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Compositions of Derivations with Central Values on Polynomials *

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Abstract

Let R be a prime ring with extended centroid C, ρ a nonzero right ideal of R, $f(X_1, \ldots, X_t)$ a non-central polynomial with zero constant term over C, not necessarily multilinear, and δ, d two nonzero derivations of R. We determine the structures of R, δ and d when $\delta d(f(X_1, \ldots, X_t))$ is central-valued on ρ. The theorem gives a generalization of several related results in the literature.

Keywords and Phrases: *Derivation, prime ring, PI, GPI.*

1. Results

Throughout this paper, let R be always a prime ring with center $\mathcal{Z}(R)$, extended centroid C, maximal ring of right quotients U and symmetric Martindale ring of quotients Q. An additive map $d\colon R \to R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. We denote by $\mathrm{ad}(b)$ the inner derivation of R induced by the element $b \in R$. That is, $\mathrm{ad}(b)(x) = [b, x] = bx - xb$ for $x \in R$. In this note we will prove a result concerning a composition of two

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derivations on a prime ring. We recall some results that have motivated our work.

Let d and δ be two derivations of R and let I be either a Lie ideal of R or a right ideal of R. A number of authors have considered the following questions in the literature: Determine the structures of R, I, δ and d when $\delta d(I) = 0$ or $\delta d(I) \subseteq \mathcal{Z}(R)$. For instance, Posner [12] proved that if $\delta d(R) = 0$ and char$R \neq 2$, then either $d = 0$ or $\delta = 0$. Other related results are [9, Theorem 4], [7, Theorem], [8, Theorem 1], [1, Theorem 2], [10, Theorem 4], [8, Main Theorem], [3, Theorem 2], and so on. In a recent paper [2] Chang obtained a common generalization of the theorems above by proving the following two theorems:

Theorem 1.1. *Let R be a prime ring, ρ a right ideal of R and δ, d two nonzero derivations of R. Suppose that $\delta d([\rho,\rho]) = 0$ and $[\rho,\rho]\rho \neq 0$. Then either $\delta = \alpha d$ for some $\alpha \in C$ and $d^2 = 0$, or there exist $p, q \in Q$ such that $\delta = \mathrm{ad}(q), d = \mathrm{ad}(p)$ with $p\rho = 0 = q\rho$ and $pq = 0$, except when $\rho C = eRC$ for some idempotent e in the socle of RC such that* $\operatorname{char} R = 2$ *and* $\dim_C eRCe = 4$.

Theorem 1.2. *Let R be a prime ring, ρ a right ideal of R and δ, d two derivations of R. If $0 \neq \delta d([\rho,\rho]) \subseteq \mathcal{Z}(R)$ and $[\rho,\rho]\rho \neq 0$, then* $\operatorname{char} R = 2$ *and* $\dim_C RC = 4$.

Since, in Theorems 1.1 and 1.2, $[\rho,\rho]$ contains the elements $xy - yx$ for all $x, y \in \rho$. It is natural to consider the above theorems by replacing $[\rho,\rho]$ with a polynomial in noncommuting indeterminates with zero constant term over C. For a right ideal ρ of R we denote by $f(\rho)$ the additive subgroup of RC generated by all elements $f(x_1,\ldots,x_t)$ for $x_1,\ldots,x_t \in \rho$. The goal of this note is to prove a common generalization of the above results:

Theorem 1.3. *Let R be a prime ring, ρ a nonzero right ideal of R, $f(X_1,\ldots,X_t)$ a polynomial in noncommuting indeterminates with zero constant term over C, and δ, d two nonzero derivations of R.*

(I) *Suppose that $\delta d(f(\rho)) = 0$. Then either $\delta = \alpha d$ for some $\alpha \in C$ and $d^2 = 0$, or there exist $p, q \in Q$ such that $\delta = \mathrm{ad}(q), d = \mathrm{ad}(p)$ with $p\rho = 0 = q\rho$ and $pq = 0$, or $\rho C = eRC$ for some idempotent e in the socle of RC such that either $f(X_1,\ldots,X_t)$ is central-valued on $eRCe$ or* $\operatorname{char} R = 2$ *and* $\dim_C eRCe = 4$.

(II) *Suppose that $0 \neq \delta d(f(\rho)) \subseteq C$. Then $\rho C = eRC$ for some idempotent*

e in the socle of RC such that $f(X_1,\ldots,X_t)$ is central-valued on $eRCe$ unless $\operatorname{char} R = 2$ *and* $\dim_C RC = 4$.

