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Optimal Redundancy Allocation in Series-Parallel System using Generalized Fuzzy Number *

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Abstract

In this paper generalized linear fuzzy numbers are used in redundancy allocation for optimum reliability of series-parallel system. Here reliability and cost of components of the system, system cost, and system weight are fuzzy numbers. We use geometric programming to solve redundancy allocation problem. The redundancy allocation problem whose aim is to find out the optimal allocation of redundancy components in such a way that maximizes the system reliability subjected to available total system cost and weight. Here it demonstrates to find a set of optimal solutions that help the decision maker to take the right decisions from the optimal solution set. Examples are displayed to illustrate the model utilizing generalized fuzzy numbers.

Keywords and Phrases: *Generalized fuzzy number, Fuzzy system reliability, Reliability optimization, Redundancy, Geometric programming.*

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1. Introduction

Reliability optimization provides a means to help the reliability engineer to achieve such an aim to find the best way to increase the systems reliability. Most methods of reliability optimization assume that systems have redundancy components in series-parallel or parallel system and that alternative designs achieve the goal of optimal system reliability by optimal allocation of redundancy components. Reliability of a multi-stage system can be improved by adding similar components as redundancy to each subsystem, may be some different components that can be considered as design alternatives in a subsystem. Thus the problem is to improve system's reliability associated with a system design under the limited available resources. Kim and Yum [8] explained how to increase the component reliability. Tian and Zuo [18] proposed multi-objective optimization model for redundancy allocation for multi-state series-parallel systems using physical programming approach and solved it by genetic algorithm. Yun and Kim [21] presented multi-level redundancy optimization in series system as a mixed integer-programming model and solved it by genetic algorithm and heuristic algorithm. Hsiesh [6] investigated the series parallel redundant reliability problems with multiple component choices by linear approximation. Zhao and Liu [24] illustrated parallel redundant and standby redundant system by the stochastic programming. Misra and Sharma [14] presented redundant components in various subsystems in the system by geometric programming formulation. Charles Elegbede et al. [2] considered the allocation of reliability and redundancy to each subsystem of parallel-series system for target reliability maintaining the minimum system cost. Tillman et al. [19] presented a comprehensive survey of previous works for system reliability with redundancy. Sinha and Misra [17], Prasad et al. [16], Kuo and Prasad [9], Kuo et al. [10], etc illustrated allocation of redundant component in a system to enhance the system reliability, which is important in reliability engineering.

In general, reliability optimization problem is solved with the assumption that the reliability, cost and weight of components are specified in an exact mode. In real life, due to hesitation in judgments, lack of confirmation or otherwise, sometimes it is not possible to get significant exact data for the reliability system. This type of imprecise data is always well represented by fuzzy number, so fuzzy reliability optimization model is needed in real life problem. Also for making a decision, decision-makers have to review the alternatives with fuzzy numbers. It can be seen that fuzzy numbers have a very important role to describe fuzzy parameters in several fuzzy reliability optimization model from the different viewpoints of decision makers. In reliability apportionment problem for a two-component series system subjecting to a single constraint, Park [15] used fuzzy set theory. Mahapatra and Roy [12] introduced fuzzy multi-objective mathematical programming technique based on generalized fuzzy set and they applied it in multi-objective reliability optimization models.

The non-linear optimization problems have been solved by different non-linear optimization techniques. Geometric Programming (GP) is an effective method among those to solve a meticulous type of non-linear programming problem. Zener [23] introduced GP technique, and Duffin et al [3] further developed the GP method. There are various mathematical programming and heuristic methods been developed to solve the single and multi-objective reliability optimization problem. GP method is rare used to solve the reliability optimization problem. Federowicz and Mazumdar [4] first used GP on reliability optimization problem. Govil [5] used GP for a 3-stage series reliability system. Now-a-days GP in fuzzy environments, a competent optimization method, is used to solve a typical fuzzy optimization problem which is called as Fuzzy Geometric Programming (FGP). In 1987, Cao [1] first introduced FGP. Mahapatra and Roy [13] used FGP with cost constraint to find optimal reliability for a series system. Fuzzy reliability optimization models with redundancy through FGP are very rare in literature.

For many practical problems, most of the parameters of an optimization model are not known exactly. Due to this imperfect and unreliability of input information, fuzzy numbers become an important aspect in the reliability design of the engineering systems.

Here we consider the problem as to find the optimum number of redundancies of similar components, which maximize the system reliability subjecting to the available system cost and weight. This paper regards the problem of geometric programming in the context of reliability and redundancy apportionment of multistage, multi-component system subject to cost and weight constraints. Here reliability, cost and weight of the components, system cost and weight are in fuzzy environment, so they are taken as generalized fuzzy number.

