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On an Inequality Concerning a Non-homogeneous Kernel and the Hypergeometric Function *

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Abstract

By introducing a non-homogeneous kernel of 0-degree with an independent parameter and estimating the weight function through the real function techniques, a new integral inequality with a best constant factor is established, where the best constant factor is made of both the beta function and the hypergeometric function. In addition, the reverse inequality and their corresponding equivalent forms are given.

Keywords and Phrases: *Hilbert's integral inequality, Weight function, Hölder's inequality, Hypergeometric Function.*

1. Introduction

One hundred years ago, D.Hilbert proved the following classic inequality [1]

$$\sum_n \sum_m \frac{a_m b_n}{m+n} \leq \pi \left(\sum_n a_n^2 \right)^{1/2} \left(\sum_n b_n^2 \right)^{1/2}. \tag{1.1}$$

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During the past century, ever since the advent the inequality (1.1), numerous related results have been obtained. The inequality(1.1) may be classified into several types (discrete and integral etc.), being the following integral form:

If f, g are real functions such that $0 < \int_0^\infty f^2(x)\mathrm{d}x < \infty, 0 < \int_0^\infty g^2(x)\mathrm{d}x < \infty$, then we have [1]

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \pi \left\{ \int_0^\infty f^2(x)\mathrm{d}x \int_0^\infty g^2(x)\mathrm{d}x \right\}^{\frac{1}{2}}, \tag{1.2}$$

where the constant factor π is the best possible. Inequality (1.2) had been generalized by Hardy-Riesz in 1925 as [1]:

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ $f, g \geq 0$ such that $0 < \int_0^\infty f^p(x)\mathrm{d}x, \int_0^\infty g^q(x)\mathrm{d}x < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \int_0^\infty f^p(x)\mathrm{d}x \right\}^{1/p} \left\{ \int_0^\infty g^q(x)\mathrm{d}x \right\}^{1/q}, \tag{1.3}$$

where the constant factor $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible. When $p = q = 2$, (1.3) reduces to (1.2), Inequality (1.3) is named as Hardy-Hilbert's integral inequality, which is of great importance in analysis and its applications [2, 3]. Its generalization can be seen in [4]-[9].

By introducing non-independent parameters p, q and λ, an inequality was given as : If $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} > 1, \lambda = 2 - \frac{1}{p} - \frac{1}{q}(0 < \lambda < 1)$, $f, g \geq 0$ such that the right hand side integrals are convergent, then [1]

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{|x+y|^\lambda}\mathrm{d}x\mathrm{d}y < K(p,q) \left\{ \int_{-\infty}^\infty f^p(x)\mathrm{d}x \right\}^{1/p} \left\{ \int_{-\infty}^\infty g^q(x)\mathrm{d}x \right\}^{1/q}, \tag{1.4}$$

where $K(p, q)$ is related to p, q. In 1998, Yang, by introduced an independent parameter $\lambda(> 0)$, has obtained the following inequality[10, 11]:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}\mathrm{d}x\mathrm{d}y < B(\frac{\lambda}{2}, \frac{\lambda}{2}) \left\{ \int_0^\infty x^{1-\lambda} f^2(x)\mathrm{d}x \int_0^\infty x^{1-\lambda} g^2(x)\mathrm{d}x \right\}^{\frac{1}{2}}, \tag{1.5}$$

where the constant factor $B(\frac{\lambda}{2}, \frac{\lambda}{2})$, known as the beta function [13] is the best possible.

Very recently, Yang [12] obtained the improved inequality of (1.4) as follows: If $p, r > 1, 0 < \lambda < 1$ and the right hand side integrals are convergent, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{|1+xy|^{\lambda}}\mathrm{d}x\mathrm{d}y$$

$$< k_{\lambda}(r)\left\{\int_{-\infty}^{\infty}|x|^{p(1-\frac{\lambda}{2})}f^{p}(x)\mathrm{d}x\right\}^{1/p}\left\{\int_{-\infty}^{\infty}|x|^{q(1-\frac{\lambda}{2})}g^{q}(x)\mathrm{d}x\right\}^{1/q}, \qquad (1.6)$$

where the constant factor $k_{\lambda}(r) = B(\frac{\lambda}{2},\frac{\lambda}{2}) + 2B(1-\lambda,\frac{\lambda}{2})$ is the best possible.

Until now, we only studied the related inequalities with $-\lambda(\lambda > 0)$ non-homogeneous kernel, now we attempt investigation for the real number non-homogeneous kernel. Motivated by (1.4)-(1.6), we establish a new inequality with the non-homogeneous kernel of 0-degree, where incidentally, the best constant factor is generated by both the hypergeometric function and the beta function.