In Theorem 1.3 (II) we cannot conclude that $f(X_1,\ldots,X_t)$ is central-valued on R in general. The following provides such an example.

Example 1.4. Let $R = \mathrm{M}_{2n}(F)$ be the $2n \times 2n$ matrix ring over F, a field of characteristic 2, where $n \geq 2$. We let $\{e_{ij} \mid 1 \leq i,j \leq 2n\}$ be the set of the usual matrix units in R, and let $\rho = eR$, where $e = e_{11} + \cdots + e_{nn}$. Choose $f(X_1,\ldots,X_t)$ to be a central polynomial for $\mathrm{M}_n(F)$ (see, for instance, [13, p.315]). Let $d = \mathrm{ad}(p)$ and $\delta = \mathrm{ad}(q)$, where $p = \sum_{i=1}^{n} e_{2n+1-i\,i}$ and $q = \sum_{i=1}^{n} e_{i\,2n+1-i}$. Note that $pe = p$, $eq = q$, $qe = 0$, $pq = \sum_{i=n+1}^{2n} e_{ii} = 1 - e$ and $qp = \sum_{i=1}^{n} e_{ii} = e$. Let $x_1,\ldots,x_t \in \rho = eR$. Then $f(x_1,\ldots,x_t)e = \beta e$ with $\beta \in F$, depending on $\{x_i\}$. We set $b = f(x_1,\ldots,x_t)$ and hence $eb = b$ and $be = \beta e$. Since $\operatorname{char} F = 2$, we have

$$\begin{aligned}\delta d(f(x_1,\ldots,x_t)) &= qpb + qbp + pbq + bpq \\ &= eb + qebp + pbeq + b(1+e) \\ &= b + \beta(1+e) + b + \beta e = \beta \in F.\end{aligned}$$

Since we can choose $x_i \in \rho$ such that $\beta \neq 0$, this implies that $0 \neq \delta d(f(\rho)) \subseteq F$. It is clear that $f(X_1,\ldots,X_t)$ is central-valued on eRe, but $f(X_1,\ldots,X_t)$ is not central-valued on R.

As an immediate consequence of the Main Theorem, we have the following:

Corollary 1.5. *Let R be a prime ring and let δ, d be two nonzero derivations of R, $f(X_1,\ldots,X_t)$ a polynomial over C, and I a nonzero ideal of R.*

(I) *Suppose that $\delta d(f(I)) = 0$. Then either $\delta = \alpha d$ for some $\alpha \in C$ and $d^2 = 0$ or $f(X_1,\ldots,X_t)$ is central-valued on RC except when* $\operatorname{char} R = 2$ *and* $\dim_C eRCe = 4$.

(II) *Suppose that $0 \neq \delta d(f(I)) \subseteq C$. Then $f(X_1,\ldots,X_t)$ is central-valued on RC except when* $\operatorname{char} R = 2$ *and* $\dim_C RC = 4$.

We remark that the exceptional case indeed exists in Corollary 1.5 (II). For instance, let $R = \mathrm{M}_2(C)$, where C is a field of characteristic 2. We set $\delta = \mathrm{ad}(e_{12})$, $d = \mathrm{ad}(e_{11})$, and $f(X,Y) = XY - YX$. Then a direct computation proves that $0 \neq \delta d(f(R)) \subseteq C$. Of course, $XY - YX$ cannot be central-valued on R.

2. Proof of Theorem 1.3

To prove Theorem 1.3 we need the following two results ([5, Theorems 1 and 2]): The first theorem investigates $f(\rho)$ the additive subgroup of RC generated by all elements $f(x_1, \ldots, x_t)$ for $x_i \in \rho$, where $f(X_1, \ldots, X_t)$ is a polynomial over C and where ρ is a right ideal of R. The second result concerns the case when ρ is an ideal of R.

Theorem 2.1. *Let R be a centrally closed prime C-algebra, ρ a nonzero right ideal of R and $f(X_1, \ldots, X_t)$ a nonzero polynomial over C.*

(I) *If ρ is a non-*PI *right ideal of R, then there exists a non-*PI *right ideal ρ_0 of R contained in ρ such that $[\rho_0, \rho] \subseteq f(\rho)$ and $\rho\rho_0 \subseteq \rho_0$.*

(II) *If ρ is a* PI *right ideal of R, then there exists an idempotent e in the socle of R such that $\rho = eR$ and the following statements hold:*

