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Some Contributions to the Theory of Generalized Orlicz Sequence Space *

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Abstract

In this article a description of the Orlicz difference sequence space $\ell_M(\Delta_{(m)})$ generated by Orlicz function M and a new generalized difference operator $\Delta_{(m)}$ is presented. We investigate some topological structures relevant to this space. It is also shown that under certain

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condition $\ell_M(\Delta_{(m)})$ is topologically isomorphic to ℓ_∞. Furthermore we define a subspace $h_M(\Delta_{(m)})$ of $\ell_M(\Delta_{(m)})$ and it is shown that under certain condition $h_M(\Delta_{(m)})$ is topologically isomorphic to c_0.

Keywords and Phrases: *Difference sequence space, Orlicz function, AK-BK space, Topological isomorphism.*

1. Introduction

Throughout this section w, ℓ_∞, ℓ_1, c and c_0 denote the space of all, bounded, absolutely summable, convergent and null sequences $x = (x_k)$ with complex terms respectively. The notion of difference sequence space was introduced by Kizmaz [4], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$.

For Z a given sequence space we have $Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$, where $\Delta x = (\Delta x_k) = (\Delta x_k - \Delta x_{k+1})$. For $Z = \ell_\infty$, c and c_0, we have the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [4].

Let m be a non-negative integer. Then Dutta [2] defined the following sequence spaces for Z a given sequence space $Z(\Delta_{(m)}) = \{x = (x_k) \in w : (\Delta_{(m)} x_k) \in Z\}$, where $\Delta_{(m)} x = (\Delta_{(m)} x_k) = (x_k - x_{k-m})$ and $\Delta_0 x_k = x_k$ for all $k \in N$. For $Z = \ell_\infty$, c and c_0, we have the spaces $\ell_\infty(\Delta_{(m)})$, $c(\Delta_{(m)})$ and $c_0(\Delta_{(m)})$ respectively.

Taking $m = 1$, we get the spaces $Z(\Delta_{(1)})$. It is obvious that $(x_k) \in Z(\Delta_{(1)})$ if and only if $(x_k) \in Z(\Delta)$.

An Orlicz function is a function $M : [0, \infty) \longrightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \longrightarrow \infty$, as $x \longrightarrow \infty$.

An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x p(t)dt$$

where p, known as kernel of M, is right differentiable for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non-decreasing, and $p(t) \longrightarrow \infty$ as $t \longrightarrow \infty$.

Consider the kernel $p(t)$ associated with the Orlicz function $M(t)$, and let $q(s) = \sup\{t : p(t) \leq s\}$. Then q possesses the same properties as the function

p. Suppose now

$$\Phi(x) = \int_0^x q(s)ds$$

Then Φ is an Orlicz function. The functions M and Φ are called mutually complementary Orlicz functions.

Now we state the following well known results.

Let M and F are mutually complementary Orlicz functions. Then we have (Young's inequality)

(i) For $x, y \geq 0$, $xy \leq M(x) + \Phi(y)$ (1.1)

We also have

(ii) For $x \geq 0$, $xp(x) = M(x) + \Phi(p(x))$ (1.2)

(iii) $M(\lambda x) < \lambda M(x)$ for all $x \geq 0$ and λ with $0 < \lambda < 1$.

An Orlicz function M is said to satisfy the Δ_2-condition for all small x or at 0 if for each $k > 0$ there exists$R_k > 0$ and $x_k > 0$ such that $M(kx) \leq R_k M(x)$ for all $x \in (0, x_k]$.

Moreover an Orlicz function M is said to satisfy the Δ_2-condition if and only if

$$\lim_{x \longrightarrow 0} \sup \frac{M(2x)}{M(x)} < \infty$$

Two Orlicz functions M_1 and M_2 are said to be equivalent if there are positive constatns α, β and x_0 such that

$$M_1(\alpha x) \leq M_2(x) \leq M_1(\beta x) \text{ for all } x \text{ with } 0 \leq x \leq x_0. \tag{1.3}$$

Lindenstrauss and Tzafriri [7] used the Orlicz function and introduced the sequence space ℓ_M as follows:

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \quad \text{for some } \rho > 0 \right\}.$$

Let m be a non-negative integer. Then we define the following spaces.

Definition 1.1. Let M be any Orlicz functions. Then we define

$$\tilde{\ell}_M(\Delta_{(m)}) = \left\{ x = (x_k) \in w : \delta(M, \Delta_{(m)}, x) = \sum_{k=1}^{\infty} M\left(|\Delta_{(m)} x_k|\right) < \infty \right\},$$

where $\Delta_{(m)} x_k = x_k - x_{k-m}$ for all $k \geq 1$.

If $m = 0$, then we write $\tilde{\ell}_M(\Delta_{(m)}) = \tilde{\ell}_M$. If $m = 1$, the space reduced to the space studied by Dutta [1].

Definition 1.2. Let M and Φ be mutually complementary functions. Then we define

$$\ell_M(\Delta_{(m)}) = \left\{ x = (x_k) \in w : \sum_{k=1}^{\infty} \left(\Delta_{(m)} x_k\right) y_k \text{ converges for all } y \in \tilde{\ell}_\Phi \right\}$$

and we call this sequence space as Orlicz $\Delta_{(m)}$-difference sequence space.

