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A New Hybrid Censoring Scheme and Some of its Properties*

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Abstract

Recently, there are several censoring schemes that have been proposed and studied. They are, among others, Chen and Bhattacharyya (1988), Childs et al. (2003), Chandrasekar et al. (2004) and Balakrishnan et al. (2008) for the case of exponential distribution. In this paper, we propose a new censoring scheme for saving time. A simulation study has been carried out and comparisons have been investigated with some known schemes. It is found that the proposed scheme is superior to others in some sense.

Keywords and Phrases: *Exponential distribution, Combined hybrid censoring, Loss function, Optimal solution.*

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1. Introduction

Consider a life testing experiment where n items are simultaneously put on test at the outset and are not replaced on failure. Let X_i denote the lifetime of component i, $i = 1, ..., n$. Suppose that X_i follows an exponential distribution with common mean lifetime θ. That is, X_i has a probability density $f(x\,|\theta) = \frac{1}{\theta}e^{\frac{-x}{\theta}}$, $x > 0$, $\theta > 0$.

Let the ordered failure times of these items be denoted by $X_{1:n}$, $X_{2:n}, ...,$ $X_{n:n}$. Epstein (1954) considered a hybrid censored sampling scheme in which the life-testing experiment is terminated at a random time $T_1^* = \min\{X_{r:n}, T\}$, where both $r \in \{1, 2, ..., n\}$ and $T \in (0, \infty)$ are fixed in advance. Chen and Bhattacharyya (1988) derived the exact distribution of the maximum likelihood estimator (MLE) as well as a lower confidence bound of the mean θ based on this scheme. Childs et al. (2003) proposed Type-I and Type-II hybrid censoring schemes (HCS) and they also derived distributions of MLE of the unknown parameter. It was shown that the derived distribution is equivalent to that of Chen and Bhattacharyya (1988) for Type-I HCS.

Chandrasekar et al. (2004) modified these schemes and introduced two new schemes which are referred to as *generalized Type-I and Type-II HCS*, respectively. These schemes may be considered as an extension of Type-I HCS and Type-II HCS in some sense. In generalized Type-I HCS, one prefixes k, $r \in \{1, 2, ..., n\}$ and $T \in (0, \infty)$ with $k < r$. If the kth failure occurs before time T, the experiment terminates at $\min\{X_{r:n}, T\}$. If the kth failure occurs after time T, the experiment terminates at $X_{k:n}$. In generalized Type-II HCS, one prefixes $r \in \{1, 2, ..., n\}$ and $T_1, T_2 \in (0, \infty)$ such that $T_1 < T_2$. If the rth failure occurs before time T_1, the experiment terminates at T_1; if the rth failure occurs between T_1 and T_2, the experiment terminates at $X_{r:n}$; otherwise, the experiment terminates at T_2. This HCS guarantees that the experiment time will not exceed T_2.

Balakrishnan et al. (2008) combined the last two schemes of sampling and introduced an *unified hybrid censored sampling* (UHCS). In this scheme, one prefixes k, $r \in \{1, 2, ..., n\}$ and T_1, $T_2 \in (0, \infty)$ such that $k < r$ and $T_1 < T_2$. If the kth failure occurs before time T_1, the experiment terminates at $\min\{\max\{X_{r:n}, T_1\}, T_2\}$. If it occurs between T_1 and T_2, the experiment terminates at $\min\{X_{r:n}, T_2\}$ and finally if it occurs after time T_2, then the experiment is terminated at $X_{k:n}$. The unified HCS can guarantee that at least k failures can be observed.

In this article, we consider a statistical decision-theoretic approach for a censoring sampling scheme. We propose a new sampling scheme for two purposes. We desire to shorten the total experimental time while we also try to estimate the parameter accurately in some sense. However, we emphasize the first purpose. For this reason, we propose a loss function which is a weighted total time length.

Under this new loss, we propose a *combined hybrid censoring sampling* (CHCS) which is formulated and studied in section 2. In section 3, we carry out some numerical and simulation study. For a given weight and some range of n and θ, several optimal solutions of the proposed sampling scheme (k, r, T_1, T_2) are given in the sense of the proposed loss. Some comparisons have also been given among some known censoring sampling schemes under our loss. From the simulation results, it is clear that our proposed optimal CHCS has greatly improved its practical applicability in the sense of saving experimental time.

