Tamsui Oxford Journal of Information and Mathematical Sciences **27(1)** (2011) 99-115
Aletheia University

# On a General Partial Integral Equation of Barbashin Type *

B. G. Pachpatte†
*57 Shri Niketan Colony, Near Abhinay Talkies,*
*Aurangabad 431 001 (Maharashtra) India*

Received April 12, 2010, Accepted August 10, 2010.

**Abstract**

The aim of the present paper is to study some basic qualitative properties of solutions of a general partial integral equation of Barbashin type which occur frequently in applications. A variant of a certain integral inequality with explicit estimate is obtained and used to establish the results.

**Keywords and Phrases:** *Partial integral equation, Barbashin type, Integral inequality, Explicit estimate, Approximate solutions, Discrete analogues.*

## 1. Introduction

E.A.Barbashin [2] was the first to attempt to solve the integrodifferential equation of the form

$$\frac{\partial}{\partial t}u(t,x) = c(t,x)\,u(t,x) + \int_a^b k(t,x,y)\,u(t,y)\,dy + f(t,x), \tag{1.1}$$

*2000 *Mathematics Subject Classification.* Primary 65R20, 45G10.
†E-mail: bgpachpatte@gmail.com

which appear in mathematical modelling of many applied problems (see [1, section 19]). The equation (1.1) attracted the attention of many researchers and is now known in the literature as integrodifferential equation of Barbashin type or simply Barbashin equation, see [1, p. 1]. The equation of the type (1.1) and the partial integral equations are connected to each other in several ways. For example, suppose we are interested in finding a solution $u$ of equation (1.1) satisfying the initial condition $u(t_0, x) = u_0(x)$ for $a \leq x \leq b$ and $t_0 \in J = [0, \infty)$ is fixed and $u_0 : [a, b] \to R$ is a given continuous function. Putting $\frac{\partial}{\partial t} u(t, x) := w(t, x)$ we arrive at the equation

$$w(t, x) = g(t, x) + \int_{t_0}^{t} c(t, x) w(\tau, x) d\tau + \int_{t_0}^{t} \int_{a}^{b} k(t, x, y) w(\tau, y) dy d\tau, \quad (1.2)$$

where

$$g(t, x) = f(t, x) + c(t, x) u_0(x) + \int_{a}^{b} k(t, x, y) u_0(y) dy. \quad (1.3)$$

The equation (1.2) may be viewed as partial integral equation of Barbashin type. In this paper we consider a general partial integral equation of the form

$$u(t, x) = h(t, x) + \int_{0}^{t} f(t, x, s, u(s, x)) ds + \int_{0}^{t} \int_{B} g(t, x, s, y, u(s, y)) dy ds, \quad (1.4)$$

for $(t, x) \in E$, where $h \in C(E, R)$, $f \in C(E_1 \times R, R)$, $g \in C(E^2 \times R, R)$ are given functions and $u$ is the unknown function to be found. Here $R = (-\infty, \infty), R_+ = [0, \infty)$, $B = \prod_{i=1}^{m} [a_i, b_i] \subset R^m$ $(a_i < b_i)$ and $E = R_+ \times B$, $E_1 = \{(t, x, s) : 0 \leq s \leq t < \infty, x \in B\}$. The partial derivative of a function $r$ defined on $E^2$ (or $E_1$) with respect to the first variable is denoted by $D_1 r$ and $C(A_1, A_2)$ denotes the class of continuous functions from the set $A_1$ to the set $A_2$. For any function $u$ defined on $B$, we denote by $\int_B u(y) dy$ the $m$-fold integral $\int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} u(y_1, \ldots, y_m) dy_m \ldots dy_1$. The problems of existence and uniqueness of solutions of equation (1.4) can be dealt with the method employed in [5,8], see also [3,4,11,12]. For a detailed account on the study of such equations, see the monograph [1] and the references cited therein.