(i) *$eR(1-e) \subseteq f(\rho)$ if $f(\rho) \neq 0$.*

(ii) *$[\rho, \rho] \subseteq f(\rho)$ except when either $f(X_1, \ldots, X_t)$ is central-valued on eRe or $eRe \cong \mathrm{M}_2(\mathrm{GF}(2))$.*

Theorem 2.2. *Let R be a prime ring with extended centroid C and I a nonzero ideal of R. Suppose that $f(X_1, \ldots, X_t)$ is a polynomial over C, which is not central-valued on RC. Then $[M, R] \subseteq f(I)$ for some nonzero ideal M of R except when $R \cong M_2 GF(2))$ and $f(R) = \{0, e_{12} + e_{21}, 1 + e_{12}, 1 + e_{21}\}$ or $\{0, 1, e_{11} + e_{12} + e_{21}, e_{22} + e_{12} + e_{21}\}$.*

We are now in a position to give the proof of our main result:

Proof of Theorem 1.3. (I) By [11, Theorem 2], R and Q satisfy the same differential identities. Thus ρ, ρR and ρQ satisfy the same differential identities. By assumption, $\delta d(f(x_1, \ldots, x_t)) = 0$ for all $x_1, \ldots, x_t \in \rho$. Replacing R, ρ with RC, ρC respectively, we may assume that R is a centrally closed prime C-algebra, so now we can assume that $\rho C \subseteq \rho$.

Consider first the case that ρ is a non-PI right ideal of R. In view of Theorem 2.1, $[\rho_0, \rho] \subseteq f(\rho)$ for some non-PI right ideal ρ_0 of R contained in ρ and $\rho\rho_0 \subseteq \rho_0$. Since $\delta d(f(\rho)) = 0$, we have $\delta d([\rho_0, \rho_0]) = 0$. Since $[\rho_0, \rho_0]\rho_o \neq 0$, applying Theorem 1.1, we have that either $\delta = \alpha d$ for some $\alpha \in C$ and $d^2 = 0$, or there exist $p, q \in Q$ such that $\delta = \mathrm{ad}(q), d = \mathrm{ad}(p)$ with $p\rho_0 = 0 = q\rho_0$ and $pq = 0$. Since $\rho\rho_0 \subseteq \rho_0$, we see that $p\rho\rho_0 = 0 = q\rho\rho_0$, implying that $p\rho = 0 = q\rho$ by the primeness of R. This proves the case.

Suppose next that ρ is a PI right ideal of R. By Theorem 1.2, $\rho = eR$ for some idempotent e in the socle of R. Suppose that $f(X_1, \ldots, X_t)$ is not central-valued on eRe. In particular, eRe is not commutative and so $[\rho,\rho]\rho \neq 0$. Moreover, we assume that either $\operatorname{char} R \neq 2$ or $\dim_C eRCe > 4$. In view of Theorem 2.1, $[\rho,\rho] \subseteq f(\rho)$ and hence $\delta d([\rho,\rho]) = 0$. Applying Theorem 1.1 we are done.

(II) By assumption, $\delta d(f(X_1,\ldots,X_t))$ is a central DI (see [4]) for ρ. It follows from [4, Theorem 1] that R is a prime PI-ring and so RC is a finite-dimensional central simple C-algebra by Posner's Theorem. By the Wedderburn-Artin theorem, $RC \cong \mathrm{M}_n(D)$ for some n and some finite-dimensional central division C-algebra D. In view of [6, Theorem 1] or [11, Theorem 2], $0 \neq \delta d(f(\rho C)) \subseteq C$. By replacing R, ρ with RC, ρC respectively, we may assume that $R = \mathrm{M}_n(D)$ and $\rho = eR$ for some idempotent $e \in R$. It follows from Theorems 2.1 and 2.2 that $[\rho,\rho] \subseteq f(\rho)$ except when either $f(X_1,\ldots,X_t)$ is central-valued on eRe, or $eRe \cong \mathrm{M}_2(\mathrm{GF}(2))$ such that $f(eRe) = \{0, e_{12}+e_{21}, e_{11}+e_{22}+e_{12}, e_{11}+e_{22}+e_{21}\}$ or $\{0, e_{11}+e_{22}, e_{11}+e_{12}+e_{21}, e_{22}+e_{12}+e_{21}\}$. Suppose that $f(X_1,\ldots,X_t)$ is not central-valued on eRe. Otherwise, we are done. In particular, $[\rho,\rho]\rho \neq 0$.

