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Properties and Characteristics of Certain Subclass of Analytic Functions with Positive Coefficients*

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Abstract

In this paper, the authors introduce and investigate the various properties and characteristics of the subclass $M_n^+(\alpha, \beta)$ of analytic functions with positive coefficients. A characteristic, several inclusion relationships, coefficient estimates, Hadamard products, distortion theorems, covering theorems and (n, δ)-neighborhoods are proven here for this function class. Furthermore, some interesting distortion theorems for the Srivastava-Saigo-Owa fractional integral operator are also obtained.

Keywords and Phrases: *Analytic functions, Inclusion relation, Coefficient estimates, Distortion theorems, Covering theorems, Integral operator, (n, δ)-neighborhood, Srivastava-Saigo-Owa fractional integral operator.*

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1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}_n$ denote the class of functions f normalized by

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (n \in \mathbb{N} := 1, 2, 3, ...), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Denote $f \in \mathcal{A}_n^+$ if $f \in \mathcal{A}_n$ and $a_k \geq 0(k \geq n+1)$. Suppose also that $\mathcal{S}^*(\beta)$ and $\mathcal{K}$ denote the subclasses of $\mathcal{A}_1$ consisting of functions which are, respectively, starlike of order β in $\mathbb{U}$ $(0 \leq \beta < 1)$ and convex in $\mathbb{U}$. Set $\mathcal{S}_n^*(\beta) := \mathcal{S}^*(\beta) \cap \mathcal{A}_n$. Then, we have

$$\mathcal{S}^*(\beta) := \{f \in \mathcal{A}_1; \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (z \in \mathbb{U})\}.$$

For functions $f \in \mathcal{A}_1$ given by (1.1) and g given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the Hadamard product(or convolution) of f and g is defined by

$$(f * g) := z + \sum_{k=2}^{\infty} a_k b_k z^k =: (g * f).$$

J. L. Li and S. Owa[3] proved the following theorem.

Theorem A. *(Li and Owa [3]) Suppose that $\alpha \geq 0$ and $f \in \mathcal{A}_1$. If*

$$Re\left(\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > -\frac{\alpha}{2} \quad (z \in \mathbb{U}),$$

then

$$f(z) \in \mathcal{S}^*(0) = \mathcal{S}^*.$$

Ravichandran et al. [6] gave the following modification of Theorem A.

Theorem B. *(Ravichandran et al. [6]) Suppose that* $\alpha \geq 0$ *and* $0 \leq \beta < 1$. *If* $f \in \mathcal{A}_1$ *and satisfies*

$$Re\left(\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > \alpha\beta(\beta + \frac{n}{2} - 1) + \beta - \frac{n\alpha}{2} \ \ (z \in \mathbb{U}), \tag{1.2}$$

then

$$f(z) \in \mathcal{S}^*(\beta).$$

Assuming that $\alpha \geq 0$, $0 \leq \beta < 1$ and $f \in \mathcal{A}_n$, Liu et al. [4] introduced the function classes $\mathcal{H}_n(\alpha, \beta)$ and $\mathcal{H}_n^+(\alpha, \beta)$. That $f(z) \in \mathcal{H}_n(\alpha, \beta)$ if and only if $f(z)$ satisfies condition (1.2), and $\mathcal{H}_n^+(\alpha, \beta)$ denotes the subset of $\mathcal{H}_n(\alpha, \beta)$ such that all functions $f(z) \in \mathcal{H}_n(\alpha, \beta)$ have the following form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \ \ (a_k \geq 0; k \geq n+1). \tag{1.3}$$

Liu et al. [4] investigated the various properties and characteristics of these two function classes. In particular, they obtained several inclusion relations, Hadamard products, coefficient estimates, distortion theorems and covering theorems of these two function classes.

S. Owa and J.Nishiwaki[5] introduced and investigated the coefficient estimates and sufficient conditions for the class $M_n(\beta)$ with $\beta > 1$. That $M_n(\beta)$ denote the subclass of $\mathcal{A}_n$ consisting of functions which satisfies

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta \ \ (z \in \mathbb{U}, \beta > 1).$$

Now we introduce the function class $M_n^+(\alpha, \beta)$.

Definition 1. Assuming that $\alpha \geq 0$, $\beta > 1$ and $f \in \mathcal{A}_n^+$, then $f(z) \in M_n^+(\alpha, \beta)$ if and only if $f(z)$ satisfies

$$\text{Re}\left(\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) < \alpha\beta(\beta + \frac{n}{2} - 1) + \beta - \frac{n\alpha}{2} \ \ (z \in \mathbb{U}). \tag{1.4}$$

It is evident that $M_n^+(0, \beta) = M_n^+(\beta)$.

In this paper, we investigate several properties and characteristics of functions belonging to the subclass $M_n^+(\alpha, \beta)$ of analytic functions. In particular, their inclusion relationships, Hadamard products, coefficient estimates, distortion theorems and covering theorems are proven here. The integral operator and (n, δ)-neighborhood of the function class $M_n^+(\alpha, \beta)$ are also considered. Furthermore, some interesting distortion theorems for the Srivastava-Saigo-Owa fractional integral operator are obtained.

In order to derive our main results, we need the following lemmas.

