a harmonic function on R^n, $n \geq 2$. Then there exists on entire function f on C such that $f(u) = h(u, 0, ..., 0)$ for all real u.This fact has been used to deduce theorems for harmonic function on R^n from classical results about entire functions. Moreover, we have considered the characterizations of lower order and lower type of h in terms of coefficients and ratio of these successive coefficients occurring in power series expansion of f.

Keywords and Phrases: *Homogeneous harmonic polynomials, Entire harmonic function, Laplace's equation, Lower order and lower type.*

1. Introduction

A twice differentiable function $h(x), x = (x_1, x_2, \cdots, x_n)$, which is a solution of Laplace's equation

$$\frac{\partial^2 h}{\partial x_1^2} + \frac{\partial^2 h}{\partial x_2^2} + \cdots + \frac{\partial^2 h}{\partial x_n^2} = 0,$$

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said to be harmonic in $R^n, n \geq 2$. If h is harmonic in $R^n, n \geq 2$,then there is a unique entire (holomorphic) function on the complex plane C such that $f(u) = h(u, 0, \cdots, 0)$ for all real u.

This fact has been used to deduce theorems for harmonic function on R^n from classical results about entire functions $[7, 11]$.The space of functions that are harmonic on R^n is denoted by $\aleph_n$ and the space of entire functions on C is denoted by $\mathcal{E}$. If $f \epsilon \aleph_n$ (respectively $\mathcal{E}$),then we write $M_\infty(f, r)$ for the maximum value of $|f|$ on the sphere of radius r centered at origin and

$$M_2(f, r) = (\int_S |f(rx)|^2 d\sigma(x))^{1/2},$$

where S is the unit sphere in R^n (or the unit circle in C) and σ is $(n-1)$ dimensional surface measure (or length measure)normalized so that $\sigma(s) = 1$.

Let $\aleph_{m,n}$ denote the vector space of all homogenous harmonic polynomials of degree m on R^n. Then $\aleph_{m,n}$ is a vector space of dimension

$$d_m = (n + 2m - 2)\frac{(n + m - 3)!}{m!(n-2)!}.$$

Suppose that $h \epsilon \aleph_n$. Then h has a unique expansion of the form

$$h = \Sigma_{m=0}^{\infty} H_m, \tag{1.1}$$

where $H_m \epsilon \aleph_{m,n}$, such that the series $\Sigma_{m=0}^{\infty} |H_m|$ is locally uniformly convergent on $R^n[1, p.84]$.We then say $\Sigma_{m=0}^{\infty} H_m$ is the polynomial expansion of h. Write e^* for the vector $(1, 0, \cdots, 0)$in R^n, we have

$$h(ue^*) = \Sigma_{m=0}^{\infty} H_m(ue^*) = \Sigma_{m=0}^{\infty} H_m(e^*)u^m \quad for \quad all \quad real \quad u.$$

Let

$$f(z) = \Sigma_{m=0}^{\infty} H_m(e^*)z^m. \tag{1.2}$$

The power series converges for all real numbers and hence all complex z,so $f \epsilon \mathcal{E}$.

Following the usual definitions of order and type of an entire function of several complex variables, the order ρ and type T of h are defined as

$$\rho = \limsup_{r \to \infty} \frac{\log \log M_\infty(h, r)}{\log r},$$

$$T = \limsup_{r \to \infty} \frac{\log M_\infty(h, r)}{r^\rho}, \qquad o < \rho < \infty,$$

where $M_\infty(h,r)$is the maximum value of $|h|$on the sphere in $R^n$of radius r centered at origin.

Using various techniques, the coefficient characterizations of order ρ and type T in R^3 were obtained by Fryant [5] and others. Srivastava [14] also obtained the coefficient characterizations of lower order and lower type in R^3. However non of them studied the growth of entire harmonic function h in $R^n, n > 3$.

In this paper we consider the characterizations of lower order and lower type in terms of $H_m(e^*)$ and ratio of these coefficients in $R^n, n \geq 2$.