In dealing with the equations like (1.4), the basic questions to be answered are : $(i)$ if solutions do exist, then what are their nature ? $(ii)$ how can we find them or closely approximate them ?. In practice one need a new insight to study such questions for equations like (1.4). In the present paper we focus our attention to study some fundamental qualitative properties of solutions of equation (1.4). A new variant of a certain integral inequality with explicit estimate has been found and used to establish the results. Discrete analogues of the main results are also given.

## 2. A Basic Integral Inequality

In this section we establish a new variant of the integral inequality given by the present author in [8] (see also [9]), which can be used as a tool in the qualitative analysis of our main results.

**Theorem 1.** *Let* $u(t,x) \in C(E, R_+)$, $q(t,x,s), D_1 q(t,x,s) \in C(E_1, R_+)$, $k(t,x,s,y)$, $D_1 k(t,x,s,y) \in C(E^2, R_+)$ *and* $c \geq 0$ *is a constant. If*

$$u(t,x) \leq c + \int_0^t q(t,x,s)\, u(s,x)\, ds + \int_0^t \int_B k(t,x,s,y)\, u(s,y)\, dy ds, \tag{2.1}$$

*for* $(t,x) \in E$, *then*

$$u(t,x) \leq c\, P(t,x) \exp\left( \int_0^t A(\sigma,x)\, d\sigma \right), \tag{2.2}$$

*for* $(t,x) \in E$, *where*

$$P(t,x) = \exp(Q(t,x)), \tag{2.3}$$

*in which*

$$Q(t,x) = \int_0^t \left[ q(\eta,x,\eta) + \int_0^\eta D_1 q(\eta,x,\xi)\, d\xi \right] d\eta, \tag{2.4}$$

*and*

$$A(t,x) = \int_B k(t,x,t,y)\, P(t,y)\, dy + \int_0^t \int_B P(s,y)\, D_1 k(t,x,s,y)\, dy ds, \tag{2.5}$$

*for* $(t,x) \in E$.

**Proof.** Define a function $n(t,x)$ by

$$n(t,x) = c + \int_0^t \int_B k(t,x,s,y)\, u(s,y)\, dyds, \tag{2.6}$$

then (2.1) can be restated as

$$u(t,x) \le n(t,x) + \int_0^t q(t,x,s)\, u(s,x)\, ds. \tag{2.7}$$

From the hypotheses, it is easy to observe that $n(t,x)$ is nonnegative for $(t,x) \in E$ and nondecreasing in $t \in R_+$ for every $x \in B$. Treating (2.7) as one-dimensional integral inequality for every $x \in B$ and a suitable application of the inequality given in [7, Theorem 1.2.1, Remark 1.2.1, p. 11] yields

$$u(t,x) \le n(t,x)\, P(t,x). \tag{2.8}$$

From (2.6) and (2.8), we obtain

$$n(t,x) \le c + \int_0^t \int_B k(t,x,s,y)\, P(s,y)\, n(s,y)\, dyds. \tag{2.9}$$

Setting

$$e(t,x,s) = \int_B k(t,x,s,y)\, P(s,y)\, n(s,y)\, dy, \tag{2.10}$$

the inequality (2.9) can be restated as

$$n(t,x) \le c + \int_0^t e(t,x,s)\, ds. \tag{2.11}$$

For every $x \in B$, define

$$z(t) = c + \int_0^t e(t,x,s)\, ds, \tag{2.12}$$

then $z(0) = c$ and

$$n(t,x) \le z(t). \tag{2.13}$$

From (2.12), (2.10), (2.13) and the fact that $z(t)$ is nondecreasing in $t \in R_+$ for every $x \in B$, we observe that

$$z'(t) = e(t,x,t) + \int_0^t D_1 e(t,x,s)\,ds$$

$$= \int_B k(t,x,t,y)\,P(t,y)\,n(t,y)\,dy + \int_0^t D_1 \left\{ \int_B k(t,x,s,y)\,P(s,y)\,n(s,y)\,dy \right\} ds$$

$$\le \int_B k(t,x,t,y)\,P(t,y)\,z(t)\,dy + \int_0^t \int_B P(s,y)\,D_1 k(t,x,s,y)\,z(s)\,dyds$$

$$\le A(t,x)\,z(t). \tag{2.14}$$

The inequality (2.14) implies

$$z(t) \le c\exp\left( \int_0^t A(\sigma,x)\,d\sigma \right). \tag{2.15}$$

The required inequality in (2.2) follows from (2.15), (2.13) and (2.8).