Lemma 1. *([2]) Let the function $f \in \mathcal{A}_n^+$ given by (1.1), then for $0 \leq \eta < 1$, $f(z) \in \mathcal{S}_n^*(\eta)$ if it satisfies*

$$\sum_{k=n+1}^{\infty} (k-\eta)a_k \leq 1-\eta.$$

Lemma 2. *(Ruscheweyh and Sheil-small [7]). Let $\varphi(z)$ be convex and $g(z)$ be starlike in $\mathbb{U}$. Then, for each function $F(z)$ analytic in $\mathbb{U}$ and satisfying the following inequality:*

$$Re(F(z)) > 0 \ (z \in \mathbb{U}),$$

$$Re\left(\frac{(\varphi * Fg)(z)}{(\varphi * g)(z)}\right) > 0 \ (z \in \mathbb{U}).$$

Lemma 3. *(Ruscheweyh and Sheil-small [7]). Let $\varphi(z)$ and $g(z)$ be starlike of order $\frac{1}{2}$ in $\mathbb{U}$. Then, for each function $F(z)$ analytic in $\mathbb{U}$ and satisfying the following inequality:*

$$Re(F(z)) > 0 \ (z \in \mathbb{U}),$$

$$Re\left(\frac{(\varphi * Fg)(z)}{(\varphi * g)(z)}\right) > 0 \ (z \in \mathbb{U}).$$

2. Properties of the Function Class $M_n^+(\alpha, \beta)$

We first derive the sufficient and necessary conditions for $f(z) \in M_n^+(\alpha, \beta)$.

Theorem 1. *Suppose that* $\alpha \geq 0$, $1 < \beta < \frac{2-n+\sqrt{17n^2+20n+4}}{4}$ *and*

$$\gamma_n = \gamma_n(\alpha, \beta) = \alpha\beta(\beta + \frac{n}{2} - 1) + \beta - \frac{n\alpha}{2}. \tag{2.1}$$

If $f(z) \in \mathcal{A}_n^+$, *then* $f(z) \in M_n^+(\alpha, \beta)$ *if and only if*

$$\sum_{k=n+1}^{\infty} [k(1 + k\alpha - \alpha) - \gamma_n]a_k \leq \gamma_n - 1. \tag{2.2}$$

Proof. First, we show that $f(z) \in M_n^+(\alpha, \beta)$ if the inequality (2.2) holds true. Since $\alpha \geq 0$ and $\beta > 1$, we have

$$\begin{aligned} \gamma_n - 1 &= \alpha\beta(\beta + \frac{n}{2} - 1) + \beta - \frac{n\alpha}{2} - 1 \\ &= \frac{n\alpha}{2}(\beta - 1) + (\alpha\beta + 1)(\beta - 1) > 0. \end{aligned}$$

Now, for the function

$$p(z) = \frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)},$$

let $q(z) = \frac{\gamma_n - p(z)}{\gamma_n - 1}$. To the end, it suffices to prove that

$$\left| \frac{q(z) - 1}{q(z) + 1} \right| < 1.$$

By the coefficient inequality (2.2), we thus obtain

$$\begin{aligned}\left|\frac{q(z)-1}{q(z)+1}\right| &= \left|\frac{1-p(z)}{2\gamma_n-1-p(z)}\right| \\ &= \left|\frac{\sum_{k=n+1}^{\infty}[1-\alpha k(k-1)-k]a_k z^{k-1}}{(2\gamma_n-2)+\sum_{k=n+1}^{\infty}[2\gamma_n-1-\alpha k(k-1)-k]a_k z^{k-1}}\right| \\ &< \frac{\sum_{k=n+1}^{\infty}[\alpha k(k-1)+k-1]a_k}{2\gamma_n-2-\sum_{k=n+1}^{\infty}[\alpha k(k-1)+k-2\gamma_n+1]a_k} \\ &= \frac{\sum_{k=n+1}^{\infty}[\alpha k(k-1)+k-\gamma_n]a_k+\sum_{k=n+1}^{\infty}(\gamma_n-1)a_k}{2\gamma_n-2-\sum_{k=n+1}^{\infty}[\alpha k(k-1)+k-\gamma_n]a_k+\sum_{k=n+1}^{\infty}(\gamma_n-1)a_k} \\ &\le \frac{\gamma_n-1+\sum_{k=n+1}^{\infty}(\gamma_n-1)a_k}{2(\gamma_n-1)-(\gamma_n-1)+\sum_{k=n+1}^{\infty}(\gamma_n-1)a_k} \\ &= 1.\end{aligned}$$

Hence we obtain

$$\operatorname{Re}(p(z)) < \gamma_n, \ (z \in \mathbb{U}),$$

that is, $f(z) \in M_n^+(\alpha, \beta)$.

Conversely, we suppose that $f(z) \in M_n^+(\alpha, \beta)$, then

$$\operatorname{Re}(p(z)) \le \gamma_n \ (z \in \mathbb{U}), \tag{2.3}$$

where

$$p(z) = \frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} = \frac{1+\sum_{k=n+1}^{\infty} k(k\alpha+1-\alpha)a_k z^{k-1}}{1+\sum_{k=n+1}^{\infty} a_k z^{k-1}}.$$

Now we choosing $z = r(0 < r < 1)$ to be real, therefore, by (2.3) we obtain that

$$1 + \sum_{k=n+1}^{\infty} k(k\alpha + 1 - \alpha)a_k r^{k-1} \leq \gamma_n(1 + \sum_{k=n+1}^{\infty} a_k r^{k-1}).$$