If $m = 0$, then we write $\ell_M(\Delta_{(m)}) = \ell_M$. If $m = 1$, the space reduced to the space studied by Dutta [1].

2. Main Results

The main aim of this section is to describe the space $\ell_M(\Delta_{(m)})$ and investigate some properties of this space as well as the subspace $h_M(\Delta_{(m)})$ of $\ell_M(\Delta_{(m)})$.

Proposition 2.1. *For any Orlicz function M, $\tilde{\ell}_M(\Delta_{(m)}) \subset \ell_M(\Delta_{(m)})$.*

Proof. Let $x \in \tilde{\ell}_M(\Delta_{(m)})$. Then $\sum_{k=1}^{\infty} M\left(|\Delta_{(m)} x_k|\right) < \infty$. Now using (1.1), we have

$$\left| \sum_{k=1}^{\infty} (\Delta_{(m)} x_k) y_k \right| \leq \sum_{k=1}^{\infty} \left| (\Delta_{(m)} x_k) y_k \right| \leq \sum_{k=1}^{\infty} M\left(|\Delta_{(m)} x_k|\right) + \sum_{k=1}^{\infty} \Phi(|y_k|) < \infty,$$

for every $y = (y_k)$ with $y \in \tilde{\ell}_\Phi$. Thus $x \in \ell_M(\Delta_{(m)})$.

Proposition 2.2. *For each $x \in \ell_M(\Delta_{(m)})$,*

$$\sup \left\{ \left| \sum_{i=1}^{\infty} (\Delta_{(m)} x_i) y_i \right| : \delta(\Phi, y) \leq 1 \right\} < \infty$$

Proof. Suppose the required result is not true. Then for each $n \geq 1$, there exists y^n with $\delta(\Phi, y^n) \leq 1$ such that

$$\left| \sum_{i=1}^{\infty} (\Delta_{(m)} x_i) y_i^n \right| > 2^n.$$

Without loss of generality we may assume that $(\Delta_{(m)}x_i)$, $y^n \geq 0$. Now, we can define a sequence $z = \{z_i\}$, where

$$z_i = \sum_{n=1}^{\infty} \frac{1}{2^n} y_i^n.$$

By the convexity of Φ,

$$\Phi\left(\sum_{n=1}^{l} \frac{1}{2^n} y_i^n\right) \leq \frac{1}{2}\left[\Phi(y_i^1) + \Phi\left(\frac{y_i^2}{2} + \cdots + \frac{y_i^l}{2^{l-1}}\right)\right] \leq \cdots\cdots \leq \sum_{n=1}^{l} \frac{1}{2^n}\Phi(y_i^n)$$

and hence, using the continuity of Φ, we have

$$\delta(\Phi, z) = \sum_{i=1}^{\infty} \Phi(z_i) \leq \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{2^n}\Phi(y_i^n) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

But for every $l \geq 1$,

$$\sum_{i=1}^{\infty}(\Delta_{(m)}x_i)z_i \geq \sum_{i=1}^{\infty}(\Delta_{(m)}x_i)\sum_{n=1}^{l}\frac{1}{2^n}y_i^n = \sum_{n=1}^{l}\sum_{i=1}^{\infty}(\Delta_{(m)}x_i)\frac{y_i^n}{2^n} \geq l.$$

Hence $\sum\limits_{i=1}^{\infty}(\Delta_{(m)}x_i)z_i$ diverges and this implies that $x \notin \ell_M(\Delta_{(m)})$. This contradiction leads us to the required result.

The preceding result encourages us to introduce the following norm $\|\bullet\|_M^{(m)}$ on $\ell_M(\Delta_{(m)})$.

Proposition 2.3. *$\ell_M(\Delta^m, \Lambda)$ is a normed linear space under the norm $\|\bullet\|_M^{\Delta^m}$ defined by*

$$\|x\|_M^{(m)} = \sup\left\{\left|\sum_{i=1}^{\infty}(\Delta_{(m)}x_i)y_i\right| : \delta(\Phi, y) \leq 1\right\} < \infty \tag{2.1}$$

Proof. It easy to verify that $\ell_M(\Delta_{(m)})$ is a linear space. Now we show that $\|\bullet\|_M^{(m)}$ is a norm on $\ell_M(\Delta_{(m)})$.

Let $x = \theta$, then obviously $\|x\|_M^{(m)} = 0$. Conversely assume $\|x\|_M^{(m)} = 0$. Then using (2.1), we have

$$\sup\left\{\left|\sum_{i=1}^{\infty}(\Delta_{(m)}x_i)y_i\right| : \delta(\Phi, y) \leq 1\right\} = 0$$

This implies $\left|\sum_{i=1}^{\infty}(\Delta_{(m)}x_i)y_i\right| = 0$ for all y such that $\delta(\Phi, y) \leq 1$. Now we consider $y = \{e_i\}$ if $\Phi(1) \leq 1$ otherwise we consider $y = \{\frac{e_i}{\Phi(1)}\}$ so that $\Delta_{(m)}x_i = 0$ for all $i \geq 1$. Taking $i = 1$, we have $\Delta_{(m)}x_1 = x_1 - x_{1-m} = 0$. This implies $x_1 = 0$, by taking $x_{1-m} = 0$. Proceeding in this way we have $x_i = 0$ for all $i \geq 1$. Thus $x = \theta$. It is easy to show $\|\alpha x\|_M^{(m)} = |\alpha| \; \|x\|_M^{(m)}$ and $\|x + y\|_M^{(m)} \leq \|x\|_M^{(m)} + \|y\|_M^{(m)}$. This completes the proof.