2. Auxiliary Results

Lemma 1. *The harmonic function $h \epsilon \aleph_n$ can be extended to an entire function if and only if*

$$\lim_{m\to\infty} |H_m(e^*)|^{1/m} = 0, \tag{2.1}$$

where $H_m(e^*)$ are defined in (1.2).

Proof. Since the power series defined by (1.2) is converges for all real and hence all complex z, so f is entire. Hence using the well known condition for entire function f we can easily prove the lemma [3,pp.27-28].

Lemma 2. *If $h \epsilon \aleph_n$ then f and g are also entire functions. Further*

$$K_1 M_\infty(f,r) \leq M_\infty(h,r) \leq M_\infty(g,r), \tag{2.2}$$

where $g(z) = \Sigma_{m=o}^{\infty} \sqrt{d_m} H_m(e^*) z^m$,

$$M_\infty(f,r) = \max_{|z|\leq r} |f(z)|, M_\infty(g,r) = \max_{|z|\leq r} |g(z)|,$$

$M_\infty(h,r)$ defined as earlier and K_1 is a constant depending on m.

Proof. Let $h \epsilon \aleph_n$ can be extended to an entire function. In view of (2.1), we have

$$\lim_{m\to\infty} (\sqrt{d_m}|H_m(e^*)|)^{1/m} = \lim_{m\to\infty} d_m^{1/2m}.|H_m(e^*)|^{1/m} = 0, \quad \left(d_m^{1/2m} \to 1 \;\; as \;\; m \to \infty\right).$$

Hence both $f$and g are entire functions. To prove inequalities in (2.2),we have by Brolet and Choquet [4, Prop.4]

$$|H_m(e^*)| \leq \sqrt{d_m} r^{-m} M_2(H,r) \quad (H \epsilon \aleph_{m,n}, r > 0), \tag{2.3}$$

where $d_m = \dim \aleph_{m,n}$. Since the series $h = \sum_{m=0}^{\infty} H_m$ converges uniformly on every sphere, we have

$$M_2^2(h,r) = \sum_{m=0}^{\infty} M_2^2(H_m,r) \quad (r > 0). \tag{2.4}$$

In view of (2.3) and (2.4), we get

$$|H_m(e^*)| \leq \sqrt{d_m} r^{-m} M_2(h,r). \tag{2.5}$$

The power series expansion of $f(z)$ with the help of (2.5), leads to

$$M_\infty(f,r) = \sum_{m=0}^{\infty} |H_m(e^*)| r^m \leq M_2(h,r) \sum_{m=0}^{\infty} \sqrt{d_m}.$$

Since $d_m \to \frac{2m^{(n-2)}}{(n-2)!}$ as $m \to \infty$, it follows that there is a constant $K_0 = K_0(m)$ such that

$$M_\infty(f,r) \leq M_2(h,r) \sum_{m=0}^{\infty} K_0 m^{(n-2)/2} \quad (m \geq 1, r > 0)$$

or

$$K_1 M_\infty(f,r) \leq M_2(h,r), \qquad K_1 = \frac{1}{\sum_{m=0}^{\infty} K_0 m^{(n-2)/2}}.$$

It can be easily prove from [6, pp.106] that

$$M_2(h,r) \leq M_\infty(h,r).$$

For right hand inequality, following on the lines of proof of [6 ,Lemma 1,pp.106] we have

$$\begin{aligned} |h(xe^*)| &= |\sum_{m=0}^{\infty} H_m(xe^*)|, \\ &= |\sum_{m=0}^{\infty}\sum_{j=1}^{d_m} H_{m_j}(e^*) r^m H_m^j(rx)| \\ &\leq \sum_{m=0}^{\infty} r^m |\sum_{j=1}^{d_m} H_{m_j}(e^*) H_m^j(rx)| \\ &\leq \sum_{m=0}^{\infty} r^m (\sum_{j=1}^{d_m} [H_{m_j}(e^*)]^2)^{\frac{1}{2}} (\sum_{j=1}^{d_m} [H_m^j(rx)]^2)^{\frac{1}{2}}. \end{aligned}$$