That is,

$$\sum_{k=n+1}^{\infty} [k(1 + k\alpha - \alpha) - \gamma_n]a_k r^{k-1} \leq \gamma_n - 1. \tag{2.4}$$

Since $\alpha \geq 0$ and $1 < \beta < \frac{2-n+\sqrt{17n^2+20n+4}}{4} < n+1$, we have

$$\begin{aligned}
& k(1 + k\alpha - \alpha) - \gamma_n \\
\geq \; & (n+1)(1+n\alpha) - [\alpha\beta(\beta + \frac{n}{2} - 1) + \beta - \frac{n\alpha}{2}] \\
= \; & -\alpha\left[\beta^2 + (\frac{n}{2} - 1)\beta - (n^2 + \frac{3n}{2})\right] + (n + 1 - \beta) \\
> \; & -\alpha\left(\beta - \frac{2 - n + \sqrt{17n^2 + 20n + 4}}{4}\right) \\
& \left(\beta + \frac{n - 2 + \sqrt{17n^2 + 20n + 4}}{4}\right) \\
> \; & 0 \;\; (k \geq n+1).
\end{aligned}$$

Hence, for every $m \geq n+1$ and $0 < r < 1$, we get

$$\sum_{k=n+1}^{m} [k(1 + k\alpha - \alpha) - \gamma_n]a_k r^{k-1} \leq \sum_{k=n+1}^{\infty} [k(1 + k\alpha - \alpha) - \gamma_n]a_k r^{k-1} \;\; \leq \;\; \gamma_n - 1,$$

which, upon letting $r \to 1^-$, immediately yields

$$\sum_{k=n+1}^{m} [k(1 + k\alpha - \alpha) - \gamma_n]a_k \leq \gamma_n - 1.$$

It follows that the inequality (2.2) holds true. This completes the proof of Theorem 1. □

Corollary 1. *Let $\alpha \geq 0$, $1 < \beta < \frac{2-n+\sqrt{17n^2+20n+4}}{4}$ and $\gamma_n = \gamma_n(\alpha, \beta)$ is defined by (2.1). Suppose that $f(z) \in M_n^+(\alpha, \beta)$, then*

$$a_k \leq \frac{\gamma_n - 1}{k(1 + k\alpha - \alpha) - \gamma_n} \quad (k \geq n+1).$$

Each of these inequalities is sharp, with the extremal function given by

$$f_k(z) = z + \frac{\gamma_n - 1}{k(1 + k\alpha - \alpha) - \gamma_n} z^k \quad (k \geq n+1). \tag{2.5}$$

Next, we prove the following inclusion relations. With the aid of Theorem 1, we have the following results.

Theorem 2. *Let $\alpha_1 > \alpha_2 \geq 0$ and $1 < \beta_1 < \beta_2 \leq 1 + \frac{n}{2}$. Then*

$$M_n^+(\alpha_1, \beta_1) \subset M_n^+(\alpha_2, \beta_2).$$

Proof. First, we show that

$$M_n^+(\alpha_1, \beta_1) \subset M_n^+(\alpha_1, \beta_2). \tag{2.6}$$

Suppose that $f \in M_n^+(\alpha_1, \beta_1)$, by Theorem 1, we have

$$\sum_{k=n+1}^{\infty} [k(1 + k\alpha_1 - \alpha_1) - \gamma_n(\alpha_1, \beta_1)]a_k \leq \gamma_n(\alpha_1, \beta_1) - 1,$$

where $\gamma_n(\alpha_1, \beta_1)$ is defined in Theorem 1. Noting $\gamma_n(\alpha_1, \beta_1) - 1 > 0$ for $\alpha \geq 0$ and $1 < \beta \leq 1 + \frac{n}{2}$, it follows that

$$\sum_{k=n+1}^{\infty} \frac{k(1 + k\alpha_1 - \alpha_1) - \gamma_n(\alpha_1, \beta_1)}{\gamma_n(\alpha_1, \beta_1) - 1} a_k \leq 1. \tag{2.7}$$

Since $\alpha_1 \geq 0$, $1 < \beta_1 < \beta_2 \leq 1 + \frac{n}{2}$ and $k \geq n+1$, direct computation yields

$$\begin{aligned} &\frac{k(1 + k\alpha_1 - \alpha_1) - \gamma_n(\alpha_1, \beta_2)}{\gamma_n(\alpha_1, \beta_2) - 1} - \frac{k(1 + k\alpha_1 - \alpha_1) - \gamma_n(\alpha_1, \beta_1)}{\gamma_n(\alpha_1, \beta_1) - 1} \\ &= \frac{(k-1)(k\alpha_1 + 1)(\beta_1 - \beta_2)[(\alpha_1(\beta_1 + \beta_2 + \frac{n}{2} - 1) + 1]}{[\gamma_n(\alpha_1, \beta_1) - 1][\gamma_n(\alpha_1, \beta_2) - 1]} \leq 0. \end{aligned} \tag{2.8}$$

Hence, connecting (2.7) and (2.8), we conclude that

$$\sum_{k=n+1}^{\infty} \frac{k(1+k\alpha_1-\alpha_1)-\gamma_n(\alpha_1,\beta_2)}{\gamma_n(\alpha_1,\beta_2)-1} a_k \leq \sum_{k=n+1}^{\infty} \frac{k(1+k\alpha_1-\alpha_1)-\gamma_n(\alpha_1,\beta_1)}{\gamma_n(\alpha_1,\beta_1)-1} a_k \leq 1, \tag{2.9}$$

which gives (2.6) by Theorem 1.