# 3. Estimates on the Solutions

First we shall give the following theorem concerning the estimate on the solution of equation (1.4).

**Theorem 2.** *Suppose that the functions $f, g, h$ in (1.4) satisfy the conditions*

$$|f(t,x,s,u) - f(t,x,s,\bar{u})| \le q(t,x,s)\,|u-\bar{u}|, \tag{3.1}$$

$$|g(t,x,s,y,u) - g(t,x,s,y,\bar{u})| \le k(t,x,s,y)\,|u-\bar{u}|, \tag{3.2}$$

*and*

$$d = \sup_{(t,x)\in E}\left[|h(t,x)| + \int_0^t |f(t,x,s,0)|\,ds + \int_0^t\int_B |g(t,x,s,y,0)|\,dyds\right] < \infty, \tag{3.3}$$

*where* $q \in C(E_1, R_+)$, $k \in C(E^2, R_+)$; $D_1 q, D_1 k$ *exist and* $D_1 q \in C(E_1, R_+)$, $D_1 k \in C(E^2, R_+)$. *If* $u(t,x)$ *is any solution of (1.4) on* $E$, *then*

$$|u(t,x)| \le d\,P(t,x)\exp\left(\int_0^t A(\sigma,x)\,d\sigma\right), \tag{3.4}$$

*for* $(t,x) \in E$, *where* $P$ *and* $A$ *are given by (2.3) and (2.5).*

**Proof.** Using the fact that $u(t,x)$ is a solution of (1.4) and hypotheses, we observe that

$$\begin{aligned}
|u(t,x)| &\le |h(t,x)| + \int_0^t |f(t,x,s,u(s,x)) - f(t,x,s,0) + f(t,x,s,0)|\,ds \\
&\quad + \int_0^t\int_B |g(t,x,s,y,u(s,y)) - g(t,x,s,y,0) + g(t,x,s,y,0)|dyds \\
&\le |h(t,x)| + \int_0^t |f(t,x,s,0)|\,ds + \int_0^t\int_B |g(t,x,s,y,0)|dyds \\
&\quad + \int_0^t |f(t,x,s,u(s,x)) - f(t,x,s,0)|\,ds \\
&\quad + \int_0^t\int_B |g(t,x,s,y,u(s,y)) - g(t,x,s,y,0)|dyds \\
&\le d + \int_0^t q(t,x,s)\,|u(s,x)|\,ds + \int_0^t\int_B k(t,x,s,y)\,|u(s,y)|dyds.
\end{aligned} \tag{3.5}$$

Now an application of Theorem 1 to (3.5) yields (3.4).

A slight variant of Theorem 2 is given in the following theorem.

**Theorem 3.** *Suppose that the functions $f, g, h$ in (1.4) satisfy the conditions (3.1), (3.2) and*

$$\bar{d} = \sup_{(t,x)\in E} \left[ \int_0^t |f(t,x,s,h(s,x))|\,ds + \int_0^t \int_B |g(t,x,s,y,h(s,y))|\,dyds \right] < \infty. \tag{3.6}$$

*If $u(t,x)$ is any solution of (1.4) on $E$, then*

$$|u(t,x) - h(t,x)| \le \bar{d}\,P(t,x) \exp\left( \int_0^t A(\sigma,x)\,d\sigma \right), \tag{3.7}$$