$H_m^j(rx)_{j=1}^{d_m}$ be an orthonormal basis for $\aleph_{m,n}$as the spaces $\aleph_{m,n}$are mutually orthogonal in the sense that

$$\int_S H_m(rx) H_i(rx) d\sigma(x) = 0,$$

$(H_m \epsilon \aleph_{m,n}, H_i \epsilon \aleph_{i,n}, m \neq i, r > 0)$. Now in the consequence of the Pythagorean identity for spherical harmonics[15 ,pp.144], that is the identity

$$\sum_{j=1}^{d_m} [H_m^j(x)]^2 = d_m,$$

for all x on the unit sphere. For, on the sphere of radius r,we get

$$|h(xe^*)| \leq \sum_{0}^{\infty} r^m |H_m(e^*)| \sqrt{d_m}.$$

Thus we have

$$M_\infty(h, r) \leq M_\infty(g, r).$$

Hence the proof is completed.

Lemma 3. *Let $f$and g be entire functions as defined above.Then orders and types of f and g are equal.*

Proof. Let $f(z)=\sum_{m=0}^{\infty} H_m(e^*)z^m$ be an entire function of order $\rho(f)$ and type $T(f)$. Then it is well known [3,pp.9-11] that

$$\rho(f)=\limsup_{m\to\infty}\frac{m\log m}{\log|H_m(e^*)|^{-1}} \tag{2.6}$$

$$T(f)=\frac{1}{e\rho}\limsup_{m\to\infty} m|H_m(e^*)|^{\rho/m} \tag{2.7}$$

Hence for the function $g(z)=\sum_{m=0}^{\infty}\sqrt{d_m}H_m(e^*)z^m$, we have

$$\begin{aligned}\frac{1}{\rho(g)} &= \lim_{m\to\infty}\inf\frac{\log\sqrt{d_m}|H_m(e^*)|^{-1}}{m\log m}\\ &= \lim_{m\to\infty}\inf\frac{\log(d_m)^{-1/2}-\log|H_m(e^*)|}{m\log m}\\ &= \lim_{m\to\infty}\inf\left[\frac{\log|H_m(e^*)|^{-1}}{m\log m}-\frac{\log d_m^{1/2m}}{\log m}\right].\end{aligned}$$

Since $d_m^{1/2m}\to 1$ as $m\to\infty$,so we get

$$\frac{1}{\rho(g)}=\lim_{m\to\infty}\inf\frac{\log|H_m(e^*)|^{-1}}{m\log m}.$$

Hence $\rho(f)=\rho(g)$. Now using (2.7) we can easily show that $T(f)=T(g)$. Hence the proof is completed.

Let $h\epsilon\aleph_n$ can be extended to an entire function of order ρ and type T. In analogy with these definitions, we define lower order λ and lower type t as

$$\begin{aligned}\lambda\equiv\lambda(h) &= \lim_{r\to\infty}\inf\frac{\log\log M_\infty(h,r)}{\log r},\\ t\equiv t(h) &= \lim_{r\to\infty}\inf\frac{\log M_\infty(h,r)}{r^\rho}, \qquad 0<\rho<\infty.\end{aligned}$$

Now we have

Lemma 4. *If* $(|H_m(e^*)|/|H_{m+1}(e^*)|)$ *forms a non-decreasing function of* m *then* (α_m/α_{m+1}) *also form a non-decreasing function of* m*, where* $\alpha_m=\sqrt{d_m}|H_m(e^*)|$.

Proof. We have

$$\frac{\alpha_m}{\alpha_{m+1}} = \left[\frac{(n+2m-2)(m+1)}{(n+2m)(n+m-2)}\right]^{1/2} |\frac{H_m(e^*)}{H_{m+1}(e^*)}|.$$

Let $f(x) = \left[\frac{(n+2x-2)(x+1)}{(n+2x)(n+x-2)}\right]^{1/2}$. By logarithmic differentiation, we have

$$\frac{F'(x)}{F(x)} = \frac{1}{2}\left[\frac{2}{n+2x-2} + \frac{1}{x+1} - \frac{1}{x+n-2} - \frac{2}{2x+n}\right] > 0 \quad for \quad any \quad x > 0.$$

Thus $F'(x) > 0$ for $x > 0$. Hence $\frac{\alpha_m}{\alpha_{m+1}}$ is nondecreasing if $|\frac{H_m(e^*)}{H_{m+1}(e^*)}|$ is nondecreasing.