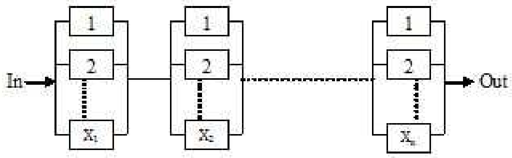

Figure 1: A schematic diagram of the n-stage series-parallel system

2. Notations

Reliability optimization model is developed and worked out under the following notations.

R_i reliability of each component of the system in the ith stage,
Q_i unreliability of each component of the system in the ith stage,
C_i cost of each component of the system in the ith stage,
W_i weight of each component of the system in the ith stage,
C available system cost of the reliability model,
W available system weight of the reliability model,
x_i number of redundancy components in the ith stage,
$R_s\left(x_1, x_2, ..., x_n\right)$ function of system reliability,
$C_s\left(x_1, x_2, ..., x_n\right)$ function of system cost,
$W_s\left(x_1, x_2, ..., x_n\right)$ function of system weight,
$\tilde{A}_{TFN}$ Triangular Fuzzy Number (TFN) $\tilde{A}$,
$\tilde{A}_{GTFN}$ Generalized Triangular Fuzzy Number (GTFN) $\tilde{A}$,
$\tilde{A}_{TrFN}$ Trapezoidal Fuzzy Number (TrFN) $\tilde{A}$,
$\tilde{A}_{GTrFN}$ Generalized Trapezoidal Fuzzy Number (GTrFN) $\tilde{A}$.

3. Mathematical Formulation of the Model

3.1 Crisp model

Consider an n stage series system and at each stage added $(x_i - 1)$ redundant components in parallel, as shown in Figure 1, our aim is to determine the number of redundant components at each stage so as the system reliability will be maximum subjecting to related cost and weight constraints.

Therefore we have to find the maximization of $R_s(x_1, x_2, ..., x_n)$ having subject to the limited available cost C and weight W.

So the problem becomes

$$Max \quad R_s(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} \{1 - (1 - R_i)^{x_i}\} \tag{3.1}$$

subject to

$$\begin{aligned} C_s(x_1, x_2, ..., x_n) &= \sum_{i=1}^{n} C_i x_i \leq C \\ W_s(x_1, x_2, ..., x_n) &= \sum_{i=1}^{n} W_i x_i \leq W \end{aligned}$$

$x_i > 1 \; for \; i = 1, 2, ..., n.$

3.2 Fuzzy model

Undoubtedly, in practical sense expressing the reliability, cost and weight of system components in the reliability optimization problem (3.1) are not transparent. While determining the system reliability; reliability, cost, weight of the components and objective goal as well as goal of the constraints can be involved in many non-stochastic uncertain factors. To make the model more flexible and adoptable to human decision process, the reliability optimization model (3.1) can be represented as fuzzy non-linear programming problems with fuzzy numbers.

Therefore in fuzzy environment the system reliability optimization problem becomes

$$Max \quad R_s(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} \left\{1 - \left(1 - \tilde{R}_i\right)^{x_i}\right\} \tag{3.2}$$

subject to

$$\begin{aligned} C_s(x_1, x_2, ..., x_n) &= \sum_{i=1}^{n} \tilde{C}_i x_i \leq \tilde{C} \\ W_s(x_1, x_2, ..., x_n) &= \sum_{i=1}^{n} \tilde{W}_i x_i \leq \tilde{W} \end{aligned}$$

$x_i > 1 \ for \ i = 1, 2, ..., n.$

Here $\tilde{R}_i, \tilde{C}_i, \tilde{W}_i \ \ (i = 1, 2, \ldots, n), \ \tilde{C}$ and $\tilde{W}$ are taken as generalized fuzzy numbers.

4. Fuzzy Mathematics Prerequisites

Zadeh [22] introduced fuzzy set in 1965 as a mathematical way of representing impreciseness or vagueness in everyday life.

Definition 1. Fuzzy Set: A fuzzy set $\tilde{A}$ in a universe of discourse X is defined as the following set of pairs $\tilde{A} = (x, \mu_{\tilde{A}}(x) : x \in X)$. Here $\mu_{\tilde{A}} : X \longrightarrow [0, 1]$ is a mapping called the membership function of the fuzzy set $\tilde{A}$ and $\mu_{\tilde{A}}(x)$ is called the membership value or degree of membership of $x \in X$ in the fuzzy set $\tilde{A}$.

Definition 2. Height: The height $h(\tilde{A})$, of a fuzzy set $\tilde{A} = (x, \mu_{\tilde{A}}(x) : x \in X)$, is the largest membership grade obtained by any element in that set i.e. $h(\tilde{A}) = \sup\limits_{x \in X} \mu_{\tilde{A}}(x)$.