*for $(t,x) \in E$, where $P$ and $A$ are given by (2.3) and (2.5).*

**Proof.** Let $e(t,x) = |u(t,x) - h(t,x)|$, $(t,x) \in E$. Using the fact that $u(t,x)$ is a solution of (1.4) and hypothses, we observe that

$$e(t,x) \le \int_0^t |f(t,x,s,u(s,x)) - f(t,x,s,h(s,x)) + f(t,x,s,h(s,x))|\,ds$$

$$+ \int_0^t \int_B |g(t,x,s,y,u(s,y)) - g(t,x,s,y,h(s,y)) + g(t,x,s,y,h(s,y))| dyds$$

$$\le \int_0^t |f(t,x,s,h(s,x))|\,ds + \int_0^t \int_B |g(t,x,s,y,h(s,y))| dyds$$

$$+ \int_0^t |f(t,x,s,u(s,x)) - f(t,x,s,h(s,x))|\,ds$$

$$+ \int_0^t \int_B |g(t,x,s,y,u(s,y)) - g(t,x,s,y,h(s,y))| dyds$$

$$\leq \bar{d} + \int_0^t q(t,x,s)\,e(s,x)\,ds + \int_0^t \int_B k(t,x,s,y)\,e(s,y)dyds. \tag{3.8}$$

Now an application of Theorem 1 to (3.8) yields (3.7).

We next prove under more restrictive conditions on the functions involved in (1.4) that the solutions tends to zero as $t \to \infty$.

**Theorem 4.** *Assume that*

$$|h(t,x)| \leq M \exp(-\alpha t), \tag{3.9}$$

$$|f(t,x,s,u) - f(t,x,s,\bar{u})| \leq q(t,x,s)\exp(-\alpha(t-s))\,|u-\bar{u}|, \tag{3.10}$$

$$|g(t,x,s,y,u) - g(t,x,s,y,\bar{u})| \leq k(t,x,s,y)\exp(-\alpha(t-s))\,|u-\bar{u}|, \tag{3.11}$$

*and* $f(t,x,s,0) = 0, g(t,x,s,y,0) = 0$, *where* $\alpha > 0, M \geq 0$ *are constants. The functions* $q, k$ *be as in Theorem 2 and*

$$\sup_{(t,x)\in E} Q(t,x) < \infty, \quad \int_0^\infty A(\sigma,x)\,d\sigma < \infty, \tag{3.12}$$

*where* $Q$ *and* $A$ *are given by (2.4) and (2.5). If* $u(t,x)$ *is any solution of (1.4) on* $E$*, then it tends exponentially toward zero as* $t \to \infty$.

**Proof.** From the hypotheses, we observe that

$$|u(t,x)| \leq |h(t,x)| + \int_0^t |f(t,x,s,u(s,x)) - f(t,x,s,0)|\,ds$$

$$+ \int_0^t \int_B |g(t,x,s,y,u(s,y)) - g(t,x,s,y,0)|dyds$$

$$\leq M \exp(-\alpha t) + \int_0^t q(t,x,s)\exp(-\alpha(t-s))\,|u(s,x)|\,ds$$

$$+\int_0^t \int_B k(t,x,s,y) \exp(-\alpha(t-s)) |u(s,y)| \, dyds. \tag{3.13}$$

From (3.13), we observe that

$$|u(t,x)| \exp(\alpha t) \le M + \int_0^t q(t,x,s) |u(s,x)| \exp(\alpha s) \, ds$$

$$+\int_0^t \int_B k(t,x,s,y) |u(s,y)| \exp(\alpha s) \, dyds. \tag{3.14}$$

Now an application of Theorem 1 to (3.14) yields

$$|u(t,x)| \exp(\alpha t) \le M \, P(t,x) \exp\left( \int_0^t A(\sigma,x) \, d\sigma \right). \tag{3.15}$$

Multiplying both sides of (3.15) by $\exp(-\alpha t)$ and in view of (3.12), the solution $u(t,x)$ tends to zero as $t \to \infty$.

# 4. Approximate Solutions

We call the function $u \in C(E,R)$ an $\varepsilon$-approximate solution of equation (1.4), if there exists a constant $\varepsilon \ge 0$ such that

$$\left| u(t,x) - h(t,x) - \int_0^t f(t,x,s,u(s,x)) \, ds - \int_0^t \int_B g(t,x,s,y,u(s,y)) \, dyds \right| \le \varepsilon, \tag{4.1}$$

for $(t,x) \in E$.