Next, it suffices to see that

$$M_n^+(\alpha_1,\beta_2) \subset M_n^+(\alpha_2,\beta_2). \tag{2.10}$$

Since $\alpha_1 > \alpha_2 \geq 0$, $1 < \beta_1 < \beta_2 \leq 1 + \frac{n}{2}$, and $k \geq n+1$, by computing easily, we have

$$\begin{aligned} &\frac{k(1+k\alpha_2-\alpha_2)-\gamma_n(\alpha_2,\beta_2)}{\gamma_n(\alpha_2,\beta_2)-1} - \frac{k(1+k\alpha_1-\alpha_1)-\gamma_n(\alpha_1,\beta_2)}{\gamma_n(\alpha_1,\beta_2)-1} \\ = &\frac{(\alpha_1-\alpha_2)(k-1)(\beta_2-1)(\beta_2+\frac{n}{2}-k)}{[\gamma_n(\alpha_2,\beta_2)-1][\gamma_n(\alpha_1,\beta_2)-1]} \leq 0. \end{aligned} \tag{2.11}$$

By Theorem 1 and (2.11), we obtain that (2.10) holds true, and this completes the proof. □

Corollary 2. *Let $\alpha \geq 0$ and $1 < \beta \leq 1 + \frac{n}{2}$. If $f(z) \in M_n^+(\alpha,\beta)$ for $z \in U$, then $f(z) \in M_n^+(\beta)$.*

By taking $\beta = \alpha/2$ and $n = 1$ in Corollary 2, we have the following corollary.

Corollary 3. *If $f(z) \in \mathcal{A}_1^+$ and satisfies*

$$Re\left(\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) < \frac{\alpha^2}{4}(\alpha-1), \ (z \in \mathbb{U})$$

for some α $(2 < \alpha \leq 3)$, then $f(z) \in M_1^+(\alpha/2)$.

Theorem 3. *Let $\alpha \geq 0$, $0 \leq \eta < 1$, and $1 < \beta < 1+\frac{n(1-\eta)}{n+2-2\eta}$. If $f \in M_n^+(\alpha,\beta)$, then $f \in \mathcal{S}_n^*(\eta)$.*

Proof. Since $0 \leq \eta < 1$, we observe that

$$1 < \beta < 1 + \frac{n(1-\eta)}{n+2-2\eta} < 1 + \frac{n}{2} < \frac{2 - n + \sqrt{17n^2 + 20n + 4}}{4},$$

then by Theorem 1, we have

$$\sum_{k=n+1}^{\infty} \frac{k(1 + k\alpha - \alpha) - \gamma_n}{\gamma_n - 1} \, a_k \leq 1,$$

where γ_n is defined in Theorem 1. Using Lemma 1, it suffices to see that

$$\frac{k-\eta}{1-\eta} - \frac{k(1 + k\alpha - \alpha) - \gamma_n}{\gamma_n - 1} < 0. \tag{2.12}$$

It is easy to know that

$$\begin{aligned} & (k-\eta)(\gamma_n - 1) - [k(1 + k\alpha - \alpha) - \gamma_n](1-\eta) \\ = \; & (k-\eta)[\alpha\beta(\beta + \frac{n}{2} - 1) + \beta - \frac{n\alpha}{2} - 1] \\ & -[k(1 + k\alpha - \alpha) - \alpha\beta(\beta + \frac{n}{2} - 1) - \beta + \frac{n\alpha}{2}](1-\eta) \\ = \; & \alpha[(k - 2\eta + 1)\beta^2 + (\frac{n}{2} - 1)(k - 2\eta + 1)\beta - \frac{n}{2}(k - 2\eta + 1) - k(k-1)(1-\eta)] \\ & +(k - 2\eta + 1)\beta - 2k + \eta + k\eta. \end{aligned}$$

Since $1 < \beta < 1 + \frac{n(1-\eta)}{n+2-2\eta} < 1 + \frac{n}{2}$, by some computation easily, for $k \geq n+1$, we have

$$(k - 2\eta + 1)\beta - 2k + \eta + k\eta = (k - 2\eta + 1)\left[\beta - 1 - \frac{(k-1)(1-\eta)}{k + 1 - 2\eta}\right] < 0,$$

and

$$\begin{aligned} & (k - 2\eta + 1)\beta^2 + (\frac{n}{2} - 1)(k - 2\eta + 1)\beta - \frac{n}{2}(k - 2\eta + 1) - k(k-1)(1-\eta) \\ & = (k - 2\eta + 1)(\beta - 1)(\beta + \frac{n}{2}) - k(k-1)(1-\eta) \\ & \leq (n + 2 - 2\eta)(\beta - 1)(\beta + \frac{n}{2}) - n(n+1)(1-\eta) \\ & < n(1-\eta)(\beta + \frac{n}{2} - n - 1) = n(1-\eta)(\beta - \frac{n}{2} - 1) < 0. \end{aligned}$$