2. A Combined Hybrid Censoring Sampling (CHCS)

Consider a life-testing experiment in which n identical units are put on test. Let X_1, $X_2, ...,$ X_n denote the respective lifetimes from a population with cumulative distribution function $F(x)$ and probability density function $f(x)$. Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ denote its order statistics. We define combined HCS as follows. Fix k, $r \in \{1, 2, ..., n\}$ and T_1, $T_2 \in (0, \infty)$ such that $k < r$ and $T_1 < T_2$. If the kth failure occurs before time T_1, the experiment terminates at $\min\{X_{r:n}, T_1\}$. However, if the kth failure occurs between T_1 and T_2, the experiment is terminated at $X_{k:n}$ and finally if the kth failure occurs after time T_2, then the experiment terminates at T_2. For our later convenience, we abbreviate this scheme as CHCS$(k, r; T_1, T_2)$.

In fact, we can categorize the following six cases, and obviously, in each case some part of data are unobservable. For our convenience, let T^* denote the stopping time of experiment.

(1) For $0 < T_1 < X_{k:n} (< T_2 < X_{r:n})$, $T^* = X_{k:n}$,
(2) For $0 < T_1 < X_{k:n} (< X_{r:n} < T_2)$, $T^* = X_{k:n}$,
(3) For $0 < T_1 < T_2 (< X_{k:n} < X_{r:n})$, $T^* = T_2$,
(4) For $0 < X_{k:n} < X_{r:n} (< T_1 < T_2)$, $T^* = X_{r:n}$,
(5) For $0 < X_{k:n} < T_1 (< X_{r:n} < T_2)$, $T^* = T_1$,
(6) For $0 < X_{k:n} < T_1 (< T_2 < X_{r:n})$, $T^* = T_1$,

where the data in parentheses are unobservable.

In those six situations, we see that except Case (3), there will be at least k failures, and for Case (3), we may have no exact life data.

Let D_j denote the number of failures until T_j, $j = 1, 2$. Obviously, $D_1 \leq D_2$. Then, the likelihood function of this combined HCS is given as follows:

$$L\left(\theta \middle| \underset{\sim}{x}\right) = \begin{cases} \frac{n!}{(n-k)!} \prod_{i=1}^{k} f(x_i) \left[1 - F(x_k)\right]^{n-k} & D_1 = 0, \cdots, k-1; D_2 = k, \\ \frac{n!}{(n-D_2)!} \prod_{i=1}^{D_2} f(x_i) \left[1 - F(T_2)\right]^{n-D_2} & D_1 = 0, \cdots, k-1; D_2 = 0, \cdots, k-1; D_1 \leq D_2, \\ \frac{n!}{(n-r)!} \prod_{i=1}^{r} f(x_i) \left[1 - F(x_r)\right]^{n-r} & D_1 = D_2 = r, \\ \frac{n!}{(n-D_1)!} \prod_{i=1}^{D_1} f(x_i) \left[1 - F(T_1)\right]^{n-D_1} & D_1 = D_2 = k, \cdots, r-1. \end{cases} \tag{2.1}$$

The MLE for the estimate of θ, denote by $\overset{\wedge}{\theta}$, is as follows:

$$\overset{\wedge}{\theta} = \begin{cases} \frac{\sum_{i=1}^{k} x_{i:n} + (n-k) x_{k:n}}{k} & D_1 = 0, \cdots, k-1; D_2 = k, \\ \frac{\sum_{i=1}^{D_2} x_{i:n} + (n-D_2) T_2}{D_2} & D_1 = 0, \cdots, k-1; D_2 = 1, \cdots, k-1; D_1 \leq D_2, \\ \frac{\sum_{i=1}^{r} x_{i:n} + (n-r) x_{r:n}}{r} & D_1 = D_2 = r, \\ \frac{\sum_{i=1}^{D_1} x_{i:n} + (n-D_1) T_1}{D_1} & D_1 = D_2 = k, \cdots, r-1, \\ nT_2 & D_2 = 0. \end{cases} \tag{2.2}$$

Remark 1. Note that when $D_2 = 0$, there is no exact observation of life time. Theoretically, according to formal definition, $\hat{\theta}$ should take infinity, however, it is not practical. For its applicable consideration, here we define its conservative MLE to be nT_2 for this special case.

Based on (2.1) and (2.2), we can compute the following.