2. Lemmas

The hypergeometric function $F(\alpha,\beta,\gamma,z)$ is defined [13] by

$${}_2F_1(\alpha,\beta,\gamma,z) = F(\alpha,\beta,\gamma,z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n}z^n, \qquad (2.1a)$$

where $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), n \geq 1$ and $(\alpha)_0 = 1, \alpha \neq 0$.

The integral form of it is [13], see also [14]

$$F(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\mathrm{d}t, \qquad (2.1b)$$

where $\mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0, |\arg(1-z)| < \pi$ and Γ is a gamma function. In particular, when $z = -1, \beta = \alpha > 0, \gamma = \alpha + 1$, we get

$$\int_0^1 t^{\alpha-1}(1+t)^{-\alpha}\mathrm{d}t = \frac{1}{\alpha}F(\alpha,\alpha,1+\alpha,-1). \qquad (2.2)$$

Lemma 2.1. *Setting* $0 < \lambda < 1$. *Define the weight function* $\varpi_{\lambda}(x)$ *as*

$$\varpi_{\lambda}(x) := \int_{-\infty}^{\infty}\left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda}\frac{1}{|y|}\mathrm{d}y, x \in (-\infty,\infty). \qquad (2.3)$$

Then for $x \in (-\infty, 0) \cup (0, \infty)$*, we get*

$$\varpi_\lambda(x) = C(\lambda) = 2\left(\frac{1}{\lambda}F(\lambda, \lambda, 1+\lambda, -1) + B(\lambda, 1-\lambda)\right). \qquad (2.4)$$

Proof. (i) If $x \in (-\infty, 0)$, then $1 + xy > 0$ for $y < -\frac{1}{x}$ and $1 + xy \le 0$ for $y \ge -\frac{1}{x}$, we have

$$\begin{aligned}
\varpi_\lambda(x) &= \int_{-\infty}^{\infty}\left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^\lambda \frac{1}{|y|}\mathrm{d}y \\
&= \int_{-\infty}^{0}\left(\frac{\min\{1,xy\}}{1+xy}\right)^\lambda \frac{1}{-y}\mathrm{d}y + \int_{0}^{-\frac{1}{x}}\left(\frac{\min\{1,-xy\}}{1+xy}\right)^\lambda \frac{1}{y}\mathrm{d}y \\
&\quad + \int_{-\frac{1}{x}}^{\infty}\left(\frac{\min\{1,-xy\}}{-1-xy}\right)^\lambda \frac{1}{y}\mathrm{d}y \\
&= \int_{-\infty}^{\frac{1}{x}}\left(\frac{1}{1+xy}\right)^\lambda \frac{1}{-y}\mathrm{d}y + \int_{\frac{1}{x}}^{0}\left(\frac{xy}{1+xy}\right)^\lambda \frac{1}{-y}\mathrm{d}y \\
&\quad + \int_{0}^{-\frac{1}{x}}\left(\frac{-xy}{1+xy}\right)^\lambda \frac{1}{y}\mathrm{d}y + \int_{-\frac{1}{x}}^{\infty}\left(\frac{1}{-1-xy}\right)^\lambda \frac{1}{y}\mathrm{d}y. \qquad (2.5)
\end{aligned}$$

Setting $t = xy$ for the first two integrals, $t = -xy$ for the last two integrals in (2.5), in view of (2.2), we obtain

$$\begin{aligned}
&\int_{-\infty}^{\frac{1}{x}}\left(\frac{1}{1+xy}\right)^\lambda \frac{1}{-y}\mathrm{d}y + \int_{\frac{1}{x}}^{0}\left(\frac{xy}{1+xy}\right)^\lambda \frac{1}{-y}\mathrm{d}y \\
&= \int_{1}^{+\infty}\frac{1}{t}\frac{1}{(1+t)^\lambda}\mathrm{d}t + \int_{0}^{1}\frac{t^{\lambda-1}}{(1+t)^\lambda}\mathrm{d}t \\
&= 2\int_{0}^{1}t^{\lambda-1}(1+t)^{-\lambda}\mathrm{d}t = \frac{2}{\lambda}F(\lambda,\lambda,1+\lambda,-1).
\end{aligned}$$

$$\begin{aligned}
&\int_{0}^{-\frac{1}{x}}\left(\frac{-xy}{1+xy}\right)^\lambda \frac{1}{y}\mathrm{d}y + \int_{-\frac{1}{x}}^{\infty}\left(\frac{1}{-1-xy}\right)^\lambda \frac{1}{y}\mathrm{d}y \\
&= \int_{0}^{1}(1-t)^{(1-\lambda)-1}t^{\lambda-1}\mathrm{d}t + \int_{1}^{+\infty}\frac{(t-1)^{(1-\lambda)-1}}{t}\mathrm{d}t \\
&= B(1-\lambda,\lambda) + B(1-\lambda,\lambda) = 2B(\lambda,1-\lambda).
\end{aligned}$$

Putting the above results into (2.5), we get (2.4).