Proposition 2.4. *$\ell_M(\Delta_{(m)})$ is a Banach space under the norm $\|\bullet\|_M^{(m)}$ as defined in (2.1).*

Proof. Let (x^i) be any Cauchy sequence in $\ell_M(\Delta_{(m)})$. Then any $\varepsilon > 0$, there exists a positive integer n_0 such that $\|x^i - x^j\|_M^{(m)} < \varepsilon$, for all $i, j > n_0$. Using the definition of norm, we get

$$\sup\left\{\left|\sum_{k=1}^{\infty}\left(\Delta_{(m)}(x_k^i - x_k^j)\right) y_k\right| : \delta(\Phi, y) \leq 1\right\} < \varepsilon, \quad \text{for all } i, j > n_0.$$

It follows that

$$\left|\sum_{k=1}^{\infty}\left(\Delta_{(m)}(x_k^i - x_k^j)\right) y_k\right| < \varepsilon, \quad \text{for all } y \text{ with } \delta(\Phi, y) \leq 1 \text{ and } i, j > n_o.$$

Now considering $y = \{e_i\}$ if $\Phi(1) \leq 1$ otherwise considering $y = \{\frac{e_i}{\Phi(1)}\}$ we have $(\Delta_{(m)}x_k^i)$ is a Cauchy sequence in C for all $k \geq 1$ and hence it is a convergent sequence in C for all $k \geq 1$. Let $\lim_{i \longrightarrow \infty} \Delta_{(m)}x_k^i = z_k$, say for all $k \geq 1$. Taking $k = 1, 2, ..., m, \ldots$ we can easily conclude that $\lim_{i \longrightarrow \infty} x_k^i = x_k$, say exists for each $k \geq 1$. Now can have

$$\sup\left\{\left|\sum_{k=1}^{\infty}\left(\Delta_{(m)}(x_k^i - x_k)\right) y_k\right| : \delta(\Phi, y) \leq 1\right\} < \varepsilon, \quad \text{for all } i \geq n_0 \text{ as } j \longrightarrow \infty.$$

It follows that $(x^i - x) \in \ell_M(\Delta_{(m)})$ and $\ell_M(\Delta_{(m)})$ is a linear space and hence $x = (x_k) \in \ell_M(\Delta_{(m)})$.

From the above proof we can easily conclude that $\|x^i\|_M^{(m)} \longrightarrow 0$ implies that $x_k^i \longrightarrow 0$ for each $i \geq 1$. Hence we have the following Proposition.

Proposition 2.5. *$\ell_M(\Delta_{(m)})$ is a BK spaces under the norm as defined in (2.1).*

Our next aim is to show that $\ell_M(\Delta_{(m)})$ can be made BK space under a different but equivalent norm.

Proposition 2.6. *$\ell_M(\Delta_{(m)})$ is a normed linear space under the norm $\|\bullet\|_{(M)}^{(m)}$ defined by*

$$\|x\|_{(M)}^{(m)} = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|\Delta_{(m)}x_k|}{\rho}\right) \leq 1\right\} \tag{2.2}$$

Proof. Clearly $\|x\|_{M}^{(m)} = 0$ if $x = \theta$. Next suppose $\|x\|_{(M)}^{(m)} = 0$. Then using (2.2) we have

$$\inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|\Delta_{(m)}x_k|}{\rho}\right) \leq 1\right\} = 0.$$

This implies that for a given $\varepsilon > 0$, there exists some ρ_ε $(0 < \rho_\varepsilon < \varepsilon)$ such that

$$\sup_k M\left(\frac{|\Delta_{(m)}x_k|}{\rho_\varepsilon}\right) \leq 1$$

This implies that

$$M\left(\frac{|\Delta_{(m)}x_k|}{\rho_\varepsilon}\right) \leq 1 \text{ for all } k \geq 1.$$

Thus

$$M\left(\frac{|\Delta_{(m)}x_k|}{\varepsilon}\right) \leq M\left(\frac{|\Delta_{(m)}x_k|}{\rho_\varepsilon}\right) \leq 1 \text{ for all } k \geq 1.$$

Suppose $\Delta_{(m)}x_{n_i} \neq 0$, for some i. Let $\varepsilon \longrightarrow 0$ then $\frac{|\Delta_{(m)}x_{n_i}|}{\varepsilon} \longrightarrow \infty$. It follows that

$$M\left(\frac{|\Delta_{(m)}x_{n_i}|}{\varepsilon}\right) \longrightarrow \infty \text{ as } \varepsilon \longrightarrow 0 \text{ for some } n_i \in N.$$

This is a contradiction. Therefore $\Delta_{(m)}x_k = 0$ for all $k \geq 1$. Considering $k = 1, 2, \ldots, m, \ldots$ it follows that $x_k = 0$ for all $k \geq 1$. Hence $x = \theta$.