Hence, we obtain that

$$(k-\eta)(\gamma_n - 1) - [k(1+k\alpha-\alpha) - \gamma_n](1-\eta) < 0,$$

which leads to (2.12). This completes the proof. □

Theorem 4. *Let* $\alpha \geq 0$, $0 \leq \eta < 1$, *and* $1 < \beta < 1 + \frac{n(1-\eta)}{n+2-2\eta}$. *If*

$$f(z) \in M_n^+(\alpha,\beta) \quad \textit{and} \quad g \in \mathcal{K} \cap \mathcal{A}_n^+,$$

then

$$(f * g)(z) \in M_n^+(\alpha,\beta).$$

Proof. Suppose that $f(z) \in M_n^+(\alpha,\beta)$. Then, by Theorem 3, we have

$$f \in \mathcal{S}_n^*(\eta) \quad \text{and} \quad \mathrm{Re}(H(z)) > 0 \ \ (z \in \mathbb{U}),$$

where

$$H(z) := \alpha\beta(\beta + \frac{n}{2} - 1) + \beta - \frac{n\alpha}{2} - \frac{\alpha z^2 f''(z)}{f(z)} - \frac{zf'(z)}{f(z)}.$$

If we set $F(z) := (f*g)(z)$, then $F(z) \in \mathcal{A}_n^+$. By Lemma 2, simple computation yields

$$\begin{aligned} & \mathrm{Re}\left(\alpha\beta(\beta + \frac{n}{2} - 1) + \beta - \frac{n\alpha}{2} - \frac{\alpha z^2 F''(z)}{F(z)} - \frac{zF'(z)}{F(z)}\right) \\ = \quad & \mathrm{Re}\left(\frac{(g * Hf)(z)}{(g*f)(z)}\right) > 0 \ \ (z \in \mathbb{U}). \end{aligned}$$

Hence

$$F(z) := (f*g)(z) \in M_n^+(\alpha,\beta),$$

which completes the proof of Theorem 4. □

With the aid of Lemma 3, if we apply the same method as in our proof of Theorem 4, we obtain the following corollary.

Corollary 4. *Let $\alpha \geq 0$, $\frac{1}{2} \leq \eta < 1$, and $1 < \beta < 1 + \frac{n(1-\eta)}{n+2-2\eta}$. If*

$$f(z) \in M_n^+(\alpha, \beta) \quad and \quad g \in \mathcal{S}_n^*(\frac{1}{2}) \cap \mathcal{A}_n^+,$$

then

$$(f * g)(z) \in M_n^+(\alpha, \beta).$$

Next, we consider the distortion theorems for the function class $M_n^+(\alpha, \beta)$.

Theorem 5. *Let $\alpha \geq 0$, $1 < \beta < \frac{2-n+\sqrt{17n^2+20n+4}}{4}$ and $\gamma_n = \gamma_n(\alpha, \beta)$ is defined by (2.1). Suppose that $f(z) \in M_n^+(\alpha, \beta)$, then*

$$r - \frac{\gamma_n - 1}{(n+1)(1+n\alpha) - \gamma_n} r^{n+1} \leq |f(z)| \leq r + \frac{\gamma_n - 1}{(n+1)(1+n\alpha) - \gamma_n} r^{n+1}, \quad (|z| = r < 1), \tag{2.13}$$

and

$$1 - \frac{(n+1)(\gamma_n - 1)}{(n+1)(1+n\alpha) - \gamma_n} r^n \leq |f'(z)| \leq 1 + \frac{(n+1)(\gamma_n - 1)}{(n+1)(1+n\alpha) - \gamma_n} r^n, \quad (|z| = r < 1). \tag{2.14}$$

Each of these inequalities is sharp, with the extremal function given by

$$f_{n+1}(z) = z + \frac{\gamma_n - 1}{(n+1)(1+n\alpha) - \gamma_n} z^{n+1}. \tag{2.15}$$

Proof. In view of Theorem 1, we get

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\gamma_n - 1}{(n+1)(1+n\alpha) - \gamma_n}.$$

Therefore, the distortion inequalities in (2.13) follow from

$$r - r^{n+1} \sum_{k=n+1}^{\infty} a_k \leq |f(z)| \leq r + r^{n+1} \sum_{k=n+1}^{\infty} a_k, \quad (|z| = r < 1).$$

Furthermore, Theorem 1 also implies

$$\sum_{k=n+1}^{\infty} ka_k \leq \frac{(n+1)(\gamma_n - 1)}{(n+1)(1+n\alpha) - \gamma_n},$$

the distortion inequalities in (2.14) follow from

$$1 - r^n \sum_{k=n+1}^{\infty} ka_k \leq |f'(z)| \leq 1 + r^n \sum_{k=n+1}^{\infty} ka_k, \quad (|z| = r < 1).$$

The proof of Theorem 5 is thus completed. □

Corollary 5. *Let $\alpha \geq 0$, $1 < \beta < 1 + \frac{n}{2}$ and $\gamma_n = \gamma_n(\alpha, \beta)$ is defined by (2.1). Suppose that $f(z) \in M_n^+(\alpha, \beta)$, then the unit disk U is mapped by $f(z)$ onto a domain that contains the disk $|w| < r_0$, where*

$$r_0 := \frac{(n+1)(1+n\alpha) - 2\gamma_n + 1}{(n+1)(1+n\alpha) - \gamma_n} > 0.$$

The result is sharp, with the extremal function $f_{n+1}(z)$ given by (2.15).

With the aid of Theorem 1, we have the following results.