Definition 3. α-Level Set or α-cut of a Fuzzy Set: The α-level set (or interval of confidence at level α or α-cut) of the fuzzy set $\tilde{A}$ of X is a crisp set A_α that contains all the elements of X that have membership values in $\tilde{A}$ greater than or equal to α i.e. $\tilde{A} = \{x, \mu_{\tilde{A}}(x) \geq \alpha, x \in X, \alpha \in [0, \ 1]\}$.

Definition 4. Generalized Fuzzy Number (GFN): Generalized Fuzzy Number $\tilde{A}$ as $\tilde{A} = (a_1, a_2, a_3, a_4; w)$, where $0 < w \leq 1$, and $a_1, a_2, a_3 \ and \ a_4$ $(a_1 < a_2 < a_3 < a_4)$ are real numbers. The generalized fuzzy number $\tilde{A}$ is a fuzzy subset of real line R, whose membership function $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

1) $\mu_{\tilde{A}} : R \to [0, 1]$
2) $\mu_{\tilde{A}}(x) = 0$ for $-\infty < x \leq a_1$
3) $\mu_{\tilde{A}}(x)$ is strictly increasing function for $a_1 \leq x \leq a_2$
4) $\mu_{\tilde{A}}(x) = w$ for $a_2 \leq x \leq a_3$
5) $\mu_{\tilde{A}}(x)$ is strictly decreasing function for $a_3 \leq x \leq a_4$
6) $\mu_{\tilde{A}}(x) = 0$ for $a_4 \leq x < \infty$

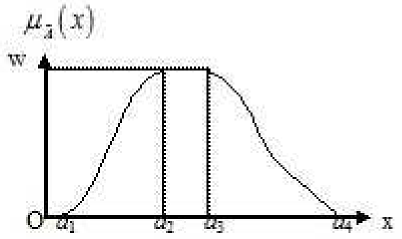

Figure 2: Generalized Fuzzy Number

Note: 4.1. $\tilde{A}$ is a convex fuzzy set and it is a non-normalized fuzzy number till $w \neq 1$. It is normalized fuzzy number for $w = 1$.

i) If $a_1 = a_2 = a_3 = a_4 = a$ (say) and $w = 1$, then $\tilde{A}$ is called a real number a

Here $\tilde{A} = (x, \mu_{\tilde{A}}(x))$ with membership function $\mu_{\tilde{A}}(x) = \begin{cases} 1 \ if \ x = a \\ 0 \ if \ x \neq a \end{cases}$

ii) If $a_1 = a_2, a_3 = a_4$ and $w = 1$ then $\tilde{A}$ is called crisp interval $[a_1, a_4]$

Here $\tilde{A} = (x, \mu_{\tilde{A}}(x))$ with membership function $\mu_{\tilde{A}}(x) = \begin{cases} 1 \ if \ a_1 \leq x \leq a_4 \\ 0 \quad\quad otherwise \end{cases}$

iii) and $a_2 = a_3$ then $\tilde{A}$ is called a GTFN as $\tilde{A} = (a_1, a_2, a_4; w) \ or \ (a_1, a_3, a_4; w)$

iv) and $a_2 = a_3, w = 1$ then $\tilde{A}$ is called a TFN as $\tilde{A} = (a_1, a_2, a_4) \ or \ (a_1, a_3, a_4)$

Here $\tilde{A} = (x, \mu_{\tilde{A}}(x))$ with membership function $\mu_{\tilde{A}}(x) = \begin{cases} w\frac{x-a_1}{a_2-a_1} \ if \ a_1 \leq x \leq a_2 \\ w\frac{a_4-x}{a_4-a_2} \ if \ a_2 \leq x \leq a_4 \\ \quad 0 \quad\quad otherwise \end{cases}$

v) and $a_2 \neq a_3$ then $\tilde{A}$ is called a GTrFN as $\tilde{A} = (a_1, a_2, a_3, a_4; w)$

vi) and $a_2 \neq a_3, w = 1$ then $\tilde{A}$ is called a TrFN as $\tilde{A} = (a_1, a_2, a_3, a_4)$

Here $\tilde{A} = (x, \mu_{\tilde{A}}(x))$ with membership function $\mu_{\tilde{A}}(x) = \begin{cases} w\frac{x-a_1}{a_2-a_1} \ if \ a_1 \leq x \leq a_2 \\ w \quad\quad\ if \ a_2 \leq x \leq a_3 \\ w\frac{a_4-x}{a_4-a_3} \ if \ a_3 \leq x \leq a_4 \\ \quad 0 \quad\quad otherwise \end{cases}$

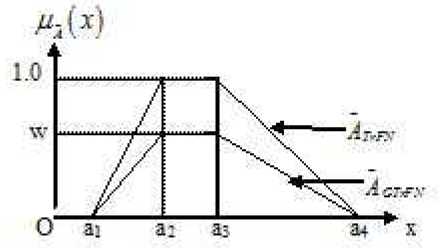

Figure 3: TrFN and GTrFN

Figure 3 shows TrFN $\tilde{A} = (a_1, a_2, a_3, a_4)$ and GTrFN $\tilde{A} = (a_1, a_2, a_3, a_4; w)$ which indicate different decision maker's opinions for different values of w $(0 < w \leq 1)$. The value of w represents the degree of confidence of the opinion of the decision maker.