(ii) If $x \in (0,\infty)$, then setting $t = -xy, t = -xy, t = xy$ and $t = xy$ respectively, by (2.2), we have

$$\begin{aligned}
\varpi_\lambda(x) &= \int_{-\infty}^{-\frac{1}{x}} \left(\frac{1}{-1-xy}\right)^\lambda \frac{1}{-y}\mathrm{d}y + \int_{-\frac{1}{x}}^{0} \left(\frac{-xy}{1+xy}\right)^\lambda \frac{1}{-y}\mathrm{d}y \\
&\quad + \int_0^{\frac{1}{x}} \left(\frac{xy}{1+xy}\right)^\lambda \frac{1}{y}\mathrm{d}y + \int_{\frac{1}{x}}^{\infty} \left(\frac{1}{1+xy}\right)^\lambda \frac{1}{y}\mathrm{d}y \\
&= \int_1^{+\infty} \frac{(t-1)^{-\lambda}}{t}\mathrm{d}t + \int_0^1 (1-t)^{-\lambda} t^{\lambda-1}\mathrm{d}t + 2\int_0^1 t^{\lambda-1}(1+t)^{-\lambda}\mathrm{d}t \\
&= 2\left(\frac{1}{\lambda}F(\lambda,\lambda,1+\lambda,-1) + B(\lambda,1-\lambda)\right) = C(\lambda).
\end{aligned}$$

Hence (2.4) is valid for $x \in (-\infty,0)\cup(0,\infty)$. □

Lemma 2.2. *Let $p > 0(\neq 1), \frac{1}{p}+\frac{1}{q} = 1, 0 < \lambda < 1, 0 < \varepsilon < \frac{|q|\lambda}{4}$. Define the weight function $\widetilde{f}(x), \widetilde{g}(x)$ as*

$$\widetilde{f}(x) = \begin{cases} x^{-1-\frac{2\varepsilon}{p}}, & x \in (1,\infty), \\ 0, & x \in [-1,1], \\ (-x)^{-1-\frac{2\varepsilon}{p}}, & x \in (-\infty,-1), \end{cases} \qquad \widetilde{g}(x) = \begin{cases} x^{-1+\frac{2\varepsilon}{q}}, & x \in (0,1), \\ 0, & x \in (-\infty,-1]\cup[1,\infty), \\ (-x)^{-1+\frac{2\varepsilon}{q}}, & x \in (-1,0). \end{cases}$$

Then we get the following inequality

$$h(\varepsilon) := \varepsilon\left\{\int_{-\infty}^{\infty} |x|^{p-1}\widetilde{f}^p(x)\mathrm{d}x\right\}^{\frac{1}{p}} \left\{\int_{-\infty}^{\infty} |x|^{q-1}\widetilde{g}^q(x)\mathrm{d}x\right\}^{\frac{1}{q}} = 1, \tag{2.6}$$

$$I(\varepsilon) := \int_{-\infty}^{\infty} \left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^\lambda \widetilde{f}(x)\widetilde{g}(y)\mathrm{d}x\mathrm{d}y = C(\lambda) + o(1),\ \varepsilon \to 0^+. \tag{2.7}$$

Proof. It is obvious that

$$h(\varepsilon) = \varepsilon\left\{2\int_1^{\infty} x^{p-1}x^{p(-1-\frac{2\varepsilon}{p})}\mathrm{d}x\right\}^{\frac{1}{p}} \left\{2\int_0^1 x^{q-1}x^{q(-1+\frac{2\varepsilon}{q})}\mathrm{d}x\right\}^{\frac{1}{q}} = 1.$$

Let $x > 0$, then setting $y = -Y$, since $\widetilde{f}(-x) = \widetilde{f}(x), \widetilde{g}(-y) = \widetilde{g}(y)$, we have

$$\widetilde{f}(-x)\int_{-\infty}^{\infty} \left(\frac{\min\{1,|-xy|\}}{|1+(-x)y|}\right)^\lambda \widetilde{g}(y)\mathrm{d}y = \widetilde{f}(x)\int_{-\infty}^{\infty} \left(\frac{\min\{1,|xY|\}}{|1+xY|}\right)^\lambda \widetilde{g}(-Y)\mathrm{d}Y$$

$$= \widetilde{f}(x)\int_{-\infty}^{\infty}\left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda}\widetilde{g}(y)\mathrm{d}y.$$