Theorem 6. *Let $\alpha \geq 0$, $1 < \beta < \frac{2-n+\sqrt{17n^2+20n+4}}{4}$ and $\gamma_n = \gamma_n(\alpha, \beta)$ is defined by (2.1). Then the class $M_n^+(\alpha, \beta)$ is a convex set.*

Theorem 7. *Suppose that $\alpha \geq 0$, $1 < \beta < \frac{2-n+\sqrt{17n^2+20n+4}}{4}$ and $\gamma_n = \gamma_n(\alpha, \beta)$ is defined by (2.1). Also let*

$$f_n(z) = z \qquad \text{and} \qquad f_k(z) = z + \frac{\gamma_n - 1}{k(1 + k\alpha - \alpha) - \gamma_n} z^k \ \ (k \geq n+1).$$

Then $f(z)$ is in the class $M_n^+(\alpha, \beta)$ if and only if is can be expressed in the following form:

$$f(z) = \sum_{k=n}^{\infty} \mu_k f_k(z),$$

where

$$\mu_k \geq 0 \, (k \geq n) \qquad \text{and} \qquad \sum_{k=n}^{\infty} \mu_k = 1.$$

Corollary 6. *Under the hypothesis of Theorem 7, the extreme points of the class* $M_n^+(\alpha, \beta)$ *are the functions* $f_k(z)(k \geq n)$ *given in Theorem 7.*

Next, we derive the integral operators for $f(z) \in M_n^+(\alpha, \beta)$.

Theorem 8. *Suppose that* $\alpha \geq 0$, $1 < \beta < \frac{2-n+\sqrt{17n^2+20n+4}}{4}$ *and* $\gamma_n = \gamma_n(\alpha, \beta)$ *is defined by (2.1). Also let* c *be a real number such that* $c > -1$. *If* $f(z) \in M_n^+(\alpha, \beta)$, *then the functions* $F(z)$ *defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \tag{2.16}$$

also belongs to the class $M_n^+(\alpha, \beta)$.

Proof. From the representation of $F(z)$, it follows that

$$F(z) = z + \sum_{k=n+1}^{\infty} b_k z^k,$$

where

$$b_k = \frac{c+1}{c+k} a_k < a_k.$$

Therefore

$$\sum_{k=n+1}^{\infty} [k(1 + k\alpha - \alpha) - \gamma_n] b_k \leq \sum_{k=n+1}^{\infty} [k(1 + k\alpha - \alpha) - \gamma_n] a_k. \tag{2.17}$$

Since $f(z) \in M_n^+(\alpha, \beta)$, by Theorem 1 we have

$$\sum_{k=n+1}^{\infty} [k(1 + k\alpha - \alpha) - \gamma_n] a_k \leq \gamma_n - 1. \tag{2.18}$$

Then $F(z) \in M_n^+(\alpha, \beta)$ follows from (2.17) and (2.18), and the proof of Theorem 8 is thus completed. □

Theorem 9. *Suppose that* $\alpha \geq 0$, $1 < \beta < \frac{2-n+\sqrt{17n^2+20n+4}}{4}$ *and* $\gamma_n = \gamma_n(\alpha, \beta)$ *is defined by (2.1). Also let* c *be a real number such that* $c > -1$. *If* $F(z) \in M_n^+(\alpha, \beta)$, *then the function defined by (2.16) is univalent in* $|z| < R^*$, *where*

$$R^* := \inf_{k \geq n+1} \left[\frac{(c+1)[k(1+k\alpha-\alpha)-\gamma_n]}{k(c+k)(\gamma_n-1)} \right]^{\frac{1}{k-1}}.$$

The result is sharp.

Proof. Let

$$F(z) = z + \sum_{k=n+1}^{\infty} a_k z^k.$$

It follows from (2.16) that

$$f(z) = \frac{z^{1-c}[z^c F(z)]'}{c+1} = z + \sum_{k=n+1}^{\infty} \frac{c+k}{c+1} a_k z^k, \quad (c > -1).$$

In order to obtain the required result, it suffices to show that $|f'(z)-1| < 1$ in $|z| < R^*$. Now

$$|f'(z) - 1| \leq \sum_{k=n+1}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| < 1$ if

$$\sum_{k=n+1}^{\infty} \frac{k(c+k)}{c+1} a_k |z|^{k-1} < 1. \tag{2.19}$$

Since $F(z) \in M_n^+(\alpha, \beta)$, we have

$$\sum_{k=n+1}^{\infty} \frac{k(1+k\alpha-\alpha)-\gamma_n}{\gamma_n-1} a_k \leq 1.$$

Hence, (2.19) will be satisfied if

$$\frac{k(c+k)}{c+1}|z|^{k-1} < \frac{k(1+k\alpha-\alpha)-\gamma_n}{\gamma_n-1},$$

or if

$$|z| < \left[\frac{(c+1)[k(1+k\alpha-\alpha)-\gamma_n]}{k(c+k)(\gamma_n-1)}\right]^{\frac{1}{k-1}} \quad (k \geq n+1).$$

Therefore $f(z)$ is univalent in $|z| < R^*$, sharpness follows if we taking

$$f_k(z) = z + \frac{k(1+k\alpha-\alpha)-\gamma_n}{\gamma_n-1} z^k \quad (k \geq n+1).$$

This completes the proof of Theorem 9. □

Now, we consider the neighborhood of the class $M_n^+(\alpha, \beta)$.

O.Altintaş, Ö.Özkan and H.M.Srivastava [1] introduced the definition of (n, δ)-neighborhood of a function $f(z) \in \mathcal{A}_n$ have the form (1.3). We give the definition of (n, δ)-neighborhood of a function $f(z) \in \mathcal{A}_n^+$ as follows.