4.1 Different methods for defuzzification of fuzzy numbers:

In real life, bulk of the information is assimilated as fuzzy numbers but there will be a need to defuzzify the fuzzy number. Actually defuzzification is the conversion of the fuzzy number to precise or crisp number. Several processes are used for such conversion. Here we have discussed three types of defuzzification, first two methods are followed by Yager [20] and later by Kaufman and Gupta [7].

4.1.1 Type-I: Center of Mass (COM) Method

Let $\tilde{A}$ be a fuzzy number then the defuzzification of $\tilde{A}$ is given by $\hat{A} = \frac{\int_{a_l}^{a_u} x\mu_{\tilde{A}}(x)dx}{\int_{a_l}^{a_u} \mu_{\tilde{A}}(x)dx}$ where a_l *and* a_u are the lower and upper limits of the support of $\tilde{A}$. The value $\hat{A}$ represents the centroid of the fuzzy number $\tilde{A}$.

4.1.1.a. Defuzzification of $\tilde{A}_{GTFN} = (a_1, a_2, a_3; w)$ by COM method $\hat{A} = \frac{1}{3}(a_1 + a_2 + a_3)$

4.1.1.b. Defuzzification of $\tilde{A}_{GTrFN} = (a_1, a_2, a_3, a_4; w)$ by COM method $\hat{A} = \frac{1}{3}\frac{a_4^2+a_3^2-a_2^2-a_1^2-a_4a_3-a_2a_1}{a_4+a_3-a_2-a_1}$

Note: 4.2. For COM method, defuzzification of GTFN and GTrFN does not depend on w. In this case, defuzzification of generalized fuzzy number and normalized fuzzy number (w=1) will be same.

4.1.2 Type-II: Mean of α-Cut (MC) Method

Let $\tilde{A}$ be a fuzzy number then the defuzzification of $\tilde{A}$ is given by $\hat{A} = \int_0^{\alpha_{\max}} m[a_\alpha^L, a_\alpha^R] d\alpha$ where α_{max} is the height of $\tilde{A}$, $A_\alpha = [a_\alpha^L, a_\alpha^R]$ is an $\alpha-$cut, $\alpha \in (0,1]$ and $m[a_\alpha^L, a_\alpha^R]$ is the mean value of the elements of that $\alpha-$cut, i.e. $m[a_\alpha^L, a_\alpha^R] = \frac{a_\alpha^L + a_\alpha^R}{2}$ where a_α^L and a_α^R are the left and right limits of the α-cut of the fuzzy number $\tilde{A}$.

4.1.2.a. Defuzzification of $\tilde{A}_{GTFN} = (a_1, a_2, a_3; w)$ by MC method $\hat{A} = \frac{w}{4}(a_1 + 2a_2 + a_3)$. Here $a_\alpha^L = a_1 + \frac{\alpha}{w}(a_2 - a_1)$ and $a_\alpha^R = a_3 - \frac{\alpha}{w}(a_3 - a_2)$

4.1.2.b. Defuzzification of $\tilde{A}_{GTrFN} = (a_1, a_2, a_3, a_4; w)$ by MC method $\hat{A} = \frac{w}{4}(a_1 + a_2 + a_3 + a_4)$. Here $a_\alpha^L = a_1 + \frac{\alpha}{w}(a_2 - a_1)$ and $a_\alpha^R = a_4 - \frac{\alpha}{w}(a_4 - a_3)$

Note: 4.3. For MC method, defuzzification of TFN and TrFN (normalized fuzzy number ($w = 1$)) obtained by putting $w = 1$ in the defuzzification rule of GTFN (4.1.2.a) and GTrFN (4.1.2.b) respectively.

4.1.3 Type-III: Removal Area (RA) Method

According to Kaufman and Gupta [7], an ordinary number $k \in R$, the left side removal of $\tilde{A}$ with respect to k, $R_l\left(\tilde{A}, k\right)$, is define as the area bounded by $x = k$ and the left side of the fuzzy number $\tilde{A}$. Similarly, the right side removal is $R_r\left(\tilde{A}, k\right)$. The removal of the fuzzy number with respect to $x = k$ is define as the mean of $R_l\left(\tilde{A}, k\right)$ and $R_r\left(\tilde{A}, k\right)$.