Thus $\widetilde{f}(x)\int_{-\infty}^{\infty}(\frac{\min\{1,|xy|\}}{|1+xy|})^{\lambda}\widetilde{g}(y)\mathrm{d}y$ is the even function of x, so

$$\begin{aligned} I(\varepsilon) &= 2\varepsilon\int_{0}^{\infty}\widetilde{f}(x)\left[\int_{-\infty}^{\infty}\left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda}\widetilde{g}(y)\mathrm{d}y\right]\mathrm{d}x \\ &= 2\varepsilon\{\int_{1}^{\infty}x^{-1-\frac{2\varepsilon}{p}}\left[\int_{-1}^{-1/x}\left(\frac{\min\{1,-xy\}}{-1-xy}\right)^{\lambda}(-y)^{-1+\frac{2\varepsilon}{q}}\mathrm{d}y\right]\mathrm{d}x \\ &\quad+\int_{1}^{\infty}x^{-1-\frac{2\varepsilon}{p}}\left[\int_{-1/x}^{0}\left(\frac{\min\{1,-xy\}}{1+xy}\right)^{\lambda}(-y)^{-1+\frac{2\varepsilon}{q}}\mathrm{d}y\right]\mathrm{d}x \\ &\quad+\int_{1}^{\infty}x^{-1-\frac{2\varepsilon}{p}}\left[\int_{0}^{1}\left(\frac{\min\{1,xy\}}{1+xy}\right)^{\lambda}y^{-1+\frac{2\varepsilon}{q}}\mathrm{d}y\right]\mathrm{d}x\} \triangleq I_1+I_2+I_3. \quad (2.8) \end{aligned}$$

Letting $t=-xy, t=-xy, t=xy$ respectively for the above three integrals, in view of (2.2), we obtain

$$\begin{aligned} I_1 &:= 2\varepsilon\int_{1}^{\infty}x^{-1-\frac{2\varepsilon}{p}}\left[\int_{-1}^{-1/x}\left(\frac{\min\{1,-xy\}}{-1-xy}\right)^{\lambda}(-y)^{-1+\frac{2\varepsilon}{q}}\mathrm{d}y\right]\mathrm{d}x \\ &= 2\varepsilon\int_{1}^{\infty}x^{-1-2\varepsilon}\left[\int_{1}^{x}\frac{1}{(t-1)^{\lambda}}t^{-1+\frac{2\varepsilon}{q}}\mathrm{d}t\right]\mathrm{d}x \\ &= 2\varepsilon\int_{1}^{\infty}\left(\int_{t}^{\infty}x^{-1-2\varepsilon}\mathrm{d}x\right)\frac{1}{(t-1)^{\lambda}}t^{-1+\frac{2\varepsilon}{q}}\mathrm{d}t \\ &= \int_{1}^{\infty}\frac{1}{(t-1)^{\lambda}}t^{-1-\frac{2\varepsilon}{p}}\mathrm{d}t=\int_{1}^{\infty}\frac{(t-1)^{(1-\lambda)-1}}{t^{1+\frac{2\varepsilon}{p}}}\mathrm{d}t=B(1-\lambda,\lambda+\frac{2\varepsilon}{p}). \end{aligned}$$

$$\begin{aligned} I_2 &:= 2\varepsilon\int_{1}^{\infty}x^{-1-\frac{2\varepsilon}{p}}\left[\int_{-1/x}^{0}\left(\frac{\min\{1,-xy\}}{1+xy}\right)^{\lambda}(-y)^{-1+\frac{2\varepsilon}{q}}\mathrm{d}y\right]\mathrm{d}x \\ &= 2\varepsilon\int_{1}^{\infty}x^{-1-2\varepsilon}\left[\int_{0}^{1}\frac{t^{\lambda-1+\frac{2\varepsilon}{q}}}{(1-t)^{\lambda}}\right]\mathrm{d}x \\ &= \int_{0}^{1}(1-t)^{(1-\lambda)-1}t^{\lambda+\frac{2\varepsilon}{q}-1}\mathrm{d}x=B(1-\lambda,\lambda+\frac{2\varepsilon}{q}). \end{aligned}$$