The following theorem deals with the estimate on the difference between the two approximate solutions of equation (1.4).

**Theorem 5.** *Suppose that the functions $f, g$ in (1.4) satisfy the conditions (3.1) and (3.2). Let $u_i(t,x)(i=1,2)$ be respectively $\varepsilon_i$-approximate solutions of (1.4) on $E$. Then*

$$|u_1(t,x)-u_2(t,x)| \leq (\varepsilon_1+\varepsilon_2) P(t,x) \exp\left(\int_0^t A(\sigma,x)\,d\sigma\right), \tag{4.2}$$

*for $(t,x)\in E$, where $P$ and $A$ are given by (2.3) and (2.5).*

**Proof.** Since $u_i(t,x)(i=1,2)$ for $(t,x)\in E$ are respectively $\varepsilon_i$-approximate solutions of (1.4), we have

$$\left|u_i(t,x)-h(t,x)-\int_0^t f(t,x,s,u_i(s,x))\,ds-\int_0^t\int_B g(t,x,s,y,u_i(s,y))\,dyds\right| \leq \varepsilon_i. \tag{4.3}$$

From (4.3) and using the elementary inequalities $|v-z|\leq|v|+|z|$, $|v|-|z|\leq |v-z|$, we observe that

$$\begin{aligned}
\varepsilon_1+\varepsilon_2 &\geq \left|u_1(t,x)-h(t,x)-\int_0^t f(t,x,s,u_1(s,x))\,ds-\int_0^t\int_B g(t,x,s,y,u_1(s,y))\,dyds\right| \\
&\quad+\left|u_2(t,x)-h(t,x)-\int_0^t f(t,x,s,u_2(s,x))\,ds-\int_0^t\int_B g(t,x,s,y,u_2(s,y))\,dyds\right| \\
&\geq \left|\left\{u_1(t,x)-h(t,x)-\int_0^t f(t,x,s,u_1(s,x))\,ds-\int_0^t\int_B g(t,x,s,y,u_1(s,y))\,dyds\right\}\right. \\
&\quad-\left.\left\{u_2(t,x)-h(t,x)-\int_0^t f(t,x,s,u_2(s,x))\,ds-\int_0^t\int_B g(t,x,s,y,u_2(s,y))\,dyds\right\}\right| \\
&\geq |u_1(t,x)-u_2(t,x)|-\left|\int_0^t f(t,x,s,u_1(s,x))\,ds-\int_0^t f(t,x,s,u_2(s,x))\,ds\right| \\
&\quad-\left|\int_0^t\int_B g(t,x,s,y,u_1(s,y))\,dyds-\int_0^t\int_B g(t,x,s,y,u_2(s,y))\,dyds\right|.
\end{aligned} \tag{4.4}$$

Let $w(t,x) = |u_1(t,x) - u_2(t,x)|$, $(t,x) \in E$. From (4.4) and using the hypotheses, we observe that

$$w(t,x) \leq \varepsilon_1 + \varepsilon_2 + \int_0^t |f(t,x,s,u_1(s,x)) - f(t,x,s,u_2(s,x))| ds$$

$$+ \int_0^t \int_B |g(t,x,s,y,u_1(s,y)) - g(t,x,s,y,u_2(s,y))| \, dyds$$

$$\leq \varepsilon_1 + \varepsilon_2 + \int_0^t q(t,x,s) \, w(s,x) ds + \int_0^t \int_B k(t,x,s,y) \, w(s,y) \, dyds. \qquad (4.5)$$

Now an application of Theorem 1 to (4.5) yields (4.2).

**Remark 1.** In case $u_1(t,x)$ is a solution of (1.4), then we have $\varepsilon_1 = 0$ and from (4.2) we see that $u_2(t,x) \to u_1(t,x)$ as $\varepsilon_2 \to 0$. Moreover, from (4.2) it follows that if $\varepsilon_1 = \varepsilon_2 = 0$, then the uniqueness of solutions of (1.4) is established.