Theorem 1. *The moment generating function (mgf) of* $\hat{\theta}$ *at* ω *is given by*

$$\begin{aligned}
M_{\hat{\theta}}(\omega) &= E(e^{\omega\hat{\theta}}) \\
&= \left(1-\frac{\omega\theta}{k}\right)^{-k}\sum_{j=0}^{k-1}\binom{n}{k}\binom{k}{j}p_{1,k}^{j}\sum_{l=k-j}^{n-j}\binom{n-j}{l}(q_{1,k}-q_{2,k})^{l}q_{2,k}^{n-j-l} \\
&\quad+\sum_{l=1}^{k-1}\left(1-\frac{\omega\theta}{l}\right)^{-l}\binom{n}{l}q_{2,l}^{n-l}\sum_{j=0}^{l}\binom{l}{j}p_{1,l}^{j}(q_{1,l}-q_{2,l})^{l-j} \\
&\quad+\left(1-\frac{\omega\theta}{r}\right)^{-r}\sum_{l=r}^{n}\binom{n}{l}p_{1,r}^{l}q_{1,r}^{n-l}+\sum_{j=k}^{r-1}\left(1-\frac{\omega\theta}{j}\right)^{-j}\binom{n}{j}p_{1,j}^{j}q_{1,j}^{n-j} \\
&\quad+q_2^{n(1-\omega\theta)}, \quad \omega<\frac{k}{\theta}, \qquad (2.3)
\end{aligned}$$

where $q_j = e^{-T_j/\theta}$ $(j = 1, 2)$ *and* $q_{a,b} = 1 - p_{a,b} = e^{-(T_a/\theta)(1-(\omega\theta/b))}$.

Proof. We condition on the values of D_1 and D_2 according to the different situations in equation (2.1).

Then, we can write

$$\begin{aligned}
M_{\hat{\theta}}(\omega) &= E(e^{\omega\hat{\theta}}) = EE(e^{\omega\hat{\theta}}|D_1,D_2) \\
&= \sum_{j=0}^{k-1}E(e^{\omega\hat{\theta}}|D_1=j)P(D_1=j)+\sum_{l=1}^{k-1}\sum_{j=0}^{l}E(e^{\omega\hat{\theta}}|D_1=j,D_2=l)P(D_1=j,D_2=l) \\
&\quad+E(e^{\omega\hat{\theta}}|D_1=r)P(D_1=r)+\sum_{j=k}^{r-1}E(e^{\omega\hat{\theta}}|D_1=j)P(D_1=j) \\
&\quad+E(e^{\omega\hat{\theta}}|D_2=0)P(D_2=0). \qquad (2.4)
\end{aligned}$$

Equivalently, we have

$$
\begin{aligned}
M_{\hat{\theta}}(\omega) &= E(e^{\omega\hat{\theta}}) \\
&= \sum_{j=0}^{k-1} \frac{n!}{(n-k)!\theta^k} \\
&\quad \times \int_{T_1}^{T_2} \cdots \int_{x_{k-1}}^{T_2} \int_0^{T_1} \cdots \int_0^{x_2} e^{-(1/\theta)(1-(\omega\theta/k))\{\sum_{i=1}^{k} x_i + (n-k)x_k\}} dx_1 \cdots dx_j dx_k \cdots dx_{j+1} \\
&\quad + \sum_{l=1}^{k-1} \sum_{j=0}^{l} \frac{n!}{(n-l)!\theta^l} \\
&\quad \times \int_{T_1}^{T_2} \cdots \int_{x_{l-1}}^{T_2} \int_0^{T_1} \cdots \int_0^{x_2} e^{-(1/\theta)(1-(\omega\theta/l))\{\sum_{i=1}^{l} x_i + (n-l)T_2\}} dx_1 \cdots dx_j dx_l \cdots dx_{j+1} \\
&\quad + \frac{n!}{(n-r)!\theta^r} \int_0^{T_1} \int_0^{x_r} \cdots \int_0^{x_2} e^{-(1/\theta)(1-(\omega\theta/r))\{\sum_{i=1}^{r-1} x_i + (n-r+1)x_r\}} dx_1 \cdots dx_r \\
&\quad + \sum_{j=k}^{r-1} \frac{n!}{(n-j)!\theta^j} \int_0^{T_1} \int_0^{x_j} \cdots \int_0^{x_2} e^{-(1/\theta)(1-(\omega\theta/j))\{\sum_{i=1}^{j} x_i + (n-j)T_1\}} dx_1 \cdots dx_j \\
&\quad + e^{\omega n T_2} e^{-nT_2/\theta}.
\end{aligned} \tag{2.5}
$$