$$\begin{aligned}
I_3 &:= 2\varepsilon\int_1^\infty x^{-1-\frac{2\varepsilon}{p}}\left[\int_0^1\left(\frac{\min\{1,xy\}}{1+xy}\right)^\lambda y^{-1+\frac{2\varepsilon}{q}}\mathrm{d}y\right]\mathrm{d}x\\
&= 2\varepsilon\int_1^\infty x^{-1-\frac{2\varepsilon}{p}}\left[\int_0^{\frac{1}{x}}\left(\frac{xy}{1+xy}\right)^\lambda y^{-1+\frac{2\varepsilon}{q}}\mathrm{d}y+\int_{\frac{1}{x}}^1\left(\frac{1}{1+xy}\right)^\lambda y^{-1+\frac{2\varepsilon}{q}}\mathrm{d}y\right]\mathrm{d}x\\
&= 2\varepsilon\int_1^\infty x^{-1-2\varepsilon}\left[\int_0^1\frac{t^{\lambda-1+\frac{2\varepsilon}{q}}}{(1+t)^\lambda}\mathrm{d}t+\int_1^x\frac{t^{-1+\frac{2\varepsilon}{q}}}{(1+t)^\lambda}\mathrm{d}t\right]\mathrm{d}x\\
&= \int_0^1\frac{t^{\lambda-1+\frac{2\varepsilon}{q}}}{(1+t)^\lambda}\mathrm{d}t+2\varepsilon\int_1^\infty\left(\int_t^\infty x^{-1-2\varepsilon}\mathrm{d}x\right)\frac{t^{-1+\frac{2\varepsilon}{q}}}{(1+t)^\lambda}\mathrm{d}t\\
&= \int_0^1\frac{t^{\lambda-1+\frac{2\varepsilon}{q}}}{(1+t)^\lambda}\mathrm{d}t+\int_1^\infty\frac{t^{-1-\frac{2\varepsilon}{p}}}{(1+t)^\lambda}\mathrm{d}t.
\end{aligned}$$

When $p>0(\neq 1)$, we have

$$\begin{aligned}
0 &\leq \int_1^\infty\frac{t^{-1}}{(1+t)^\lambda}\mathrm{d}t-\int_1^\infty\frac{t^{-1-\frac{2\varepsilon}{p}}}{(1+t)^\lambda}\mathrm{d}t=\int_1^\infty\frac{t^{-1}}{(1+t)^\lambda}(1-t^{-1-\frac{2\varepsilon}{p}})\mathrm{d}t\\
&\leq \int_1^\infty(t^{-1-\lambda}-t^{-1-\lambda-\frac{2\varepsilon}{p}})\mathrm{d}t=\frac{1}{\lambda}-\frac{1}{\lambda+2\varepsilon/p}.
\end{aligned}$$

When $q>1$, we get

$$\begin{aligned}
0 &\leq \int_0^1\frac{t^{\lambda-1}}{(1+t)^\lambda}\mathrm{d}t-\int_0^1\frac{t^{\lambda-1+\frac{2\varepsilon}{q}}}{(1+t)^\lambda}\mathrm{d}t=\int_0^1\frac{t^{\lambda-1}}{(1+t)^\lambda}(1-t^{\frac{2\varepsilon}{q}})\mathrm{d}t\\
&\leq \int_0^1(t^{\lambda-1}-t^{\lambda-1+\frac{2\varepsilon}{q}})\mathrm{d}t=\frac{1}{\lambda}-\frac{1}{\lambda+2\varepsilon/q}.
\end{aligned}$$

Similarly, the above inequalities are justified by exchanging the minuend with subtrahend. Thus

$$\begin{aligned}
I_3 &= \int_1^\infty\frac{t^{-1}}{(1+t)^\lambda}\mathrm{d}t+o_1(1)+\int_0^1\frac{t^{\lambda-1}}{(1+t)^\lambda}\mathrm{d}t+o_2(1)\\
&= 2\int_0^1 t^{\lambda-1}(1+t)^{-\lambda}\mathrm{d}t+o_3(1)=\frac{1}{\lambda}F(\lambda,\lambda,1+\lambda,-1)+o_3(1),\ \varepsilon\to 0^+.
\end{aligned}$$

In view of the results of I_1, I_2, I_3 and the continuous of Beta function, by (2.8), we obtain (2.7.) □

3. Main Results and Applications

Theorem 3.1. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \lambda < 1, f, g \geq 0$ *such that* $0 < \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x < \infty$ *and* $0 < \int_{-\infty}^{\infty} |x|^{q-1} g^q(x)\mathrm{d}x < \infty$. *Then we have the following equivalent inequalities*

$$I := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda} f(x)g(y)\mathrm{d}x\mathrm{d}y$$

$$< C(\lambda)\left\{\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x\right\}^{1/p}\left\{\int_{-\infty}^{\infty} |x|^{q-1} g^q(x)\mathrm{d}x\right\}^{1/q}, \tag{3.1}$$

$$J := \int_{-\infty}^{\infty} |y|^{-1}\left[\int_{-\infty}^{\infty} \left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda} f(x)\mathrm{d}x\right]^p \mathrm{d}y < C^p(\lambda)\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x, \tag{3.2}$$

where the constant factors $C(\lambda) = 2(\frac{1}{\lambda}F(\lambda,\lambda,1+\lambda,-1) + B(\lambda, 1-\lambda))$ *and* $C^p(\lambda)$ *are both the best possible.*