Let $x = (x_k)$ and $y = (y_k)$ be any two elements of $\|\bullet\|_{(M)}^{\Delta^m}$. Then there exist $\rho_1, \rho_2 > 0$ such that

$$\sup_k M\left(\frac{|\Delta_{(m)}x_k|}{\rho_1}\right) \leq 1 \quad \text{and} \quad \sup_k M\left(\frac{|\Delta_{(m)}y_k|}{\rho_2}\right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by convexity of M, we have

$$\sup_k M\left(\frac{|\Delta_{(m)}(x_k + y_k)|}{\rho}\right) \leq \frac{\rho_1}{\rho_1+\rho_2}\sup_k M\left(\frac{|\Delta_{(m)}x_k|}{\rho_1}\right) + \frac{\rho_2}{\rho_1+\rho_2}\sup_k M\left(\frac{|\Delta_{(m)}y_k|}{\rho_2}\right) \leq 1.$$

Hence we have

$$\|x+y\|_{(M)}^{(m)} = \inf\left\{\rho > 0 : \sup_k M\left(\frac{|\Delta_{(m)}(x_k + y_k)|}{\rho}\right) \leq 1\right\}$$

$$\leq \inf\left\{\rho_1 > 0 : \sup_k M\left(\frac{|\Delta_{(m)}x_k|}{\rho_1}\right) \leq 1\right\} + \inf\left\{\rho_2 > 0 : \sup_k M\left(\frac{|\Delta_{(m)}y_k|}{\rho_2}\right) \leq 1\right\}$$

This implies that $\|x+y\|_{(M)}^{(m)} \leq \|x\|_{(M)}^{(m)} + \|y\|_{(M)}^{(m)}$. Finally,let ν be any scalar. Then

$$\|\nu x\|_{(M)}^{(m)} = \inf\left\{\rho > 0 : \sup_k M\left(\frac{|\Delta_{(m)}\nu x_k|}{\rho}\right) \leq 1\right\}$$

$$= \inf\left\{r|\nu| > 0 : \sup_k M\left(\frac{|\Delta_{(m)}x_k|}{r}\right) \leq 1\right\} \qquad \text{where} \quad r = \frac{\rho}{|\nu|}$$

$$= |\nu|\, \|x\|_{(M)}^{(m)}.$$ This completes the proof.

Proposition 2.7. *For $x \in \ell_M(\Delta_{(m)})$, we have*

$$\sum_{k=1}^{\infty} M\left(\frac{|\Delta_{(m)}x_k|}{\|x\|_{(M)}^{(m})}\right) \leq 1.$$

Proof. Proof is immediate from (2.2).

Now we show that the norms $\|\bullet\|_{(M)}^{(m)}$ and $\|\bullet\|_{M}^{(m)}$ are equivalent. To prove this some other results are required. First we prove those results.

Proposition 2.8. *Let $x \in \ell_M(\Delta_{(m)})$ with $\|x\|_M^{(m)} \leq 1$. Then $\{p(|\Delta_{(m)}x_n|)\} \in \tilde{\ell}_\Phi$ and $\delta(\Phi, \{p(|\Delta_{(m)}x_n|)\}) \leq 1$.*

Proof. For any $z \in \tilde{\ell}_\Phi$, we may write

$$\left|\sum_{i=1}^{\infty}(\Delta_{(m)}x_i)z_i\right| \leq \|x||_M^{(m)} \quad \text{if } \delta(\Phi, z) \leq 1 \tag{2.3}$$

$$= \delta(\Phi, z)\|x||_M^{(m)} \quad \text{if } \delta(\Phi, z) > 1$$

Let now $x \in \ell_M(\Delta_{(m)})$ with $\|x||_M^{(m)} \leq 1$. Also $x^{(n)} = (x_1, x_2, \dots, x_n, 0, 0, \dots) \in \ell_M(\Delta_{(m)})$ for $n \geq 1$. We observe that

$$\|x||_M^{(m)} \geq \left| \sum_{i=1}^{\infty} (\Delta_{(m)} x_i) y_i^{(n)} \right| = \left| \sum_{i=1}^{\infty} (\Delta_{(m)} x_i^{(n)}) y_i \right|,$$

$n \geq 1$ for every $y \in \tilde{\ell}_\Phi$ with $\delta(\Phi, y) \leq 1$ and thus $\|x^{(n)}||_M^{(m)} \leq \|x||_M^{(m)} \leq 1$. Since

$$\sum_{i=1}^{\infty} \Phi\left(p\left(\left|\Delta_{(m)} x_i\right|\right)\right) = \sum_{i=1}^{\infty} \Phi\left(p\left(\left|\Delta_{(m)} x_i^{(n)}\right|\right)\right)$$

We find that $\left\{p\left(\left|\Delta_{(m)} x_i^{(n)}\right|\right)\right\} \in \tilde{\ell}_\Phi$ for each $n \geq 1$. Let $l \geq 1$ be an integer such that

$$\sum_{i=1}^{l} \Phi\left(p\left(\left|\Delta_{(m)} x_i\right|\right)\right) > 1. \quad \text{Then} \quad \sum_{i=1}^{\infty} \Phi\left(p\left(\left|\Delta_{(m)} x_i^{(l)}\right|\right)\right) > 1$$