Consider the equation (1.4) together with the following integral equation

$$v(t,x) = \bar{h}(t,x) + \int_0^t \bar{f}(t,x,s,v(s,x)) \, ds + \int_0^t \int_B \bar{g}(t,x,s,y,v(s,y)) \, dyds, \qquad (4.6)$$

for $(t,x) \in E$, where $\bar{h} \in C(E,R)$, $\bar{f} \in C(E_1 \times R, R)$, $\bar{g} \in C(E^2 \times R, R)$.

In the following theorem we provide conditions concerning the closeness of solutions of (1.4) and (4.6).

**Theorem 6.** *Suppose that the functions $f, g$ in (1.4) satisfy the conditions (3.1), (3.2) and there exist constants $\bar{\varepsilon}_i \geq 0$ $(i = 1,2)$, $\bar{\delta} \geq 0$ such that*

$$\left| f(t,x,s,u) - \bar{f}(t,x,s,u) \right| \leq \bar{\varepsilon}_1, \qquad (4.7)$$

$$|g(t,x,s,y,u) - \bar{g}(t,x,s,y,u)| \leq \bar{\varepsilon}_2, \qquad (4.8)$$

$$\left|h(t,x)-\bar{h}(t,x)\right| \leq \bar{\delta}, \tag{4.9}$$

*where $f, g, h$ and $\bar{f}, \bar{g}, \bar{h}$ are functions involved in (1.4) and (4.6) and let*

$$\bar{M} = \sup_{t \in R_+} \left[\bar{\delta} + t\left\{\bar{\varepsilon}_1 + \bar{\varepsilon}_2 \prod_{i=1}^{m}(b_i - a_i)\right\}\right]. \tag{4.10}$$

*Let $u(t,x)$ and $v(t,x)$ be respectively the solutions of (1.4) and (4.6) for $(t,x) \in E$. Then*

$$|u(t,x)-v(t,x)| \leq \bar{M}\, P(t,x) \exp\left(\int_0^t A(\sigma,x)\,d\sigma\right), \tag{4.11}$$

*for $(t,x) \in E$, where $P$ and $A$ are given by (2.3) and (2.5).*

**Proof.** Let $z(t,x) = |u(t,x)-v(t,x)|$, $(t,x) \in E$. Using the hypotheses, we observe that

$$z(t,x) \leq \left|h(t,x)-\bar{h}(t,x)\right| + \int_0^t |f(t,x,s,u(s,x)) - f(t,x,s,v(s,x))$$

$$+ f(t,x,s,v(s,x)) - \bar{f}(t,x,s,v(s,x))\Big|\, ds$$

$$+ \int_0^t \int_B |g(t,x,s,y,u(s,y)) - g(t,x,s,y,v(s,y))$$

$$+ g(t,x,s,y,v(s,y)) - \bar{g}(t,x,s,y,v(s,y))|\, dyds$$

$$\leq \bar{\delta} + \int_0^t |f(t,x,s,u(s,x)) - f(t,x,s,v(s,x))|\, ds$$

$$+ \int_0^t \left|f(t,x,s,v(s,x)) - \bar{f}(t,x,s,v(s,x))\right| ds$$

$$+ \int_0^t \int_B |g(t,x,s,y,u(s,y)) - g(t,x,s,y,v(s,y))| dyds$$

$$+\int_0^t \int_B |g(t,x,s,y,v(s,y)) - \bar{g}(t,x,s,y,v(s,y))| dyds$$

$$\leq \bar{\delta} + \int_0^t q(t,x,s)\, z(s,x)\, ds + \int_0^t \bar{\varepsilon}_1 ds$$

$$+\int_0^t \int_B k(t,x,s,y)\, z(s,y) dyds + \int_0^t \int_B \bar{\varepsilon}_2 dyds$$

$$\leq \bar{M} + \int_0^t q(t,x,s)\, z(s,x)\, ds + \int_0^t \int_B k(t,x,s,y)\, z(s,y) dyds. \tag{4.12}$$

Now an application of Theorem 1 to (4.12) yields (4.11).

**Remark 2.** The result given in Theorem 6 relates the solutions of (1.4) and (4.6) in the sense that if $f, g, h$ are respectively close to $\bar{f}, \bar{g}, \bar{h}$; then the solutions of (1.4) and (4.6) are also close together.