Thus $R\left(\tilde{A}, k\right) = \frac{1}{2}\left(R_l\left(\tilde{A}, k\right) + R_r\left(\tilde{A}, k\right)\right)$.

For example here we take $k = 0$ for the trapezoidal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$, left and right removal area are shown in figure 4(a) and 4(b).

4.1.3.a. Defuzzification of $\tilde{A}_{GTFN} = (a_1, a_2, a_3; w)$ by RA method

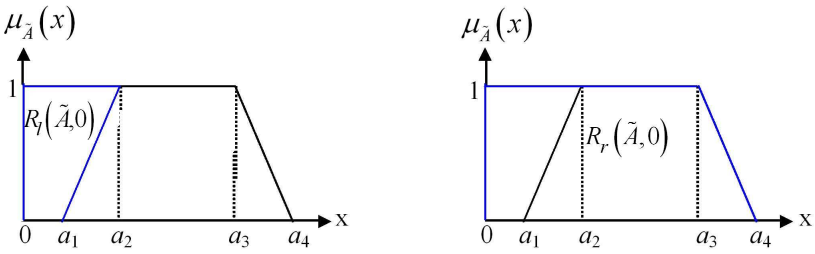

Figure 4: (a): Left removal area of $R_l\left(\tilde{A},0\right)$ **of** $\tilde{A}$ (b): Right removal area of $R_r\left(\tilde{A},0\right)$ **of** $\tilde{A}$

The removal number of $\tilde{A}$ with respect to origin is defined as the mean of two areas, $R_l\left(\tilde{A},0\right) = w\frac{a_1+a_2}{2}$ and $R_r\left(\tilde{A},0\right) = w\frac{a_2+a_3}{2}$ So $\hat{A} = R\left(\tilde{A},0\right) = \frac{w}{4}(a_1 + 2a_2 + a_3)$.

4.1.3.b. Defuzzification of $\tilde{A}_{GTrFN} = (a_1, a_2, a_3, a_4; w)$ by RA method

The removal number of $\tilde{A}$ with respect to origin is defined as the mean of two areas, $R_l\left(\tilde{A},0\right) = w\frac{a_1+a_2}{2}$ and $R_r\left(\tilde{A},0\right) = w\frac{a_4+a_3}{2}$ So $\hat{A} = R\left(\tilde{A},0\right) = \frac{w}{4}(a_1 + a_2 + a_3 + a_4)$.

Note:4.4. For RA method, defuzzification of TFN and TrFN are obtained by putting w=1 in the defuzzification rule of GTFN (4.1.3.a), GTrFN (4.1.3.b) respectively.

Note: 4.5. Defuzzification of GTFN and GTrFN by type-II and type-III method are same but these are different with type-I

5. Geometric Programming

Geometric programming (GP) had its beginning in 1961 by Zener [23]. Later Duffin, Peterson and Zener [3] developed the theory with its application.

Primal Geometric Programming (PGP):

$$Min \quad g_0(t) = \sum_{k=1}^{T_0} c_{0k} \prod_{j=1}^{m} t_j^{\alpha_{0kj}} \tag{5.1}$$

$$subject\ to\ g_r(t) = \sum_{k=1+T_{r-1}}^{T_r} c_{rk} \prod_{j=1}^{m} t_j^{\alpha_{rkj}} \leq 1$$

$t_j > 0, j = 1, 2, \ldots, m$

where $c_{rk}(> 0)$ and $\alpha_{rkj}(k = 1, 2, ..., l + T_{r-1}, .., T_r; r = 0, 1, 2, ..., l; j = 1, 2, \ldots, m)$ are real numbers.

It is a constrained posynomial PGP problem. The number of terms in each posynomial constraint function varies and it is denoted by T_r for each $r = 0, 1, 2, \ldots, l$. Let $T = T_0 + T_1 + T_2 + \ldots + T_l$ be the total number of terms in the primal program. The Degree of Difficulty (DD) $= T - (m + 1)$.