Using (1.2), we have

$$\begin{aligned} \Phi\left(p\left(\left|\Delta_{(m)} x_i^{(l)}\right|\right)\right) &< M\left(\left|\Delta_{(m)} x_i^{(l)}\right|\right) + \Phi\left(p\left(\left|\Delta_{(m)} x_i^{(l)}\right|\right)\right) \\ &= \left|\Delta_{(m)} x_i^l\right| p\left(\left|\Delta_{(m)} x_i^l\right|\right) \quad \text{for all } i, l \geq 1. \end{aligned}$$

So by (2.3), we get

$$\sum_{i=1}^{\infty} \Phi\left(p\left(\left|\Delta_{(m)} x_i^{(l)}\right|\right)\right) < \left\|x^{(l)}\right\|_M^{(m)} \delta\left(\Phi, \left\{p\left(\left|\Delta_{(m)} x_i^l\right|\right)\right\}\right)$$

This implies that $\left\|x^{(l)}\right\|_M^{(m)} > 1$, a contradiction. This contradiction implies that

$$\sum_{i=1}^{l} \Phi\left(p\left(\left|\Delta_{(m)} x_i\right|\right)\right) \leq 1 \quad \text{for all } l \geq 1.$$

Hence

$$\left\{p\left(\left|\Delta_{(m)} x_n\right|\right)\right\} \in \tilde{\ell}_\Phi \quad \text{and} \quad \delta\left(\Phi, \left\{p\left(\left|\Delta_{(m)} x_n\right|\right)\right\}\right) \leq 1.$$

Proposition 2.9. *Let* $x \in \ell_M(\Delta_{(m)})$ *with* $\|x\|_M^{(m)} \leq 1$. *Then* $x \in \tilde{\ell}_M(\Delta_{(m)})$ *and* $\delta\left(M, \Delta_{(m)}, x\right) \leq \|x\|_M^{(m)}$

Proof. Let $y = \left\{p\left(\left|(\Delta_{(m)}x_i\right|\right)/sgn(\Delta_{(m)}x_i)\right\}$. Then the Proposition 2.8, $y \in \tilde{\ell}_\Phi$ and $\delta(\Phi, y) \leq 1$. By (1.2), we get

$$\sum_{i=1}^{\infty} M\left(\left|\Delta_{(m)}x_i\right|\right) \leq \sum_{i=1}^{\infty} M\left(\left|\Delta_{(m)}x_i\right|\right) + \sum_{i=1}^{\infty} \Phi\left(p\left(\left|\Delta_{(m)}x_i\right|\right)\right)$$

$$= \sum_{i=1}^{\infty} \left|\Delta_{(m)}x_i\right| p\left(\left|\Delta_{(m)}x_i\right|\right) = \left|\sum_{i=1}^{\infty}(\Delta_{(m)}x_i)y_i\right| \leq \|x\|_M^{(m)}$$

This implies that $\delta(M, \Delta_{(m)}, x) \leq \|x\|_M^{(m)}$.

Proposition 2.10. *For $x \in \ell_M(\Delta_{(m)})$, we have*

$$\sum_{k=1}^{\infty} M\left(\frac{\left|\Delta_{(m)}x_k\right|}{\|x\|_M^{(m)}}\right) \leq 1.$$

Proof. Proof is immediate from Proposition 2.9.

Theorem 2.11. *For $x \in \ell_M(\Delta_{(m)})$,*

$$\|x\|_{(M)}^{(m)} \leq \|x\|_M^{(m)} \leq 2\,\|x\|_{(M)}^{(m)}.$$

Proof. We have

$$\|x\|_{(M)}^{(m)} = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|\Delta_{(m)}x_k|}{\rho}\right) \leq 1\right\}.$$

Then using Proposition 2.10, we get $\|x\|_{(M)}^{(m)} \leq \|x\|_M^{(m)}$. Let us observe that if $x \in \ell_M(\Delta_{(m)})$ with $\|x\|_{(M)}^{(m)} \leq 1$. Then $x \in \tilde{\ell}_M(\Delta_{(m)})$ and $\delta\left(M, \Delta_{(m)}, x\right) \leq 1$. Indeed,

$$\frac{1}{\|x\|_{(M)}^{(m)}} \sum_{i=1}^{\infty} M\left(\left|\Delta_{(m)}x_i\right|\right) \leq \sum_{i=1}^{\infty} M\left(\frac{|\Delta_{(m)}x_i|}{\|x\|_{(M)}^{(m)}}\right) \leq 1,$$

by Proposition 2.7. Thus

$$\frac{1}{\|x\|_{(M)}^{(m)}} \in \tilde{\ell}_M(\Delta_{(m)}) \quad \text{with} \quad \delta\left(M, \frac{x}{\|x\|_{(M)}^{(m)}}\right) \leq 1.$$

We further observe that for an arbitrary $z \in \tilde{\ell}_M(\Delta_{(m)})$,

$$\|z\|_M^{(m)} = \sup\left\{\left|\sum_{i=1}^{\infty}\left(\Delta_{(m)} z_i\right) y_i\right| : \delta(\Phi, y) \leq 1\right\} < 1+\delta\left(M, \Delta_{(m)}, z\right), \quad \text{using (1.1).}$$

Hence taking $z = \dfrac{x}{\|x\|_{(M)}^{(m)}}$, we have

$$\left\|\frac{x}{\|x\|_{(M)}^{(m)}}\right\|_M^{(m)} \leq 1 + \sum_{i=1}^{\infty} M\left(\frac{|x|}{\|x\|_{(M)}^{\Delta^{(m)}}}\right) \leq 2, \quad \text{by Proposition 2.7.}$$

Thus $\|x\|_M^{(m)} \leq 2\,\|x\|_{(M)}^{(m)}$. This completes the proof.