Upon carrying out the necessary integration in equation (2.5) and simplifying the resulting expression, we obtain

$$
\begin{aligned}
M_{\hat{\theta}}(\omega) &= E(e^{\omega\hat{\theta}}) \\
&= \left(1-\frac{\omega\theta}{k}\right)^{-k} \sum_{j=0}^{k-1} \binom{n}{k} \binom{k}{j} p_{1,k}^{j} \sum_{l=k-j}^{n-j} \binom{n-j}{l} (q_{1,k}-q_{2,k})^l q_{2,k}^{n-j-l} \\
&\quad + \sum_{l=1}^{k-1} \left(1-\frac{\omega\theta}{l}\right)^{-l} \binom{n}{l} q_{2,l}^{n-l} \sum_{j=0}^{l} \binom{l}{j} p_{1,l}^{j} (q_{1,l}-q_{2,l})^{l-j} \\
&\quad + \left(1-\frac{\omega\theta}{r}\right)^{-r} \sum_{l=r}^{n} \binom{n}{l} p_{1,r}^{l} q_{1,r}^{n-l} + \sum_{j=k}^{r-1} \left(1-\frac{\omega\theta}{j}\right)^{-j} \binom{n}{j} p_{1,j}^{j} q_{1,j}^{n-j} \\
&\quad + q_2^{n(1-\omega\theta)}, \quad \omega < \frac{k}{\theta},
\end{aligned}
$$

which completes the proof of the theorem. □

3. New Loss Function and its Optimal Solution

As it has been mentioned previously, we desire to shorten its total experimental time for reduction of cost.

Let c denote the cost of unit time, we define the loss function associated with CHCS$(k, r; T_1, T_2)$ as follows.

$$L(\theta; \mathrm{CHCS}(k, r; T_1, T_2)) = cT^*, \tag{3.1}$$

where T^* is the stopping time of the combined HCS.

For notational simplicity, we use $L(\theta; k, r, T_1, T_2)$ henceforth for our presentation.

Theorem 2. *The expected loss is given by*

$$\begin{aligned}
&EL(\theta; k, r, T_1, T_2)\\
&= cET^* = cEE(T^*|D_1, D_2)\\
&= c\left\{\sum_{j=0}^{k-1}\frac{n!p_1^j}{(n-k)!j!(k-j)!}\sum_{l=k-j}^{n-j}\binom{n-j}{l}\right.\\
&\qquad\times\left[T_2(q_1-q_2)^l q_2^{n-j-l} - \frac{\theta}{l+1}\sum_{s=l+1}^{n-j}\binom{n-j}{s}(q_1-q_2)^s q_2^{n-j-s}\right]\\
&\quad + T_2\sum_{l=1}^{k-1}\binom{n}{l}q_2^{n-l}\sum_{j=0}^{l}\binom{l}{j}p_1^j(q_1-q_2)^{l-j}\\
&\quad \left.+\sum_{l=r}^{n}\binom{n}{l}\left[T_1p_1^lq_1^{n-l} - \frac{\theta}{l+1}\sum_{s=l+1}^{n}\binom{n}{s}p_1^sq_1^{n-s}\right] + T_1\sum_{j=k}^{r-1}\binom{n}{j}p_1^jq_1^{n-j} + T_2q_2^n\right\}.
\end{aligned} \tag{3.2}$$

Proof. We condition on the values of D_1 and D_2 according to the different situations in equation (2.1).