Dual Program (DP):

The dual programming of (5.1) is as follows:

$$Max\ v(\delta) = \prod_{r=0}^{l} \prod_{k=1}^{T_r} \left(\frac{c_{rk}}{\delta_{rk}}\right)^{\delta_{rk}} \left(\sum_{s=1+T_{r-1}}^{T_r} \delta_{rs}\right)^{\delta_{rk}} \quad (5.2)$$

$$subject\ to \sum_{k=1}^{T_0} \delta_{0k} = 1, \quad \text{(Normality condition)}$$

$$\sum_{r=0}^{l} \sum_{k=1}^{T_r} \alpha_{rkj} \delta_{rk} = 0,\ j = 1, 2, ..., m, \text{(Orthogonality conditions)}$$

$\delta_{rk} > 0$, $(r = 0, 1, 2, \ldots, l; k = 1, 2, ..., T_r)$. (Non-negativity conditions)

Ones optimal dual variable vector δ^* is known, the corresponding values of the primal variable vector t is found from the following relations:

$$c_k \prod_{j=1}^{n} t_j^{\alpha_{kj}} = \delta_k^* v^*(\delta^*), \quad (k = 1, 2, \ldots, T_0) \quad (5.3)$$

Taking logarithms in (5.3), T_0 log-linear simultaneous equations are transformed as

$$\sum_{j=1}^{n} \alpha_{kj} (\log t_j) = \frac{\delta_k^* v^*(\delta^*)}{c_k}, \quad (k = 1, 2, \ldots, T_0) \quad (5.4)$$

It is a system of T_0 linear equations in x_j $(= logt_j)$ for $j = 1, 2, ..., n$.

Note: 5.1. If there are more primal variables tj than the number of terms T_0 (> 1), many solutions t_j $(j = 1, 2, ..., n)$ may exist. Therefore to find the optimal primal variables t_j $(j = 1, 2, ..., n)$, it remains to minimize the primal objective function with respect to reduced $n - T_0 (\neq 0)$ variables. When $n - T_0 = 0$ i.e. number of primal variables = number of log-linear equations, primal variables can be determined uniquely from log-linear equations.

6. Solution Procedure of Fuzzy Reliability Model through Geometric Programming

The problem (3.2) can be written as follows taking logarithm of the objective function

$$\tilde{M}ax \quad \log\left(R_s\left(x_1, x_2, ..., x_n\right)\right) = Max \prod_{i=1}^{n} \log\left(1 - \left(1 - \tilde{R}_i\right)^{x_i}\right) \tag{6.1}$$

The above problem (6.1) can be reduced by the approximation (see Appendix–I) where as $\log\left(R_s\left(x_1, x_2, ..., x_n\right)\right) = -R_s^{'}$ as follows

$$\tilde{M}ax \quad R_s^{'}\left(x_1, x_2, ..., x_n\right) = \sum_{i=1}^{n} \left(1 - \tilde{R}_i\right)^{x_i} \tag{6.2}$$

subject to

$$\begin{aligned} \sum_{i=1}^{n} \tilde{C}_i x_i &\leq \tilde{C} \\ \sum_{i=1}^{n} \tilde{W}_i x_i &\leq \tilde{W} \end{aligned}$$

$x_i > 1 \ for \ i = 1, 2, ..., n.$

After defuzzification of the fuzzy parameters (6.2) reduces to

$$Max \quad R_s^{'}\left(x_1, x_2, ..., x_n\right) = \sum_{i=1}^{n} \left(1 - \hat{R}_i\right)^{x_i}$$

subject to

$$\begin{aligned}\sum_{i=1}^{n} \hat{C}_i x_i &\leq \hat{C} \\ \sum_{i=1}^{n} \hat{W}_i x_i &\leq \hat{W}\end{aligned}$$

$x_i > 1 \ for \ i = 1, 2, ..., n.$
The above problem reduced (Tillman et al [19]) as

$$Min \ R_s^{'} = \sum_{i=1}^{n} Q_i^{x_i} = \sum_{i=1}^{n} q_i \quad where \ \ 1 - \hat{R}_i = Q_i \ and \ Q_i^{x_i} = q_i$$

subject to

$$\begin{aligned}\sum_{i=1}^{n} \frac{\hat{C}_i \log q_i}{\log Q_i} &\leq \hat{C} \\ \sum_{i=1}^{n} \frac{\hat{W}_i \log q_i}{\log W_i} &\leq \hat{W}\end{aligned}$$

Where $q_i > 0$ for $i = 1, 2, \ldots, n$.
The above problem can be reduced as

$$Min \ R_s^{'} = \sum_{i=1}^{n} q_i$$

subject to

$$\begin{aligned}e^{-1} \prod_{i=1}^{n} q_i^{-k_{1i}} &\leq 1 \\ e^{-1} \prod_{i=1}^{n} q_i^{-k_{2i}} &\leq 1\end{aligned}$$

Where $q_i > 0$ for $i = 1, 2, \ldots, n$.
Where $k_{1i} = -\frac{\hat{C}_i}{\hat{C} \log Q_i}$ and $k_{2i} = -\frac{\hat{W}_i}{\hat{W} \log Q_i}$ for $i = 1, 2, \ldots, n$.