Hence we have the following Corollary.

Corollary 2.12. *$\ell_M(\Delta_{(m)})$ is a BK space under both the norms $\|x\|_M^{(m)}$ and $\|x\|_{(M)}^{(m)}$.*

Proposition 2.13. *For any Orlicz function M, $\ell_M(\Delta_{(m)}) = \ell'_M(\Delta_{(m)})$, where*

$$\ell'_M(\Delta_{(m)}) = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{\left|\Delta_{(m)} x_k\right|}{\rho}\right) < \infty, \quad \textit{for some } \rho > 0\right\}.$$

Proof. Proof follows from Proposition 2.10.

In view of above Proposition we give the following definition:

Definition 2.14. For any Orlicz function M,

$$h_M(\Delta_{(m)}) = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{\left|\Delta_{(m)} x_k\right|}{\rho}\right) < \infty, \quad \text{for each } \rho > 0\right\}.$$

Clearly $h_M(\Delta_{(m)})$ is a subspace of $\ell_M(\Delta_{(m)})$.

Henceforth we shall write $\|\bullet\|$ instead of $\|\bullet\|_{(M)}^{(m)}$ provided it does not lead to any confusion. The topology of $h_M(\Delta_{(m)})$ is the one it inherits from $\|\bullet\|$.

Proposition 2.15. *Let M be an Orlicz function. Then $\left(h_M(\Delta_{(m)}), \|\bullet\|\right)$ is an AK-BK space.*

Proof. First we show that $h_M(\Delta_{(m)})$ is an AK-space. Let $x \in h_M(\Delta_{(m)})$. Then for each ε, $0 < \varepsilon < 1$, we can find an n_0 such that

$$\sum_{n \geq n_0} M\left(\frac{|\Delta_{(m)} x_i|}{\varepsilon}\right) \leq 1$$

Hence for $n \geq n_0$,

$$\begin{aligned}\|x - x^{(n)}\| &= \inf\left\{\rho > 0 : \sum_{i \geq n+1} M\left(\frac{|\Delta_{(m)} x_i|}{\rho}\right) \leq 1\right\} \\ &\leq \inf\left\{\rho > 0 : \sum_{i \geq n} M\left(\frac{|\Delta_{(m)} x_i|}{\rho}\right) \leq 1\right\} \varepsilon\end{aligned}$$

Thus we can conclude that $h_M(\Delta_{(m)})$ is an AK space.

Next to show $h_M(\Delta_{(m)})$ is an BK-space it is enough to show $h_M(\Delta_{(m)})$ is a closed subspace of $\ell_M(\Delta_{(m)})$. For this let $\{x^n\}$ be a sequence in $h_M(\Delta_{(m)})$ such that $\|x^n - x\| \longrightarrow 0$, where $x \in \ell_M(\Delta_{(m)})$. To complete the proof we need to show that $x \in h_M(\Delta_{(m)})$, i.e.,

$$\sum_{i \geq 1} M\left(\frac{|\Delta_{(m)} x_i|}{\rho}\right) < \infty, \quad \text{for every } \rho > 0.$$

To $\rho > 0$ there corresponds an l such that $\|x^l - x\| \leq \frac{\rho}{2}$. Then using the convexity of M,

$$\begin{aligned}\sum_{i \geq 1} M\left(\frac{|\Delta_{(m)} x_i|}{\rho}\right) &= \sum_{i \geq 1} M\left(\frac{2\left|\Delta_{(m)} x_i^l\right| - 2\left(\left|\Delta_{(m)} x_i^l\right| - \left|\Delta_{(m)} x_i\right|\right)}{2\rho}\right) \\ &\leq \frac{1}{2}\sum_{i \geq 1} M\left(\frac{2|\Delta_{(m)} x_i^l|}{\rho}\right) + \frac{1}{2}\sum_{i \geq 1} M\left(\frac{2|\Delta_{(m)}(x_i^l - x_i)|}{\rho}\right) \\ &\leq \frac{1}{2}\sum_{i \geq 1} M\left(\frac{2|\Delta_{(m)} x_i^l|}{\rho}\right) + \frac{1}{2}\sum_{i \geq 1} M\left(\frac{2|\Delta_{(m)}(x_i^l - x_i)|}{\|x^l - x\|}\right)\end{aligned}$$

Thus $x \in h_M(\Delta_{(m)})$ and consequently $h_M(\Delta_{(m)})$ is a BK space.