Proof. By Hölder's inequality with weight[15], we find

$$\begin{aligned} I &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda} \left[\frac{|x|^{1/q}}{|y|^{1/p}} f(x)\right]\left[\frac{|y|^{1/p}}{|x|^{1/q}} g(y)\right]\mathrm{d}x\mathrm{d}y \\ &\leq \left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda} \frac{|x|^{p-1}}{|y|} f^p(x)\mathrm{d}x\mathrm{d}y\right\}^{1/p} \\ &\quad \times\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda} \frac{|y|^{q-1}}{|x|} g^q(y)\mathrm{d}x\mathrm{d}y\right\}^{1/q}. \end{aligned} \tag{3.3}$$

If (3.3) takes the form of the equality, then there exist constants a and b such that they are not all zero and

$$\frac{a|x|^{p-1}}{|y|} \cdot f^p(x) = \frac{b|y|^{q-1}}{|x|} \cdot g^q(y) \text{ a.e. in } (-\infty,\infty)\times(-\infty,\infty).$$

Then we have $a|x|^p f^p(x) = b|y|^q g^q(y)$ a.e. in $(-\infty,\infty)\times(-\infty,\infty)$. Hence there exists a constant c, such that

$$a|x|^p f^p(x) = b|y|^q g^q(y) = c \text{ a.e. in } (-\infty,\infty)\times(-\infty,\infty).$$

Without losing the generality, suppose $a \neq 0$, we may get $a|x|^{p-1}f^p(x) = \frac{c}{a|x|}$ a.e. in$(-\infty,\infty)$, which contradicts the facts that $0 < \int_{-\infty}^{\infty}|x|^{p-1}f^p(x)\mathrm{d}x < \infty$. Hence by (2.3), (3.3) takes a strict inequality

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda} f(x)g(y)\mathrm{d}x\mathrm{d}y \\ < \;& \left\{\int_{-\infty}^{\infty}\varpi_\lambda(x)|x|^{p-1}f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}}\left\{\int_{-\infty}^{\infty}\varpi_\lambda(y)|y|^{q-1}g^q(y)\mathrm{d}y\right\}^{\frac{1}{q}} \end{aligned}$$

In view of (2.4), we obtain (3.1).

Assume that the constant factor $C(\lambda)$ in (3.1) is not the best possible, then there exists a positive number k, $0 < k < C(\lambda)$. Applying Lemma 2 and (2.7), we have $C(\lambda) + o(1) = I(\varepsilon) < k \cdot h(\varepsilon) = k$ for $0 < \varepsilon < \frac{q\lambda}{4}$. It follows that $C(\lambda) \leq k$ ($\varepsilon \to 0^+$). Hence the constant factor $C(\lambda)$ in (3.1) is the best possible.

In what follows, we will prove the equivalence between (3.1) and (3.2). In fact, if $J = 0$, then (3.2) is valid; if $J > 0$, for $x \in (-\infty,\infty)$. Setting

$$[f(x)]_n = f(x), f(x) \leq n; [f(x)]_n = n, f(x) > n$$

and $E_n = \left[-n, -\frac{1}{n}\right]\bigcup\left[\frac{1}{n}, n\right]$, then there exist $n, n_0 \in \mathbb{N}, n \geq n_0$ such that

$$\int_{E_n}|x|^{p-1}[f(x)]_n^p\mathrm{d}x > 0,\; J(n) := \int_{E_n}\frac{1}{|y|}\left[\int_{E_n}\left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda}[f(x)]_n\mathrm{d}x\right]^p\mathrm{d}y > 0.$$

Let

$$g_n(y) := \frac{1}{|y|}\left[\int_{E_n}\left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda}[f(x)]_n\mathrm{d}x\right]^{p-1},$$

$$I(n) = \int_{E_n}\int_{E_n}\left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda}[f(x)]_n g_n(y)\mathrm{d}x\mathrm{d}y\;(n \geq n_0).$$

By (3.1), we obtain

$$\begin{aligned} 0 \;&< \int_{E_n}|y|^{q-1}g_n^q(y)\mathrm{d}y = J(n) = I(n) \\ &< C(\lambda)\left\{\int_{E_n}|x|^{p-1}[f(x)]_n^p\mathrm{d}x\right\}^{1/p}\left\{\int_{E_n}|y|^{q-1}g_n^q(y)\mathrm{d}y\right\}^{1/q} < \infty, \end{aligned} \tag{3.5}$$

$$0 < \int_{E_n} |y|^{q-1} g_n^q(y)\mathrm{d}y < C^p(\lambda) \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x < \infty. \tag{3.6}$$

This shows that $0 < \int_{-\infty}^{\infty} |y|^{q-1} g_\infty^q(y)\mathrm{d}y < \infty$, when $n \to \infty$. Applying (3.1), (3.5) and (3.6) still take the form of strict inequality, which is a contradiction. Hence (3.2) is correct.