Then, we can write

$$\begin{aligned}
&EL(\theta; k, r, T_1, T_2)\\
&= cET^* = cEE(T^*|D_1, D_2)\\
&= c\left\{\sum_{j=0}^{k-1}E(T^*|D_1=j)P(D_1=j) + \sum_{l=1}^{k-1}\sum_{j=0}^{l}E(T^*|D_1=j, D_2=l)P(D_1=j, D_2=l)\right.\\
&\quad \left.+E(T^*|D_1=r)P(D_1=r) + \sum_{j=k}^{r-1}E(T^*|D_1=j)P(D_1=j) + E(T^*|D_2=0)P(D_2=0)\right\}.
\end{aligned} \tag{3.3}$$

So, an alternative expression of the risk function in equation (3.3) is given by

$$
\begin{aligned}
&EL(\theta; k, r, T_1, T_2) \\
&= cET^* = cEE(T^*|D_1, D_2) \\
&= c\Bigg\{\sum_{j=0}^{k-1} E(T^*|D_1 = j)P(D_1 = j) + \sum_{l=1}^{k-1}\sum_{j=0}^{l} E(T^*|D_1 = j, D_2 = l)P(D_1 = j, D_2 = l) \\
&\qquad + E(T^*|D_1 = r)P(D_1 = r) + \sum_{j=k}^{r-1} E(T^*|D_1 = j)P(D_1 = j) + E(T^*|D_2 = 0)P(D_2 = 0)\Bigg\} \\
&= c\Bigg\{\sum_{j=0}^{k-1} \frac{n!}{(n-k)!\theta^k} \\
&\qquad \times \int_{T_1}^{T_2} \cdots \int_{x_{k-1}}^{T_2} \int_0^{T_1} \cdots \int_0^{x_2} x_k e^{-(1/\theta)\{\sum_{i=1}^{k-1} x_i + (n-k+1)x_k\}} dx_1 \cdots dx_j dx_k \cdots dx_{j+1} \\
&\qquad + \sum_{l=1}^{k-1}\sum_{j=0}^{l} \frac{n!}{(n-l)!\theta^l} \int_{T_1}^{T_2} \cdots \int_{x_{l-1}}^{T_2} \int_0^{T_1} \cdots \int_0^{x_2} T_2 e^{-(1/\theta)\{\sum_{i=1}^{l} x_i + (n-l)T_2\}} dx_1 \cdots dx_j dx_l \cdots dx_{j+1} \\
&\qquad + \frac{n!}{(n-r)!\theta^r} \int_0^{T_1} \int_0^{x_r} \cdots \int_0^{x_2} x_r e^{-(1/\theta)\{\sum_{i=1}^{r-1} x_i + (n-r+1)x_r\}} dx_1 \cdots dx_r \\
&\qquad + \sum_{j=k}^{r-1} \frac{n!}{(n-j)!\theta^j} \int_0^{T_1} \int_0^{x_j} \cdots \int_0^{x_2} T_1 e^{-(1/\theta)\{\sum_{i=1}^{j} x_i + (n-j)T_1\}} dx_1 \cdots dx_j \\
&\qquad + T_2 e^{-nT_2/\theta}\Bigg\}. \qquad (3.4)
\end{aligned}
$$

Upon carrying out the necessary integration in equation (3.4) and simplifying the resulting expression, we can finally conclude that

$$
\begin{aligned}
&EL(\theta; k, r, T_1, T_2) \\
&= c\Bigg\{\sum_{j=0}^{k-1} \frac{n! p_1^j}{(n-k)! j! (k-j)!} \sum_{l=k-j}^{n-j} \binom{n-j}{l} \\
&\qquad \times \left[T_2 (q_1 - q_2)^l q_2^{n-j-l} - \frac{\theta}{l+1} \sum_{s=l+1}^{n-j} \binom{n-j}{s} (q_1 - q_2)^s q_2^{n-j-s}\right] \\
&\qquad + T_2 \sum_{l=1}^{k-1} \binom{n}{l} q_2^{n-l} \sum_{j=0}^{l} \binom{l}{j} p_1^j (q_1 - q_2)^{l-j} \\
&\qquad + \sum_{l=r}^{n} \binom{n}{l} \left[T_1 p_1^l q_1^{n-l} - \frac{\theta}{l+1} \sum_{s=l+1}^{n} \binom{n}{s} p_1^s q_1^{n-s}\right] + T_1 \sum_{j=k}^{r-1} \binom{n}{j} p_1^j q_1^{n-j} + T_2 q_2^n\Bigg\}. \\
&\qquad (3.5)
\end{aligned}
$$

For a given θ, and prefixed c, computationally, we can find value k_0, r_0, T_{10} and T_{20} so that

$$EL(\theta; k_0, r_0, T_{10}, T_{20}) = \min_{1 \le k < r} \min_{r \le n} \min_{T_1 < T_2} EL(\theta; k, r, T_1, T_2) \tag{3.6}$$

Obviously, if θ were known, CHCS$(k_0, r_0; T_{10}, T_{20})$ is the best we can choose in the sense of our loss. Thus, it is our optimal solution to our censoring sampling problem.