# 5. Discrete Analogues

Let $N = \{1, 2, ...\}, N_0 = \{0, 1, 2, ...\}, N_i[\alpha_i, \beta_i] = \{\alpha_i, \alpha_i + 1, ..., \beta_i\}\ (\alpha_i < \beta_i), \alpha_i \in N_0, \beta_i \in N$ for i=1,2,...,m, $H = \prod_{i=1}^{m} N_i[\alpha_i, \beta_i] \subset R^m, G = N_0 \times H, G_1 = \{(n, x, \sigma) : \sigma \leq n, \sigma, n \in N_0, x \in H\}$. For any function $r$ defined on $G^2$ (or $G_1$), we define the operator $\Delta_1$ by $\Delta_1 r(n,x,\sigma,y) = r(n+1,x,\sigma,y) - r(n,x,\sigma,y)$ (or $\Delta_1 r(n,x,\sigma) = r(n+1,x,\sigma) - r(n,x,\sigma)$) and for any function $w$ defined on $H$ we denote the $m$-fold sum over $H$ for $y \in H$ by $\sum_H w(y) = \sum_{y_1=\alpha_1}^{\beta_1} ... \sum_{y_m=\alpha_m}^{\beta_m} w(y_1, ..., y_m)$. We denote by $D(S_1, S_2)$ the class of discrete functions from the set $S_1$ to the $S_2$ and use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. The sum-difference equation that constitutes the discrete analogue of equation (1.4) can be written as

$$u(n,x) = h(n,x) + \sum_{\sigma=0}^{n-1} f(n,x,\sigma,u(\sigma,x)) + \sum_{\sigma=0}^{n-1}\sum_{H} g(n,x,\sigma,y,u(\sigma,y)), \tag{5.1}$$

for $(n,x) \in G$, where $h \in D(G,R)$, $f \in D(G_1 \times R, R)$, $g \in D(G^2 \times R, R)$. In this section, we formulate in brief the discrete analogues of Theorems 1 and 2 only. One can formulate results similar to those given above in Theorems 3-6 for solutions of equation (5.1). For detailed account on the study of various types of sum-difference equations, see [6,7].

**Theorem 7.** *Let* $u(n,x) \in D(G,R_+)$, $q(n,x,\sigma)$, $\Delta_1 q(n,x,\sigma) \in D(G_1,R_+)$, $k(n,x,\sigma,y)$, $\Delta_1 k(n,x,\sigma,y) \in D(G^2,R_+)$ *and* $c \geq 0$ *is a constant. If*

$$u(n,x) \leq c + \sum_{\sigma=0}^{n-1} q(n,x,\sigma)\, u(\sigma,x) + \sum_{\sigma=0}^{n-1}\sum_{H} k(n,x,\sigma,y)\, u(\sigma,y), \tag{5.2}$$

*for* $(n,x) \in G$, *then*

$$u(n,x) \leq c\,\bar{P}(n,x) \prod_{\sigma=0}^{n-1}\left[1 + \bar{A}(\sigma,x)\right], \tag{5.3}$$

*for* $(n,x) \in G$, *where*

$$\bar{P}(n,x) = \prod_{\xi=0}^{n-1}\left[1 + q(\xi+1,x,\xi) + \sum_{s=0}^{\xi-1} \Delta_1 q(\xi,x,s)\right], \tag{5.4}$$

$$\bar{A}(n,x) = \sum_{H} k(n+1,x,n,y)\,\bar{P}(n,y) + \sum_{s=0}^{n-1}\sum_{H} \Delta_1 k(n,x,s,y)\,\bar{P}(s,y). \tag{5.5}$$

*The proof can be completed by closely looking at the proof of Theorem 1 given above and also the proofs of similar inequalities given in [6,7].*