On the other hand, suppose that (3.2) is valid. By Hölder's inequality, we find

$$I = \int_{-\infty}^{\infty} \left[|y|^{\frac{1}{q}-1} \int_{-\infty}^{\infty} \left(\frac{\min\{1,|xy|\}}{|1+xy|} \right)^\lambda f(x)\mathrm{d}x \right] \left[|y|^{1-\frac{1}{q}} g(y) \right] \mathrm{d}y$$

$$\leq J^{1/p} \left\{ \int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y \right\}^{1/q}. \tag{3.7}$$

Then by (3.2), we have (3.1). Thus (3.1) and (3.2) are equivalent.

If the constant $C^p(\lambda)$ in (3.2) is not the best possible, then by (3.7), we may get a contradiction that the constant factor in (3.1) is not the best possible. This completes the proof. □

Theorem 3.2. *If* $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \lambda < 1, f, g \geq 0$ *such that* $0 < \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x < \infty$ *and* $0 < \int_{-\infty}^{\infty} |x|^{q-1} g^q(x)\mathrm{d}x < \infty$, *then we have the following equivalent inequalities*

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{\min\{1,|xy|\}}{|1+xy|} \right)^\lambda f(x)g(y)\mathrm{d}x\mathrm{d}y$$

$$> C(\lambda) \left\{ \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x \right\}^{1/p} \left\{ \int_{-\infty}^{\infty} |x|^{q-1} g^q(x)\mathrm{d}x \right\}^{1/q}, \tag{3.8}$$

$$J = \int_{-\infty}^{\infty} |y|^{-1} \left[\int_{-\infty}^{\infty} \left(\frac{\min\{1,|xy|\}}{|1+xy|} \right)^\lambda f(x)\mathrm{d}x \right]^p \mathrm{d}y > C^p(\lambda) \int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x, \tag{3.9}$$

$$L := \int_{-\infty}^{\infty} |x|^{-1} \left[\int_{-\infty}^{\infty} \left(\frac{\min\{1,|xy|\}}{|1+xy|} \right)^\lambda g(y)\mathrm{d}y \right]^p \mathrm{d}x < C^q(\lambda) \int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y, \tag{3.10}$$

where the constant factors $C(\lambda) = 2(\frac{1}{\lambda} F(\lambda, \lambda, 1+\lambda, -1) + B(\lambda, 1-\lambda))$, $C^p(\lambda)$ *and* $C^q(\lambda)$ *are both the best possible.*

Proof. Following the analysis of that of the proof of Theorem 3.1, by the reverse Hölder's inequality with weight[15], we obtain

$$I > \left\{\int_{-\infty}^{\infty} \varpi_\lambda(x)|x|^{p-1} f^p(x)\mathrm{d}x\right\}^{\frac{1}{p}} \left\{\int_{-\infty}^{\infty} \varpi_\lambda(y)|y|^{q-1} g^q(y)\mathrm{d}y\right\}^{\frac{1}{q}}. \tag{3.11}$$

In view of (2.4), we get (3.8).

Assume that the constant factor $C(\lambda)$ in (3.8) is not the best possible, then there exists a positive number k with $k \geq C(\lambda)$, then applying Lemma 2 and (2.7), when $0 < \varepsilon < \frac{|q|\lambda}{4}$, we have $C(\lambda) + o(1) = I(\varepsilon) > k \cdot h(\varepsilon) = k$. It follows that $C(\lambda) \geq k$ ($\varepsilon \to 0^+$). Hence the constant factor $k = C(\lambda)$ in (3.8) is the best possible.