3. Main Results

Theorem 1. *Let $h \epsilon \aleph_n$ can be extended to an entire function of order ρ, lower order λ, type T and lower type t. If f and g are entire functions as defined above, then*

$$\rho(f) = \rho(g) = \rho(h) \tag{3.1}$$

$$T(f) = T(g) = T(h) \tag{3.2}$$

$$\lambda(f) \leq \lambda(h) \leq \lambda(g) \tag{3.3}$$

$$t(f) \leq t(h) \leq t(g). \tag{3.4}$$

Proof. Using (2.2), we have

$$\begin{aligned}\limsup_{r\to\infty}(\inf)\frac{\log\log(K_1 M_\infty(f,r))}{\log r} &\leq \limsup_{r\to\infty}(\inf)\frac{\log\log M_\infty(h,r)}{\log r} \\ &\leq \limsup_{r\to\infty}(\inf)\frac{\log\log M_\infty(g,r)}{\log r}.\end{aligned}$$

It is well known [3, p-13] that for an entire function f of finite order,

$$\log M_\infty(f,r) \simeq \log m(f,r) \quad as \quad r \to \infty.$$

Where $m(f,r)$ is the maximum term in the power series expansion of $f(z)$. Hence in view of above inequalities we get

$$\rho(f) \leq \rho(h) \leq \rho(g); \lambda(f) \leq \lambda(h) \leq \lambda(g).$$

Since $\rho(f) = \rho(g)$,we thus obtain (3.1) and (3.3). Denoting by ρ the common value of orders of f, g and h, we have from (2.2),

$$\limsup_{r\to\infty} \frac{\log m(f,r)}{r^\rho} \leq \limsup_{r\to\infty} \frac{\log M_\infty(h,r)}{r^\rho} \leq \limsup_{r\to\infty} \frac{\log M_\infty(g,r)}{r^\rho}.$$

Hence using Lemma 3 we get (3.2). Similarly we can prove (3.4).

Theorem 2. *Let* $h\epsilon\aleph_n$ *can be extended to an entire function of order* $\rho(0 < \rho < \infty)$, *type* T *and lower type* t. *Then*

$$\lim_{m\to\infty}\inf \frac{m}{\rho}\left(\frac{|H_{m+1}(e^*)|}{|H_m(e^*)|}\right)^\rho \leq t \leq T \leq \limsup_{m\to\infty} \frac{m}{\rho}\left(\frac{|H_{m+1}(e^*)|}{|H_m(e^*)|}\right)^\rho. \tag{3.5}$$

Further, if $(|H_m(e^*)|/|H_{m+1}(e^*)|)$ *forms a non-decreasing function of* m *for all* $m > m_0$, *then*

$$\limsup_{m\to\infty} \frac{m}{\rho}\left(\frac{|H_{m+1}(e^*)|}{|H_m(e^*)|}\right)^\rho \leq eT, \tag{3.6}$$

$$\limsup_{m\to\infty} \frac{\log m}{\log(|H_m(e^*)|/|H_{m+1}(e^*)|)} = \rho. \tag{3.7}$$

Proof. If $F(z) = \sum_{m=0}^{\infty} H_m(e^*)z^m$ is an entire function of order ρ, type T and lower type t , then we have [8, Thm.1]

$$\lim_{m\to\infty}\inf \frac{m}{\rho}\left|\frac{H_{m+1}(e^*)}{H_m(e^*)}\right|^\rho \leq t \leq T \leq \limsup_{m\to\infty} \frac{m}{\rho}\left|\frac{H_{m+1}(e^*)}{H_m(e^*)}\right|^\rho. \tag{3.8}$$

Applying left hand inequality to $f(z) = \sum_{m=0}^{\infty} H_m(e^*)z^m$, we get

$$\lim_{m\to\infty}\inf \frac{m}{\rho}\left[\left|\frac{H_{m+1}(e^*)}{H_m(e^*)}\right|\right]^\rho \leq t(f), \quad i.e, t(f) \leq t(h) = t.$$