To compute the optimal solution $(k_0, r_0, T_{10}, T_{20})$, we can apply equation (3.5) to obtain its solutions. However, here we apply Monte Carlo method to find the optimal solutions when some various values of θ, n and c are given. These results will be explained in next section.

4. Some Simulated Optimal Solutions and Numerical Comparisons

For given θ, n and c, we take a sample of size n from exponential population with mean θ. For this complete data, we simultaneously apply the proposed Combined, Generalized Type-I, Generalized Type-II and Unified HCS, respectively to obtain the observed loss associated with each censoring sampling plan. This process repeats 5000 times and take its average as its risk for each censoring scheme. By this way, for each censoring scheme we can find the corresponding $(k_0, r_0, T_{10}, T_{20})$ so that its associated risk is minimized.

Taking $c = 1$, for cases of $n = 10$, 20 and for $\theta = 1(1)10(5)30(10)50$, the optimal solution of $(k_0, r_0, T_{10}, T_{20})$ coupling with its risk corresponding respectively to Combined, Generalized Type-I, Generalized Type-II and Unified HCS are tabulated in Table 1 and Table 2.

Let R(A) denote the simulated minimum risk of censoring scheme A. We define relative efficiency of censoring scheme B with respect to CHCS by eff(B)=R(B)/R(CHCS). In Fig. 1 and Fig. 2, it plots for value of $n = 10$ and 20 with $c = 1$ respectively. It is readily seen that for those cases, CHCS clearly shows its superiority against others.

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Table 1. Optimal solutions of $(k_0, r_0, T_{10}, T_{20})$ and its risk $(n = 10)$.

θ	HCS	Minimum risk	T_{10}	T_{20}	k_0	r_0
1	**combined**	**0.098714**	**0.109800**	**0.114192**	**1**	**2**
	generalized Type-I	0.128781	0.109800	–	1	2
	generalized Type-II	0.111188	0.109800	0.114192	1	–
	unified	0.142953	0.109800	0.114192	1	2
2	**combined**	**0.197813**	**0.219600**	**0.228384**	**1**	**2**
	generalized Type-I	0.261799	0.219600	–	1	2
	generalized Type-II	0.222432	0.219600	0.228384	1	–
	unified	0.289841	0.219600	0.228384	1	2
3	**combined**	**0.298199**	**0.329400**	**0.342576**	**1**	**2**
	generalized Type-I	0.399882	0.329400	–	1	2
	generalized Type-II	0.333813	0.329400	0.342576	1	–
	unified	0.440544	0.329400	0.342576	1	2
4	**combined**	**0.397380**	**0.439200**	**0.456768**	**1**	**2**
	generalized Type-I	0.518074	0.439200	–	1	2
	generalized Type-II	0.444693	0.439200	0.456768	1	–
	unified	0.572478	0.439200	0.456768	1	2
5	**combined**	**0.493712**	**0.549000**	**0.570960**	**1**	**2**
	generalized Type-I	0.655800	0.549000	–	1	2
	generalized Type-II	0.555962	0.549000	0.570960	1	–
	unified	0.726611	0.549000	0.570960	1	2
6	**combined**	**0.592260**	**0.658800**	**0.685152**	**1**	**2**
	generalized Type-I	0.781235	0.658800	–	1	2
	generalized Type-II	0.667263	0.658800	0.685152	1	–
	unified	0.866353	0.658800	0.685152	1	2
7	**combined**	**0.696264**	**0.768600**	**0.799344**	**1**	**2**
	generalized Type-I	0.912497	0.768600	–	1	2
	generalized Type-II	0.778539	0.768600	0.799344	1	–
	unified	1.006880	0.768600	0.799344	1	2
8	**combined**	**0.796329**	**0.878400**	**0.913536**	**1**	**2**
	generalized Type-I	1.053634	0.878400	–	1	2
	generalized Type-II	0.889769	0.878400	0.913536	1	–
	unified	1.160693	0.878400	0.913536	1	2
9	**combined**	**0.893355**	**0.988199**	**1.027728**	**1**	**2**
	generalized Type-I	1.162537	0.988199	–	1	2
	generalized Type-II	1.000936	0.988199	1.027728	1	–
	unified	1.285751	0.988199	1.027728	1	2
10	**combined**	**0.991686**	**1.098000**	**1.141920**	**1**	**2**
	generalized Type-I	1.327536	1.098000	–	1	2
	generalized Type-II	1.112665	1.098000	1.141920	1	–
	unified	1.465210	1.098000	1.141920	1	2
15	**combined**	**1.494521**	**1.647000**	**1.712880**	**1**	**2**
	generalized Type-I	1.973882	1.647000	–	1	2
	generalized Type-II	1.668271	1.647000	1.712880	1	–
	unified	2.173867	1.647000	1.712880	1	2
20	**combined**	**1.971385**	**2.196000**	**2.283840**	**1**	**2**
	generalized Type-I	2.603023	2.196000	–	1	2
	generalized Type-II	2.224693	2.196000	2.283840	1	–
	unified	2.889557	2.196000	2.283840	1	2
25	**combined**	**2.482993**	**2.745000**	**2.854800**	**1**	**2**
	generalized Type-I	3.270078	2.745000	–	1	2
	generalized Type-II	2.780472	2.745000	2.854800	1	–
	unified	3.610295	2.745000	2.854800	1	2
30	**combined**	**2.978349**	**3.294000**	**3.425760**	**1**	**2**
	generalized Type-I	3.943298	3.294000	–	1	2
	generalized Type-II	3.336302	3.294000	3.425760	1	–
	unified	4.353028	3.294000	3.425760	1	2
40	**combined**	**3.960017**	**4.392000**	4.567680	**1**	**2**
	generalized Type-I	5.204969	4.392000	–	1	2
	generalized Type-II	4.449993	4.392000	4.567680	1	–
	unified	5.761470	4.392000	4.567680	1	2
50	**combined**	**5.004714**	**5.490000**	**5.709600**	**1**	**2**
	generalized Type-I	6.665269	5.490000	–	1	2
	generalized Type-II	5.565708	5.490000	5.709600	1	–
	unified	7.309605	5.490000	5.709600	1	2