Since $\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x > 0$, it is obvious that $J > 0$. If $J = \infty$, then (3.9) is valid naturally. We assume $0 < J < \infty$ and $g(y) = |y|^{-1}\left[\int_{-\infty}^{\infty}\left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda} f(x)\mathrm{d}x\right]^{p-1}, y \in (-\infty,\infty)$. By (3.8), we get

$$\begin{aligned}
\infty &> \int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y = J = I \\
&> C(\lambda)\left\{\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x\right\}^{1/p} \left\{\int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y\right\}^{1/q} > 0,
\end{aligned}$$

$$J^{1/p} = \left\{\int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y\right\}^{1/p} > C(\lambda)\left\{\int_{-\infty}^{\infty} |x|^{p-1} f^p(x)\mathrm{d}x\right\}^{1/p}.$$

Hence (3.9) is valid.

On the other hand, suppose that (3.9) is valid. By reverse Hölder's inequality, we find

$$\begin{aligned}
I &= \int_{-\infty}^{\infty}\left[|y|^{\frac{1}{q}-1}\int_{-\infty}^{\infty}\left(\frac{\min\{1,|xy|\}}{|1+xy|}\right)^{\lambda} f(x)\mathrm{d}x\right]\left[|y|^{1-\frac{1}{q}} g(y)\right]\mathrm{d}y \\
&\geq J^{1/p}\left\{\int_{-\infty}^{\infty} |y|^{q-1} g^q(y)\mathrm{d}y\right\}^{1/q}.
\end{aligned} \tag{3.12}$$

Then by (3.9), we obtain (3.8). Thus (3.8) and (3.9) are equivalent.

If $L = 0$, then (3.10) is valid naturally; If $L > 0$, for $y \in (-\infty,\infty)$, then setting

$$[g(y)]_n = \begin{cases} \frac{1}{n}, & g(y) < \frac{1}{n}, \\ g(y), & \frac{1}{n} \le g(y) \le n, \\ n, & g(y) > n, \end{cases}$$

and $E_n = \left[-n, -\frac{1}{n}\right] \cup \left[\frac{1}{n}, n\right]$, then there exist $n, n_0 \in \mathbb{N}, n \ge n_0$, such that

$$\int_{E_n} |y|^{q-1}[g(y)]_n^q \mathrm{d}y > 0,\ L(n) := \int_{E_n} \frac{1}{|x|} \left(\int_{E_n} \left(\frac{\min\{1, |xy|\}}{|1+xy|} \right)^\lambda [g(y)]_n \mathrm{d}y \right)^q \mathrm{d}x > 0.$$

Further, let

$$f_n(x) := \frac{1}{|x|} \left(\int_{E_n} \left(\frac{\min\{1, |xy|\}}{|1+xy|} \right)^\lambda [g(y)]_n \mathrm{d}y \right)^{q-1},$$

$$I(n) := \int_{E_n} \int_{E_n} \left(\frac{\min\{1, |xy|\}}{|1+xy|} \right)^\lambda f_n(x)[g(y)]_n \mathrm{d}x\mathrm{d}y \ (n \ge n_0).$$

By (3.8), in view of $q < 0$, we have

$$\begin{aligned} \infty &> \int_{E_n} |x|^{p-1} f_n^p(x) \mathrm{d}x = L(n) = I(n) \\ &> C(\lambda) \left\{ \int_{E_n} |x|^{p-1} [f(x)]_n^p \mathrm{d}x \right\}^{1/p} \left\{ \int_{E_n} |y|^{q-1} g_n^q(y) \mathrm{d}y \right\}^{1/q} > 0, \end{aligned}$$

$$0 < \int_{E_n} |x|^{p-1} f_n^p(x) \mathrm{d}x = L(n) < C^q(\lambda) \int_{-\infty}^{\infty} |y|^{q-1} g^q(y) \mathrm{d}y < \infty.$$

This shows that $0 < \int_{-\infty}^{\infty} |x|^{p-1} f_\infty^p(x) \mathrm{d}x < \infty$, when $n \to \infty$, applying (3.8), the above two inequalities still take the form of strict inequality. Hence (3.10) is correct.

On the other hand, assume that (3.10) is valid. By the reverse Hölder's inequality, we find

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \left[|x|^{\frac{1}{q}} f(x) \right] \left[|x|^{-\frac{1}{q}} \int_{-\infty}^{\infty} \left(\frac{\min\{1, |xy|\}}{|1+xy|} \right)^\lambda g(y) \mathrm{d}y \right] \mathrm{d}x \\ &\ge L^{1/q} \left\{ \int_{-\infty}^{\infty} |x|^{p-1} f^p(x) \mathrm{d}x \right\}^{1/p}. \qquad (3.13) \end{aligned}$$

Then by (3.10), we have (3.8). Thus (3.10) and (3.8) are equivalent.

It is of course that the constant factors in (3.9) and (3.10) are the best possible. Otherwise, by (3.12) and (3.13), we may get a contradiction that the constant factor in (3.8) is not the best possible. This completes the proof. □

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