To prove right hand inequality of (3.5),we consider the entire function $g(z) = \sum_{m=0}^{\infty} \sqrt{d_m} H_m(e^*)z^m$. Then we obtain

$$T(g) \leq \limsup_{m\to\infty} \frac{m}{\rho}\left[\left|\frac{H_{m+1}(e^*)}{H_m(e^*)}\right| \left(\frac{(n+m-2)(n+2m)}{(n+2m-2)(m+1)}\right)^{1/2}\right]^\rho.$$

Now we consider

$$\begin{aligned} & \frac{1}{2}[\log(n+m-2)+\log(n+2m)-\log(n+2m-2)-\log(m+1)] \\ = & \frac{1}{2}[\log m+\log(1+\frac{n-2}{m})+\log 2m+\log(1+\frac{n}{2m})-\log 2m \\ - & \log(1+\frac{n-2}{2m})-\log m-\log(1+1/m)]. \end{aligned}$$

Since $n \geq 2$, we have

$$\begin{aligned} = & \frac{1}{2}[(\frac{n-2}{m}-\frac{(n-2)^2}{2m^2}+\frac{(n-2)^3}{3m^3}...)+(\frac{n}{2m}-\frac{n^2}{8m^2}+...) \\ - & (\frac{n-2}{2m}-\frac{(n-2)^2}{8m^2}+...)-(\frac{1}{m}-\frac{1}{2m^2}+...)] \to 0 \quad as \quad m \to \infty. \end{aligned}$$

Hence

$$\left[\frac{(n+m-2)(n+2m)}{(n+2m-2)(m+1)}\right]^{1/2} \to 1 \quad as \quad m \to \infty.$$

Since $T(g)=T(h)=T$,We get from above inequality

$$T \leq \limsup_{m\to\infty} \frac{m}{\rho}|\frac{H_{m+1}(e^*)}{H_m(e^*)}|^{\rho}.$$

Hence the proof of (3.5) is completed.

To prove (3.6) and (3.7), let us consider an entire function $f(z)=\sum_{m=0}^{\infty} H_m(e^*)z^m$ of order $\rho(f)$ and type $T(f)$. If $|\frac{H_m(e^*)}{H_{m+1}(e^*)}|$ forms a non-decreasing function of m for $m > m_0$ then we know ([12],[2,Thm.2]) that

$$\rho(f)=\limsup_{m\to\infty} \frac{\log m}{\log |H_m(e^*)/H_{m+1}(e^*)|}. \tag{3.9}$$

Further we know [8, Thm.3] that

$$\limsup_{m\to\infty} \frac{m}{\rho(f)}|\frac{H_{m+1}(e^*)}{H_m(e^*)}|^{\rho(f)} \leq eT(f). \tag{3.10}$$

Let us suppose that $|H_m(e^*)/H_{m+1}(e^*)|$ forms a non-decreasing function of m for $m > m_0$. From Lemma 4, (α_m/α_{m+1}) also forms a non-decreasing function of m for $m > m_0$. Applying (3.9) to $g(z) = \sum_{m=0}^{\infty} \sqrt{d_m} H_m(e^*) z^m$, we get

$$\rho(g) = \limsup_{m\to\infty} \frac{\log m}{\log(\alpha_m/\alpha_{m+1})}$$

$$= \limsup_{m\to\infty} \frac{\log m}{\log|H_m(e^*)/H_{m+1}(e^*)| + \frac{1}{2}[\log(n+2m-2) + \log(m+1) - \log(n-2m) - \log(n+m-2)]}$$

$$= \limsup_{m\to\infty} \frac{\log m}{\log|H_m(e^*)/H_{m+1}(e^*)|},$$

using the calculations of the first part. Since $\rho(g) = \rho(h) = \rho$, we obtain (3.7). Now using (3.10) for $g(z) = \sum_{m=0}^{\infty} \sqrt{d_m} H_m(e^*) z^m$, we get

$$\limsup_{m\to\infty} \frac{m}{\rho(g)} \left[|\frac{H_{m+1}(e^*)}{H_m(e^*)}| \left[\frac{(n+2m)(n+m-2)}{(n+2m-2)(m+1)} \right]^{1/2} \right]^{\rho(g)} \leq eT(g).$$

Since $\rho(g) = \rho(h) = \rho, T(g) = T(h) = T$, we thus obtain

$$\limsup_{m\to\infty} \frac{m}{\rho} |\frac{H_{m+1}(e^*)}{H_m(e^*)}|^{\rho} \leq eT.$$

Hence the proof of Theorem 2 is completed.