This is the primal form of the GP with DD $= n+2-n-1 = 1$
Now the DP of this PGP is

$$\begin{aligned} Max\ v(\delta) &= \left(\frac{e^{-1}}{\delta_{11}}\right)^{\delta_{11}} \left(\frac{e^{-1}}{\delta_{21}}\right)^{\delta_{21}} (\delta_{11})^{\delta_{11}} (\delta_{21})^{\delta_{21}} \prod_{i=1}^{n} \left(\frac{1}{\delta_{0i}}\right)^{\delta_{0i}} \\ subject\ to \sum_{i=1}^{n} \delta_{0i} &= 1, \\ \delta_{0i} - (k_{1i}\delta_{11} + k_{2i}\delta_{21}) &= 0 \quad for \quad j = 1,2,...,n, \end{aligned}$$

$\delta_{0i}, \delta_{1i}, \delta_{2i} > 0$ for $i = 1, 2, \ldots, n$
Solving the above equations in terms of δ_{11} we get
$\delta_{21} = 1 - \frac{1}{B}(1 - A\delta_{11}),\ \delta_{0i} = k_{1i}\delta_{11} + \frac{k_{2i}}{B}(1 - A\delta_{11})\ \ for\ i = 1, 2, \ldots, n$
$where\ A = \sum_{i=1}^{n} k_{1i}\ \ and\ B = \sum_{i=1}^{n} k_{2i}$
substituting the dual variables into the dual function we get

$$Max\ v(\delta) = \prod_{i=1}^{n} \left(\frac{1}{k_{1i}\delta_{11} + \frac{k_{2i}}{B}(1 - A\delta_{11})}\right)^{k_{1i}\delta_{11} + \frac{k_{2i}}{B}(1-A\delta_{11})} \left(e^{-1}\right)^{\delta_{11}\left(1 - \frac{1}{B}(1-A\delta_{11})\right)}$$

To obtain the optimal values first differentiate the log dual function with respect to δ_{11} and then set to zero, we get
$\prod_{i=1}^{n} \left(k_{1i}\delta_{11} + \frac{k_{2i}}{B}(1 - A\delta_{11})\right)^{k_{1i} - \frac{Ak_{2i}}{B}} = e^{\frac{A}{B} - 1}$
Solving the equation by Newton Raphson or any other method we get the optimal value δ_{11}^* and hence, we get the optimal value of $\delta_{01}^*, \delta_{02}^*, ..., \delta_{0n}^*\ and\ \delta_{21}^*$ by the relations
$\delta_{21}^* = 1 - \frac{1}{B}(1 - A\delta_{11}^*),\quad \delta_{0i}^* = k_{1i}\delta_{11}^* + \frac{k_{2i}}{B}(1 - A\delta_{11}^*)\ \ for\ i = 1, 2, \ldots, n$
Then we get the optimal value of the objective function of DP $v^*(\delta^*)$
Now we can find the solution of PGP according to primal-dual relation
$q_i^* = \delta_{0i}^* v^*(\delta^*)\ \ for\ \ i = 1, 2, ..., n$

$$i.e.\ x_i^* = \log_{\left(1 - \hat{R}_i\right)} \delta_{0i}^* v^*(\delta^*)\ \ for\ \ i = 1, 2, ..., n \tag{6.3}$$

Hence we get the number of optimal redundancy components for each ith-stage from (6.3)

7. Numerical Expose

For numerical explanation here we consider the four stages of reliability optimization model and assume that reliability and cost of each component, system cost and system weight of the DP (6.2) are fuzzy in nature. We take two types of fuzzy generalized, GTFN, GTrFN as input data instead of crisp coefficient.

Table-1

Input data table for fuzzy model (3.2) as TFN

R_1	$(0.75, 0.80, 0.85; w)$	C_1	$(1, 1.15, 1.30; w)$	W_1	$(0.95, 1, 1.2; w)$
R_2	$(0.60, 0.75, 0.90; w)$	C_2	$(2, 2.2, 2.5; w)$	W_2	$(0.9, .95, 1.5; w)$
R_3	$(0.70, 0.80, 0.85; w)$	C_3	$(3, 3.3, 3.6; w)$	W_3	$(0.9, 1, 1.3; w)$
R_4	$(0.75, 0.80, 0.90; w)$	C_4	$(4, 4.4, 4.8; w)$	W_4	$(0.95, 1.2, 1.5; w)$
		C	$(50, 55, 60; w)$	W	$(26, 32, 38; w)$