**Theorem 8.** *Suppose that the functions $f, g, h$ in (5.1) satisfy the conditions*

$$|f(n,x,\sigma,u) - f(n,x,\sigma,\bar{u})| \leq q(n,x,\sigma)|u-\bar{u}|, \tag{5.6}$$

$$|g(n,x,\sigma,y,u) - g(n,x,\sigma,y,\bar{u})| \leq k(n,x,\sigma,y)|u-\bar{u}|, \tag{5.7}$$

*and*

$$\gamma = \sup_{(n,x)\in G}\left[|h(n,x)| + \sum_{\sigma=0}^{n-1}|f(n,x,\sigma,0)| + \sum_{\sigma=0}^{n-1}\sum_{H}|g(n,x,\sigma,y,0)|\right] < \infty, \tag{5.8}$$

*where $q \in D(G_1, R_+)$, $k \in D(G^2, R_+)$; $\Delta_1 q, \Delta_1 k$ exist and $\Delta_1 q \in D(G_1, R_+)$, $\Delta_1 k \in D(G^2, R_+)$. If $u(n,x)$ is any solution of (5.1) on $G$, then*

$$|u(n,x)| \leq \gamma \bar{P}(n,x) \prod_{\sigma=0}^{n-1}\left[1 + \bar{A}(\sigma,x)\right], \tag{5.9}$$

*for $(n,x) \in G$, where $\bar{P}$ and $\bar{A}$ are given by (5.4) and (5.5).*

The proof follows by the arguments as in the proof of Theorem 2 given above and using Theorem 7. We omit the details.

**Remark 3.** We note that the idea used in this paper can be extended to study the integral equation of the form

$$u(x,y,z) = h(x,y,z) + \int_0^x\int_0^y f(x,y,z,s,t,u(s,t,z))\,dtds + \int_0^x\int_0^y\int_\Omega g(x,y,z,s,t,r,u(s,t,r))drdtds, \tag{5.10}$$

and its discrete analogue, which can be considered as a generalization of the equation recently studied by the present author in [10]. Here we do not discuss the details. We strongly believe that the results obtained here may be a source of an extensive and fruitful research in the future.

# References


[1] J. M. Appell, A. S. Kalitvin, and P. P. Zabrejko, *Partial integral operators and integrodifferential equations*, **230**, Pure and applied mathematics monographs, Marcel and Dekker, Inc., New York, 2000.

[2] E. A. Barbashin, On conditions for the conservation of stability of solutions to integrodifferential equations [in Russian], *Izv.VUZov Mat.,* **1**(1957), 25-34.

[3] H. Brunner, *Collocation methods for Volterra integral and related functional differential equations*, Cambridge University Press, Cambridge, 2004.

[4] O. Diekmann, Thresholds and travelling waves for the geogrephical spread of infection, *J. Math. Biol.,* **6**(1978), 109-130.

[5] B. G. Pachpatte, On mixed Volterra-Fredholm type integral equations, *Indian J. Pure Appl. Math.*, **17**(1986), 488-496.

[6] B. G. Pachpatte, *Inequalities for Finite Difference Equations*, **247**, Pure and Applied mathematics monographs, Marcel and Dekker, Inc., New York, 2002.

[7] B. G. Pachpatte, *Integral and Finite Difference Inequalities and Applications*, North-Holland Mathematics Studies, **205**, Elsevier Science B.V., Amerstdam, 2006 .

[8] B. G. Pachpatte, On a parabolic type Fredholm integrodifferential equation, *Numer. Funct. Anal. Optim.,* **30**(2009), 136-147.

[9] B. G. Pachpatte, Growth estimates on mixed Volterra-Fredholm type integral inequalities, *Fasc. Math.,* **42** (2009), 63-72.

[10] B. G. Pachpatte, New integral and finite difference inequalities in three variables, *Demonstratio Mathematica*, **XLII**(2009), 339-349.

[11] H. R. Thieme, On the boundedness and asymptotic behavior of the nonnegative solutions of Volterra-Hammerstein integral equations, *Manuscr. Math.,* **31**(1980), 379-412.

[12] A. M. Wazwaz, A reliable treatment for mixed Volterra-Fredholm integral equations, *App. Math. Comput.,* **127**(2002), 405-414.