Suppose that $eRe \ncong \mathrm{M}_2(\mathrm{GF}(2))$. Thus $[\rho,\rho] \subseteq f(\rho)$ and so $\delta d([\rho,\rho]) \subseteq \delta d(f(\rho)) \subseteq C$. If $0 \neq \delta d([\rho,\rho]) \subseteq C$, then applying Theorem 1.2 yields $\operatorname{char} R = 2$ and $\dim_C R = 4$. We are done in this case. Suppose that $\delta d([\rho,\rho]) = 0$. By Theorem 1.1, either $\delta = \alpha d$ for some $\alpha \in C$ and $d^2 = 0$, or there exist $p, q \in Q$ such that $\delta = \mathrm{ad}(q), d = \mathrm{ad}(p)$ with $p\rho = 0 = q\rho$ and $pq = 0$, except when $\operatorname{char} R = 2$ and $\rho C = eRC$ for some idempotent e in the socle of RC such that $\dim_C eRCe = 4$. For the first two cases, a direct computation proves $\delta d(f(\rho)) = 0$, contrary to our assumption. Thus $\operatorname{char} R = 2$ and $\dim_C eRCe = 4$. From the proof of Theorem 1.1 (see [2]), we see that both δ and d must be inner. In this case we have $[\rho,\rho] \subset f(\rho) \subseteq \rho$. Since $\dim_C \rho = 1 + \dim_C[\rho,\rho]$, this implies that $Cf(\rho) = f(\rho)$. Hence $\delta d(\rho) = \delta d(Cf(\rho)) = C\delta d(f(\rho)) \subseteq C$ as δ and d are both inner. So we conclude that $0 \neq \delta d(\rho) \subseteq C$. It follows from Theorem 1.2 that $\operatorname{char} R = 2$ and $\dim_C R = 4$.

Thus we see that $eRe \cong \mathrm{M}_2(\mathrm{GF}(2))$. In this case, we see $C = \mathrm{GF}(2)$. Moreover, $D = C$ since $\dim_C D < \infty$, so now $R \cong \mathrm{M}_2(\mathrm{GF}(2))$. Now, $d(C) = 0 = \delta(C)$ and so δ and d are both inner. Suppose that $d = \mathrm{ad}(p)$ and $\delta = \mathrm{ad}(q)$ for some $p, q \in R$. We may assume, without loss of generality, that $e = e_{11} + e_{22}$. We describe $f(\rho)$. Using $eR = eRe + eR(1-e)$, a direct computation proves that $f(eR) \subseteq f(eRe) + eR(1-e)$. On the other hand, applying Theorem 2.1 (II) (i) yields $eR(1-e) \subseteq f(\rho)$ since $f(eR) \neq 0$. Thus we have

$f(eR) = f(eRe) + eR(1-e)$. Also, in view of Theorem 2.2, $f(eRe) = \{0, e_{12} + e_{21}, e_{11}+e_{22}+e_{12}, e_{11}+e_{22}+e_{21}\}$ or $\{0, e_{11}+e_{22}, e_{11}+e_{12}+e_{21}, e_{22}+e_{12}+e_{21}\}$. So we obtain that either $f(\rho) = C(e_{11}+e_{22}+e_{12})+C(e_{11}+e_{22}+e_{21})+\sum_{i\leq 2, j>2} Ce_{ij}$ or $f(\rho) = C(e_{11}+e_{12}+e_{21})+C(e_{12}+e_{21}+e_{22})+\sum_{i\leq 2, j>2} Ce_{ij}$. We will derive a contradiction by proving $\delta d(f(\rho)) = 0$ in either case. Since the two cases have analogous arguments, we only prove the second case. Suppose $n \geq 3$. Moreover, we may assume that $n \geq 4$ since $\operatorname{char} R = 2$ and $\operatorname{trace}(\delta d(f(\rho))) = 0$. Write $p = \sum_{1\leq i,j\leq n} p_{ij}e_{ij}$ and $q = \sum_{1\leq i,j\leq n} q_{ij}e_{ij}$, where $p_{ij}, q_{ij} \in C$.

If $p_{st} = 0 = q_{st}$ for all $s > 2$ and $t \leq 2$, then $0 \neq \delta d(f(\rho)) \subseteq \rho \cap C$. So $\rho = R$, a contradiction. Thus we may assume that either $p_{st} \neq 0$ or $q_{st} \neq 0$ for some $s > 2$ and $t \leq 2$. Without loss of generality, we may assume that $s = 3$ and $t = 1$. We separate the argument into three cases.