Proposition 2.16. *Let M be an Orlicz function. If M satisfies the Δ_2-condition at 0, then $\ell_M(\Delta_{(m)})$ is an AK space.*

Proof. In fact we shall show that if M satisfies the Δ_2-condition at 0, then $\ell_M(\Delta_{(m)}) = h_M(\Delta_{(m)})$ and the results follows. Therefore it is enough to show that $\ell_M(\Delta_{(m)}) \subset h_M(\Delta_{(m)})$. Let $x \in \ell_M(\Delta_{(m)})$, then $\rho > 0$,

$$\sum_{i\geq 1} M\left(\frac{|\Delta_{(m)}x_i|}{\rho}\right) < \infty$$

This implies that

$$M\left(\frac{|\Delta_{(m)}x_i|}{\rho}\right) \longrightarrow 0, \quad \text{as} \quad i \longrightarrow \infty \tag{2.4}$$

Choose an arbitrary $l > 0$. If $\rho \leq l$, then

$$\sum_{i\geq 1} M\left(\frac{|\Delta_{(m)}x_i|}{l}\right) < \infty.$$

Let now $l < \rho$ and put $k = \frac{\rho}{l}$. Since M satisfies the Δ_2-condition at 0, there exist $R = R_k > 0$ and $r = r_k > 0$ with $M(kx) \leq RM(x)$ for all $x \in (0, r]$. By (2.4) there exists a positive integer n_1 such that

$$M\left(\frac{|\Delta_{(m)}x_i|}{\rho}\right) < \frac{1}{2}rp\left(\frac{r}{2}\right) \quad \text{for all} \quad i \geq n_1$$

We claim that $\frac{|\Delta_{(m)}x_i|}{\rho} \leq r$ for all $i \geq n_1$. Otherwise, we can find $j > n_1$ with $\frac{|\Delta_{(m)}x_j|}{\rho} > r$, and thus

$$M\left(\frac{|\Delta_{(m)}x_j|}{\rho}\right) \geq \int_{\frac{r}{2}}^{|\Delta_{(m)}x_j|} p(t)dt > \frac{1}{2}rp\left(\frac{r}{2}\right), \quad \text{a contadiction.}$$

Hence our claim is true. Then we can find that

$$\sum_{i\geq n_1} M\left(\frac{|\Delta_{(m)}x_i|}{l}\right) \leq \sum_{i\geq n_1} M\left(\frac{|\Delta_{(m)}x_i|}{\rho}\right)$$

and hence

$$\sum_{i\geq 1} M\left(\frac{|\Delta_{(m)}x_i|}{l}\right) < \infty, \quad \text{for every} \quad l > 0$$

This completes the proof.

Proposition 2.17. *Let M_1 and M_2 be two Orlicz functions. If M_1 and M_2 are equivalent then $\ell_{M_1}(\Delta_{(m)}) = \ell_{M_2}(\Delta_{(m)})$ and the identity map*

$$I : \left(\ell_{M_1}(\Delta_{(m)}), \|\bullet\|_{M_1}^{(m)}\right) \longrightarrow \left(\ell_{M_2}(\Delta_{(m)}), \|\bullet\|_{M_2}^{(m)}\right)$$

is a topological isomorphism.

Proof. Let M_1 and M_2 are equivalent and so satisfy (1.3). Suppose $x \in \ell_{M_2}(\Delta_{(m)})$, then

$$\sum_{i=1}^{\infty} M_2\left(\frac{|\Delta_{(m)}x_i|}{\rho}\right) < \infty \quad \text{for some } \rho > 0.$$

Hence for some $l \geq 1$,

$$\frac{|\Delta_{(m)}x_i|}{l\rho} \leq x_0, \quad \text{for all} \;\; i \geq 1.$$

Therefore

$$\sum_{i=1}^{\infty} M_1\left(\frac{\alpha|\Delta_{(m)}x_i|}{l\rho}\right) \leq \sum_{i=1}^{\infty} M_2\left(\frac{|\Delta_{(m)}x_i|}{\rho}\right) < \infty.$$

Thus $\ell_{M_2}(\Delta_{(m)}) \subset \ell_{M_1}(\Delta_{(m)})$. Similarly $\ell_{M_1}(\Delta_{(m)}) \subset \ell_{M_2}(\Delta_{(m)})$. Let us abbreviate here $\|\bullet\|_{M_1}^{(m)}$ and $\|\bullet\|_{M_2}^{(m)}$ respectively. For $x \in \ell_{M_2}(\Delta_{(m)})$,

$$\sum_{i=1}^{\infty} M_2\left(\frac{|\Delta_{(m)}x_i|}{\|x\|_2}\right) \leq 1.$$

One can find $\mu > 1$ with

$$\left(\frac{x_0}{2}\right)\mu p_2\left(\frac{x_0}{2}\right) \geq 1,$$

where p_2 is the kernel associated with M_2. Hence

$$M_2\left(\frac{|\Delta_{(m)}x_i|}{\|x\|_2}\right) \leq \left(\frac{x_0}{2}\right)\mu p_2\left(\frac{x_0}{2}\right) \quad \text{for all } i \geq 1.$$

This implies that

$$\frac{|\Delta_{(m)}x_i|}{\mu\|x\|_2} \leq x_0 \;\; \text{for all} \;\; i \geq 1.$$

Therefore
$$\sum_{i=1}^{\infty} M_1\left(\frac{\alpha|\Delta_{(m)}x_i|}{\mu\|x\|_2}\right) < 1$$
and so
$$\|x\|_1 \leq \left(\frac{\mu}{\alpha}\right)\|x\|_2.$$
Similarly we can show $\|x\|_2 \leq \beta\gamma\|x\|_1$ by choosing γ with $\gamma\beta > 1$ such that
$$\gamma\beta\left(\frac{x_0}{2}\right)p_1\left(\frac{x_0}{2}\right) \geq 1.$$
Thus $\alpha\mu^{-1}\|x\|_1 \leq \|x\|_2 \leq \beta\gamma\|x\|_1$ which establishes the topological isomorphism of I.