Table 2. Optimal solutions of $(k_0, r_0, T_{10}, T_{20})$ and its risk $(n = 20)$.

θ	HCS	Minimum risk	T_{10}	T_{20}	k_0	r_0
1	**combined**	**0.078426**	**0.109800**	**0.114192**	**1**	**2**
	generalized Type-I	0.083392	0.109800	–	1	2
	generalized Type-II	0.110271	0.109800	0.114192	1	–
	unified	0.116374	0.109800	0.114192	1	2
2	**combined**	**0.156601**	**0.219600**	**0.228384**	**1**	**2**
	generalized Type-I	0.166479	0.219600	–	1	2
	generalized Type-II	0.220509	0.219600	0.228384	1	–
	unified	0.232618	0.219600	0.228384	1	2
3	**combined**	**0.238527**	**0.329400**	**0.342576**	**1**	**2**
	generalized Type-I	0.254287	0.329400	–	1	2
	generalized Type-II	0.330871	0.329400	0.342576	1	–
	unified	0.350141	0.329400	0.342576	1	2
4	**combined**	**0.311088**	**0.439200**	**0.456768**	**1**	**2**
	generalized Type-I	0.332022	0.439200	–	1	2
	generalized Type-II	0.441104	0.439200	0.456768	1	–
	unified	0.466496	0.439200	0.456768	1	2
5	**combined**	**0.390674**	**0.549000**	**0.570960**	**1**	**2**
	generalized Type-I	0.414435	0.549000	–	1	2
	generalized Type-II	0.551208	0.549000	0.570960	1	–
	unified	0.580970	0.549000	0.570960	1	2
6	**combined**	**0.470030**	**0.658800**	**0.685152**	**1**	**2**
	generalized Type-I	0.500076	0.658800	–	1	2
	generalized Type-II	0.661475	0.658800	0.685152	1	–
	unified	0.698293	0.658800	0.685152	1	2
7	**combined**	**0.539775**	**0.768600**	**0.799344**	**1**	**2**
	generalized Type-I	0.574270	0.768600	–	1	2
	generalized Type-II	0.771700	0.768600	0.799344	1	–
	unified	0.813760	0.768600	0.799344	1	2
8	**combined**	**0.622739**	**0.878400**	**0.913536**	**1**	**2**
	generalized Type-I	0.665276	0.878400	–	1	2
	generalized Type-II	0.882081	0.878400	0.913536	1	–
	unified	0.933658	0.878400	0.913536	1	2
9	**combined**	**0.702709**	**0.988199**	**1.027728**	**1**	**2**