Definition 2. Let $f(z) \in \mathcal{A}_n^+$ and $\delta > 0$, then the (n, δ)-neighborhood of a function $f(z)$ defined by

$$N_{n,\,\delta}(f) = \{g \in \mathcal{A}_n^+ : g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta\}.$$

In particular, for the identity of function

$$e(z) = z$$

immediately have

$$N_{n,\,\delta}(e) = \{g \in \mathcal{A}_n^+ : g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \text{ and } \sum_{k=n+1}^{\infty} k b_k \leq \delta\}.$$

Theorem 10. *Suppose that* $\alpha \geq 0$, $1 < \beta < \frac{2-n+\sqrt{17n^2+20n+4}}{4}$ *and* $\gamma_n = \gamma_n(\alpha, \beta)$ *is defined by (2.1). Then*

$$M_n^+(\alpha, \beta) \subset N_{n,\,\delta}(e),$$

where

$$\delta := \frac{(n+1)(\gamma_n - 1)}{(n+1)(1+n\alpha) - \gamma_n}.$$

Proof. For $f(z) \in M_n^+(\alpha, \beta)$, by Theorem 1, we have

$$\sum_{k=n+1}^{\infty} k a_k \leq \frac{(n+1)(\gamma_n - 1)}{(n+1)(1+n\alpha) - \gamma_n},$$

which, in view of definition 2, proves Theorem 10. □

Next, we discuss the neighborhood of the class $M_n^+(\alpha, \beta)$ which is defined as follows. A function $f(z) \in \mathcal{A}_n^+$ is said to be in the class $M_{n,\gamma}^+(\alpha, \beta)$ if there exists a function $g(z) \in M_n^+(\alpha, \beta)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1 - \gamma, \quad (z \in \mathbb{U}; 0 \leq \gamma < 1).$$

Theorem 11. *Let $\alpha \geq 0$, $1 < \beta < 1 + \frac{n}{2}$ and $\gamma_n = \gamma_n(\alpha, \beta)$ is defined by (2.1). If $g(z) \in M_n^+(\alpha, \beta)$ and $\delta > 0$ such that*

$$\gamma := 1 - \frac{\delta[(n+1)(1+n\alpha) - \gamma_n]}{(n+1)[(n+1)(1+n\alpha) + 1 - 2\gamma_n]} \geq 0, \tag{2.20}$$

then

$$N_{n,\,\delta}(g) \subset M_{n,\gamma}^+(\alpha, \beta).$$

Proof. Suppose that $f \in N_{n,\,\delta}(g)$. From definition 2, we have

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \leq \delta,$$

which readily implies the coefficient inequality

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \leq \frac{\delta}{n+1} \quad (n \in \mathbb{N}).$$

Next, since $g(z) \in M_n^+(\alpha, \beta)$ and $1 < \beta < 1 + \frac{n}{2}$, from Theorem 1, we have

$$\sum_{k=n+1}^{\infty} b_k \leq \frac{\gamma_n - 1}{(n+1)(1+n\alpha) - \gamma_n} < 1,$$

so that

$$
\begin{aligned}
\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum_{k=n+1}^{\infty}|a_k-b_k|}{1-\sum_{k=n+1}^{\infty}b_k}\\
&\leq \frac{\delta}{n+1}\cdot\frac{1}{1-\frac{\gamma_n-1}{(n+1)(1+n\alpha)-\gamma_n}}\\
&= \frac{\delta[(n+1)(1+n\alpha)-\gamma_n]}{(n+1)[(n+1)(1+n\alpha)+1-2\gamma_n]}\\
&= 1-\gamma,
\end{aligned}
$$

provided that γ is given precisely by (2.20). Thus, by definition, $f(z) \in M_{n,\gamma}^{+}(\alpha,\beta)$, which evidently completes the proof of Theorem 11. □

Srivastava et al.[9] introduced the following definition of a fractional integral operator, which is popularly referred to as the Srivastava-Saigo-Owa fractional integral operator.

Definition 3. (See, for details, Srivastava et al.[9, 8]). For real numbers $\eta > 0$, γ and δ, the fractional integral operator $I_{0,z}^{\eta,\gamma,\delta}$ is defined by

$$I_{0,z}^{\eta,\gamma,\delta}f(z) := \frac{z^{-\eta-\gamma}}{\Gamma(\eta)}\int_0^z (z-t)^{\eta-1}{}_2F_1(\eta+\gamma,-\delta;\eta;1-\frac{t}{z})f(t)dt, \tag{2.21}$$

where $f(z)$ is an analytic function in a simply-connected region of the complex z-plane, containing the origin, with the following order:

$$f(z) = O(|z|^{\varepsilon}) \ \ (z \to 0; \varepsilon > \max\{0,\gamma-\delta\}-1),$$

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

denotes the Gauss hypergeometric function in terms of the Pochhammer symbol $(\lambda)_k$ given by

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & (k=0),\\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k\in\mathbb{N}),\end{cases}$$

and the multiplicity of $(z-t)^{\eta-1}$ is removed by requiring $\log(z-t)$ to be real when $z-t>0$.