Proposition 2.18. *Let M be an Orlicz function and p the corresponding kernel. If $p(x) = 0$ for all x in $[0, x_0]$ where x_0 is some positive number, then $\ell_M(\Delta_{(m)})$ is topologically isomorphic to $\ell_\infty(\Delta_{(m)})$ and $h_M(\Delta_{(m)})$ is topologically isomorphic to $c_0(\Delta_{(m)})$, where*
$$\ell_\infty(\Delta_{(m)}) = \left\{x = (x_k) : \sup_k \left|\Delta_{(m)}x_k\right| < \infty\right\}$$
and
$$c_0(\Delta_{(m)}) = \left\{x = (x_k) : \lim_{k\longrightarrow\infty}(\Delta_{(m)}x_k) = 0\right\}.$$

Proof. Let $p(x) = 0$ for all x in $[0, x_0]$. If $y \in \ell_\infty(\Delta_{(m)})$, then we can find a $\rho > 0$ such that $\frac{|\Delta_{(m)}y_i|}{\rho} \leq x_0$ for $i \geq 1$ and so $\sum_{i=1}^{\infty} M\left(\frac{|\Delta_{(m)}y_i|}{\rho}\right) < \infty$, giving thus $y \in \ell_M(\Delta_{(m)})$. On the other hand let $y \in \ell_M(\Delta_{(m)})$, then $\sum_{i=1}^{\infty} M\left(\frac{|\Delta_{(m)}y_i|}{\rho}\right) < \infty$, for some $\rho > 0$ and so $|\Delta_{(m)}y_i| < \infty$ for all $i \geq 1$, giving thus $y \in \ell_\infty(\Delta_{(m)})$. Hence $y \in \ell_\infty(\Delta_{(m)})$ if and only if $y \in \ell_M(\Delta_{(m)})$. We can easily find an x_1 with $M(x_1) \geq 1$. Let $y \in \ell_\infty(\Delta_{(m)})$ and $\alpha = \|y\|_\infty = \sup_i \left(|\Delta^{(m)}\lambda_i y_i|\right) > 0$. (it is easy to show that $\|y\|_\infty = \sup_i \left(|\Delta_{(m)}y_i|\right)$ is a norm on $\ell_\infty(\Delta_{(m)})$). For every ε, $0 < \varepsilon < \alpha$, we can determine y_j with $|\Delta_{(m)}y_j| > \alpha - \varepsilon$ and so
$$\sum_{i=1}^{\infty} M\left(\frac{|\Delta_{(m)}y_i|x_i}{\alpha}\right) \geq M\left(\frac{(\alpha-\varepsilon)x_1}{\alpha}\right).$$

As M is continuous, we find

$$\sum_{i=1}^{\infty} M\left(\frac{|\Delta_{(m)}y_i|x_1}{\alpha}\right) \geq 1,$$

and so $\|y\|_\infty \leq x_1\|y\|$, for otherwise

$$\sum_{i=1}^{\infty} M\left(\frac{|\Delta_{(m)}y_i|}{\|y\|}\right) > 1, \quad \text{a contradiction by Proposition 2.7.}$$

Again

$$\sum_{i=1}^{\infty} M\left(\frac{|\Delta_{(m)}y_i|x_0}{\alpha}\right) = 0$$

and it follows that

$$\|y\| \leq \left(\frac{1}{x_0}\right)\|y\|_\infty.$$

Thus the identity map

$$I : \left(\ell_M(\Delta_{(m)}), \|\bullet\|\right) \longrightarrow \left(\ell_\infty(\Delta_{(m)}), \|\bullet\|\right)$$

is a topological isomorphism.

For the last part, let $y \in h_M(\Delta_{(m)})$, then for any $\varepsilon > 0$, $|\Delta_{(m)}y_i| \leq \varepsilon x_1$, for all sufficiently large, where x_1 is some positive number with $p(x_1) > 0$. Hence $y \in c_0(\Delta_{(m)})$. Next let $y \in c_0(\Delta_{(m)})$. Then for any $\rho > 0$, $\frac{|\Delta_{(m)}y_i|}{\rho} < \frac{1}{2}x_0$ for all sufficiently large i. Thus $M\left(\frac{|\Delta_{(m)}y_i|}{\rho}\right) < \infty$ for all $\rho > 0$ and $y \in h_M(\Delta_{(m)})$. Hence $h_M(\Delta_{(m)}) = c_0(\Delta_{(m)})$ and we are done.

Proposition 2.19. *Let M be an Orlicz function and p the corresponding kernel. If $p(x) = 0$ for all x in $[0, x_0]$ where x_0 is some positive number, then $\ell_M(\Delta_{(m)})$ is topologically isomorphic to ℓ_∞ and $h_M(\Delta_{(m)})$ is topologically isomorphic to c_0.*

Proof. For $Z = \ell_\infty$ and c_0, $Z(\Delta_{(m)})$ and Z are equivalent as topological spaces, since $T : Z(\Delta_{(m)}) \longrightarrow Z$, defined by $Tx = y = (\Delta_{(m)}x_k)$, is a linear homeomorphism. This completes the proof.

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