Finally we obtain the coefficient characterizations of the lower order λ and lower type t of $h\epsilon\aleph_n$.

Theorem 3. *Let $h\epsilon\aleph_n$ can be extended to an entire function of order $\rho, 0 < \rho < \infty$, lower order λ ,and lower type t. If $|H_m(e^*)/H_{m+1}(e^*)|$ forms a non-decreasing function of m for $m > m_0$, then*

$$\lambda = \liminf_{m\to\infty} \frac{m \log m}{\log|H_m(e^*)|^{-1}} \tag{3.11}$$

$$t = \liminf_{m\to\infty} \frac{m}{e\rho} |H_m(e^*)|^{\rho/m}. \tag{3.12}$$

Proof. For entire function $f(z) = \sum_{m=0}^{\infty} H_m(e^*)z^m$, if $|H_m(e^*)/H_{m+1}(e^*)|$ forms a non-decreasing function of m for $m > m_0$ then we have ([9,Thm.2],[11]),

$$\lambda(f) = \liminf_{m\to\infty} \frac{m \log m}{\log |H_m(e^*)|^{-1}}. \tag{3.13}$$

Let $|H_m(e^*)/H_{m+1}(e^*)|$ forms a non-decreasing function of m for $m > m_0$. Applying Lemma 4 and (3.13) to $g(z) = \sum_{m=0}^{\infty} \sqrt{d_m} H_m(e^*)z^m$,

$$\begin{aligned}\lambda(g) &= \liminf_{m\to\infty} \frac{m \log m}{\log H_m(e^*)|^{-1} - \log(d_m)^{1/2}} \\ &= \liminf_{m\to\infty} \frac{m \log m \log H_m(e^*)^{-1}}{.}\end{aligned} \tag{3.14}$$

In view of (3.3) with the relations (3.13) and (3.14), the proof of (3.11) is completed.

If $f(z) = \sum_{m=0}^{\infty} H_m(e^*)z^m$ is an entire function of order $\rho(f)$, lower type $t(f)$ and $|H_m(e^*)/H_{m+1}(e^*)|$ forms a non-decreasing function of m for $m > m_0$, then by a result of Shah [13], we get

$$t(f) = \liminf_{m\to\infty} \frac{m}{e\rho(f)} |H_m(e^*)|^{\rho(f)/m},$$

$$t(g) = \liminf_{m\to\infty} \frac{m}{e\rho(g)} |H_m(e^*)|^{\rho(g)/m}.$$

(3.12) now follows in view of (3.1) and (3.4). This completes the proof of Theorem 3.

Theorem 4. *Let $h\epsilon\aleph_n$ can be extended to an entire function of lower order λ, and let $|H_m(e^*)/H_{m+1}(e^*)|$ forms a non-decreasing function of m for $m > m_0$. Then*

$$\lambda = \liminf_{m\to\infty} \frac{\log m}{\log |H_m(e^*)/H_{m+1}(e^*)|} \tag{3.15}$$

Proof . For an entire function $f(z) = \sum_{m=0}^{\infty} H_m(e^*)z^m$, from [10,corollary,p.312], we have

$$\lambda(f) = \liminf_{m\to\infty} \frac{\log m}{\log |H_m(e^*)/H_{m+1}(e^*)|} \tag{3.16}$$

provided $|H_m(e^*)/H_{m+1}(e^*)|$ forms a non-decreasing function of m for $m > m_0$. Using the condition on $\{\alpha_m\}$ we can easily prove, that

$$\lambda(g) = \liminf_{m\to\infty} \frac{\log m}{\log |H_m(e^*)/H_{m+1}(e^*)|} \tag{3.17}$$

The relation (3.15) follows in views in view of (3.3) with (3.16) and (3.17).

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