Case 1. $p_{31} \neq 0$ and $q_{31} = 0$. By replacing p, q with $p - p_{11}I_n, q - q_{11}I_n$ respectively, we may assume $p_{31} = 1$ and $p_{11} = q_{11} = 0$. For $j \neq 1$, the $(j,1)$-entry of $\delta d(e_{13}) \in C$ equals $-q_{j1}p_{31} - p_{j1}q_{31}$, so $q_{j1} = 0$ since $q_{31} = 0$ and $p_{31} = 1$. For $j \neq 1, 2$, the $(j,1)$-entry of $\delta d(e_{23}) \in C$ equals $-q_{j2}p_{31} - p_{j2}q_{31}$, so $q_{j2} = 0$. Computing the $(3,4)$-entry of $\delta d(e_{13}) \in C$, we have $q_{34} = 0$. Now comparing the $(1,1)$-entry and $(4,4)$-entry of $\delta d(e_{23}) \in C$, we can get that $q_{12} = 0$. For $s \geq 3$, the $(1,1)$-entry of $\delta d(e_{1s}) \in C$ equals 0, so $\delta d(e_{1s}) = 0$. Similarly, the $(1,1)$-entry of $\delta d(e_{2s}) \in C$ equals 0, so $\delta d(e_{2s}) = 0$. We have showed that $\delta d(x) = 0$ for all $x \in \rho\cap\ell(\rho)$. The $(3,4)$-entries of $\delta d(e_{11}+e_{12}+e_{21})$ and $\delta d(e_{12}+e_{21}+e_{22})$ are $p_{31}q_{14}+p_{31}q_{24}+p_{32}q_{14} = 0$ and $p_{31}q_{24}+p_{32}q_{14}+p_{32}q_{24} = 0$. Since $p_{32} = 0$ or $p_{32} = 1$, for any case we can get that $q_{14} = q_{24} = 0$. Now the $(4,4)$-entries of $\delta d(e_{11} + e_{12} + e_{21})$ and $\delta d(e_{12} + e_{21} + e_{22})$ equal 0. Thus $\delta d(e_{11}+e_{12}+e_{21}) = \delta d(e_{12}+e_{21}+e_{22}) = 0$. Then $\delta d(f(\rho)) = 0$, a contradiction.

Case 2. $p_{31} = 0$ and $q_{31} \neq 0$. Applying the same argument given in Case 1, we can get that $\delta d(f(\rho)) = 0$. We omit its details.

Case 3. $p_{31} \neq 0$ and $q_{31} \neq 0$. By replacing p, q with $p - p_{11}I_n, q - q_{11}I_n$ respectively, we may assume that $p_{31} = q_{31} = 1, p_{11} = q_{11} = 0$. We can get that $p_{i1} = q_{i1}$ for all i, $p_{j2} = q_{j2}$ for all $j \neq 1, 2$, $p_{34} = q_{34}$ and $p_{12} = q_{12}$ as in Case 1. For $s \geq 3$, the $(2,2)$-entry of $\delta d(e_{1s}) \in C$ equals $q_{21}p_{s2} + p_{21}q_{s2} = 0$, so $\delta d(e_{1s}) = 0$. Similarly, the $(1,1)$-entry of $\delta d(e_{2s}) \in C$ equals $q_{12}p_{s1}+p_{12}q_{s1} = 0$, so $\delta d(e_{2s}) = 0$. We have showed that $\delta d(x) = 0$ for all $x \in \rho\cap\ell_R(\rho)$. The $(3,4)$-entries of $\delta d(e_{11} + e_{12} + e_{21})$ and $\delta d(e_{12} + e_{21} + e_{22})$ are $q_{31}p_{14} + q_{31}p_{24} + q_{32}p_{14} + p_{31}q_{14}+p_{31}q_{24}+p_{32}q_{14} = 0$ and $q_{31}p_{24}+q_{32}p_{14}+q_{32}p_{24}+p_{31}q_{24}+p_{32}q_{14}+p_{32}q_{24} = 0$. Since $p_{32} = q_{32} = 0$ or $p_{32} = q_{32} = 1$, in either case we can get that $p_{14} = q_{14}$ and $p_{24} = q_{24}$. Now the $(4,4)$-entries of $\delta d(e_{11}+e_{12}+e_{21})$ and $\delta d(e_{12}+e_{21}+e_{22})$

equal 0, implying that $\delta d(e_{11}+e_{12}+e_{21})=0=\delta d(e_{12}+e_{21}+e_{22})$. Hence $\delta d(f(\rho))=0$, a contradiction again. The theorem is thus proved.

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