Lemma 4. *(See [9]). The following formula holds true for the Srivastava-Saigo-Owa fractional integral operator* $I_{0,z}^{\eta,\gamma,\,\delta}$ *is defined by (2.21):*

$$I_{0,\,z}^{\eta,\gamma,\,\delta}z^k = \frac{\Gamma(k+1)\Gamma(k-\gamma+\delta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\eta+\delta+1)}z^{k-\gamma}.$$

Theorem 12. *Let*

$$\eta > 0, \gamma < 2, \eta+\delta > -2, \gamma-\delta < 2 \text{ and } \gamma(\eta+\delta) \leq \eta(n+2).$$

Suppose also that $\alpha \geq 0$, $1 < \beta < \frac{2-n+\sqrt{17n^2+20n+4}}{4}$ *and* $\gamma_n = \gamma_n(\alpha,\beta)$ *is defined by (2.1). If the function* $f(z)$ *is in the class* $M_n^+(\alpha,\beta)$*, then*

$$\begin{aligned}|I_{0,\,z}^{\eta,\gamma,\,\delta}f(z)| \ &\leq\ \frac{\Gamma(2-\gamma+\delta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\eta+\delta)} \\ &\quad\cdot\left(1+\frac{(2)_n(2-\gamma+\delta)_n(\gamma_n-1)}{(2-\gamma)_n(2+\eta+\delta)_n[(n+1)(1+n\alpha)-\gamma_n]}|z|^n\right), \quad (z\in\mathbb{U}_0)\end{aligned}$$

and

$$\begin{aligned}|I_{0,\,z}^{\eta,\gamma,\,\delta}f(z)| \ &\geq\ \frac{\Gamma(2-\gamma+\delta)|z|^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\eta+\delta)} \\ &\quad\cdot\left(1-\frac{(2)_n(2-\gamma+\delta)_n(\gamma_n-1)}{(2-\gamma)_n(2+\eta+\delta)_n[(n+1)(1+n\alpha)-\gamma_n]}|z|^n\right), \quad (z\in\mathbb{U}_0)\end{aligned}$$

where

$$\mathbb{U}_0 = \begin{cases} \mathbb{U} & (\gamma \leq 1), \\ \mathbb{U}-\{0\} & (\gamma > 1). \end{cases}$$

Each of these inequalities is sharp, with the extremal function $f_{n+1}(z)$ given by (2.15).

Proof. By Lemma 4, we have

$$I_{0,\,z}^{\eta,\gamma,\,\delta}f(z) := \frac{\Gamma(2-\gamma+\delta)z^{1-\gamma}}{\Gamma(2-\gamma)\Gamma(2+\eta+\delta)} + \sum_{k=n+1}^{\infty}\frac{\Gamma(k+1)\Gamma(k-\gamma+\delta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\eta+\delta+1)}a_k z^{k-\gamma}.$$

If we set

$$G(z) = \frac{\Gamma(2-\gamma)\Gamma(2+\eta+\delta)}{\Gamma(2-\gamma+\delta)} z^{\gamma} I_{0,z}^{\eta,\gamma,\ \delta} f(z) = z + \sum_{k=n+1}^{\infty} g(k) a_k z^k, \tag{2.22}$$

where

$$g(k) = \frac{k!(2-\gamma+\delta)_{k-1}}{(2-\gamma)_{k-1}(2+\eta+\delta)_{k-1}}, \ \ (k \geq n+1),$$

then, since

$$\eta > 0,\ \gamma < 2,\ \eta+\delta > -2,\ \gamma-\delta < 2 \text{ and } \gamma(\eta+\delta) \leq \eta(n+2),$$

we find that

$$\begin{aligned} \frac{g(k+1)}{g(k)} &= \frac{(k+1)(k+1-\gamma+\delta)}{(k+1-\gamma)(k+1+\eta+\delta)} \\ &= \frac{(k+1)^2-(k+1)(\gamma-\delta)}{(k+1)^2-(k+1)(\gamma-\delta)+\eta(k+1)-\gamma(\eta+\delta)} \\ &\leq 1, \ \ (k \geq n+1). \end{aligned}$$

Therefore, $g(k)$ is a non-increasing function for integers $k \geq n+1$, and we have

$$0 < g(k) \leq g(n+1) = \frac{(2)_n(2-\gamma+\delta)_n}{(2-\gamma)_n(2+\eta+\delta)_n}, \ \ (k \geq n+1). \tag{2.23}$$

Hence, applying Theorem 5 and (2.23), we obtain

$$\begin{aligned} |G(z)| &\leq |z| + g(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} a_k \\ &\leq |z| + \frac{(2)_n(2-\gamma+\delta)_n(\gamma_n-1)}{(2-\gamma)_n(2+\eta+\delta)_n[(n+1)(1+n\alpha)-\gamma_n]}|z|^{n+1} \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} |G(z)| &\geq |z| - g(n+1)|z|^{n+1} \sum_{k=n+1}^{\infty} a_k \\ &\geq |z| - \frac{(2)_n(2-\gamma+\delta)_n(\gamma_n-1)}{(2-\gamma)_n(2+\eta+\delta)_n[(n+1)(1+n\alpha)-\gamma_n]}|z|^{n+1}. \end{aligned} \tag{2.25}$$

Now the inequalities in the above-mentioned assertions of Theorem 12 follows when we make use of (2.24) and (2.25) in the definition (2.22).

The inequalities in the above-mentioned assertions of Theorem 12 are easily seen to be attained by the function $f_{n+1}(z)$ defined by (2.15). This evidently completes the proof of Theorem 12. □

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