	generalized Type-I	0.747127	0.988199	–	1	2
	generalized Type-II	0.992251	0.988199	1.027728	1	–
	unified	1.046793	0.988199	1.027728	1	2
10	**combined**	**0.780363**	**1.098000**	**1.141920**	**1**	**2**
	generalized Type-I	0.828235	1.098000	–	1	2
	generalized Type-II	1.102635	1.098000	1.141920	1	–
	unified	1.161604	1.098000	1.141920	1	2
15	**combined**	**1.170085**	**1.647000**	**1.712880**	**1**	**2**
	generalized Type-I	1.247372	1.647000	–	1	2
	generalized Type-II	1.653489	1.647000	1.712880	1	–
	unified	1.748256	1.647000	1.712880	1	2
20	**combined**	**1.554688**	**2.196000**	**2.283840**	**1**	**2**
	generalized Type-I	1.660507	2.196000	–	1	2
	generalized Type-II	2.205911	2.196000	2.283840	1	–
	unified	2.333811	2.196000	2.283840	1	2
25	**combined**	**1.937463**	**2.745000**	**2.854800**	**1**	**2**
	generalized Type-I	2.066556	2.745000	–	1	2
	generalized Type-II	2.757007	2.745000	2.854800	1	–
	unified	2.912672	2.745000	2.854800	1	2
30	**combined**	**2.335625**	**3.294000**	**3.425760**	**1**	**2**
	generalized Type-I	2.484987	3.294000	–	1	2
	generalized Type-II	3.307773	3.294000	3.425760	1	–
	unified	3.490579	3.294000	3.425760	1	2
40	**combined**	**3.134853**	**4.392000**	**4.567680**	**1**	**2**
	generalized Type-I	3.353591	4.392000	–	1	2
	generalized Type-II	4.410420	4.392000	4.567680	1	–
	unified	4.675210	4.392000	4.567680	1	2
50	**combined**	**3.934242**	**5.490000**	**5.709600**	**1**	**2**
	generalized Type-I	4.186464	5.490000	–	1	2
	generalized Type-II	5.512789	5.490000	5.709600	1	–
	unified	5.824028	5.490000	5.709600	1	2

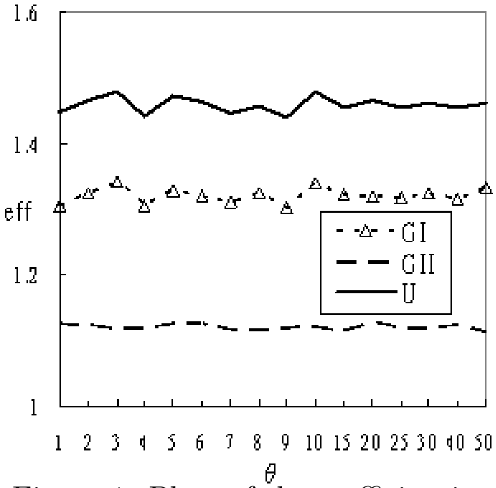

Figure 1. Plots of three efficiencies with $n = 10$.

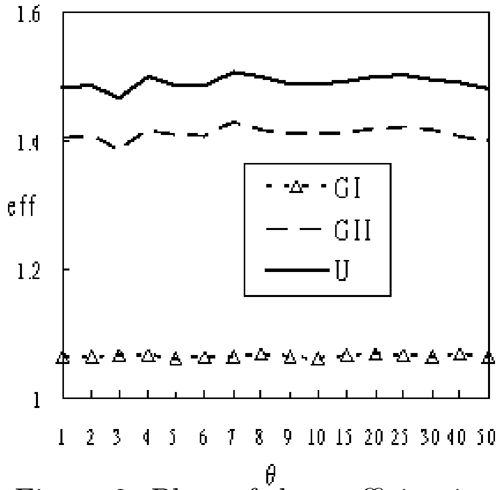

Figure 2. Plots of three efficiencies with $n = 20$.