Table-2

Input data table for fuzzy model (3.2) as TrFN

R_1	$(0.70, 0.75, 0.80, 0.85; w)$	C_1	$(1, 1.15, 1.20, 1.30; w)$	W_1	$(0.92, 0.98, 1, 1.3; w)$
R_2	$(0.60, 0.70, 0.85, 0.90; w)$	C_2	$(2, 2.2, 2.4, 2.6; w)$	W_2	$(0.95, 0.98, 1.2, 1.4; w)$
R_3	$(0.65, 0.73, 0.80, 0.84; w)$	C_3	$(3, 3.2, 3.35, 3.5; w)$	W_3	$(0.94, 1, 1.2, 1.3; w)$
R_4	$(0.72, 0.78, 0.85, 0.90; w)$	C_4	$(4, 4.25, 4.4, 4.6; w)$	W_4	$(0.95, 0.98, 1, 1.2; w)$
		C	$(50, 54, 58, 62; w)$	W	$(26, 30, 32, 38; w)$

Numerical result by GP technique for different weights of generalized fuzzy numbers which are exhibited in the table-3 and 5. As redundancy must be integer, so after approximating the optimal fractional value of the number of redundancy for the optimal system reliability as follows

Table-3

Optimal redundancy for model (3.2) by GP method when input data are GTFN

Weights	x_1^*	x_2^*	x_3^*	x_4^*	R_s^*	Defuzzification Type
$w = 1$	5	6	5	4	0.997830	Type-I
$w = 0.2$	11	7	5	3	0.133367	Type-II&III
$w = 0.5$	7	6	5	4	0.725076	Type-II&III
$w = 0.8$	6	6	5	4	0.972085	Type-II&III
$w = 1$	5	6	5	4	0.997768	Type-II&III

The table 3 gives the result of redundancy for optimum system reliability using generalized triangular fuzzy number by the defuzzification rule of COM method, MC method and RA method. For MC method and RA method the outcome are same.

Table-4

Optimal redundancy for model (3.2) by INLP method when input data are GTFN

Weights	x_1^*	x_2^*	x_3^*	x_4^*	R_s^*	Defuzzification Type
$w = 1$	5	6	5	4	0.996493	Type-I
$w = 0.2$	9	6	4	4	0.124312	Type-II&III
$w = 0.5$	6	6	5	4	0.720801	Type-II&III
$w = 0.8$	6	6	5	4	0.972036	Type-II&III
$w = 1$	6	6	5	4	0.998023	Type-II&III

Table 4 displays the result of series-parallel model by integer non-linear programming (INLP) by Lingo [11] software. It is notice that GP method gives better result for some case otherwise almost same. So our approximation of the model (3.2) to the model (6.2) does not diverge from the original result.

Table-5

Optimal redundancy for model (3.2) by GP method when input data are GTrFN

Weights	x_1^*	x_2^*	x_3^*	x_4^*	R_s^*	Defuzzification Type
$w = 1$	5	5	5	4	0.996493	Type-I
$w = 0.2$	11	7	5	3	0.133367	Type-II&III
$w = 0.5$	7	6	5	4	0.725076	Type-II&III
$w = 0.8$	6	5	5	4	0.963690	Type-II&III
$w = 1$	6	5	5	4	0.996999	Type-II&III

The table 5 gives the result of redundancy for optimum system reliability using generalized trapezoidal triangular fuzzy number by the defuzzification rule of center of mass method, mean of $\alpha-$ cut method and removal area method. Here also the outcome are same for mean of $\alpha-$ cut method and removal area method.

8. Conclusion

Here we have considered the problem so as to find out the optimum number of redundancies, which maximizes the system reliability subject to the available system cost and system weight. Geometric programming technique is used to solve the problem with the coefficients, which are fuzzy number for reliability and cost of components. Here the system cost and system weight are taken as fuzzy number also. In many situations, problem parameters are more competent to take as GFN for real life examples. Hence this work gives more significant for reliability engineer for decision-making. For practical situation, based on decision maker's choice, several combination of different type of fuzzy number may be considered in the reliability model.

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Appendix-I

The explanation of approximation of the model (3.2) for the standard form of the primal geometric programming problem is given below as fellows

$$Max \quad R_s\left(x_1, x_2, ..., x_n\right) = \prod_{i=1}^{n} \left\{1 - \left(1 - \tilde{R}_i\right)^{x_i}\right\}$$

Let $\tilde{R}_i = (R_{i1}, R_{i2}, ..., R_{i3}; w_i)$, so its $\alpha-$cut is
$R_i\left(\alpha\right) = \left[R_{i1} + \frac{\alpha}{w_i}\left(R_{i2} - R_{i1}\right), R_{i3} - \frac{\alpha}{w_i}\left(R_{i3} - R_{i2}\right)\right]$

So $R_s\left(\alpha\right) = \left[R_s^L\left(\alpha\right), R_s^U\left(\alpha\right)\right]$ where
$R_s^L\left(\alpha\right) = \min\left\{\prod_{i=1}^{n}\left(1 - \left(1 - R_i\right)^{x_i}\right) : R_i \in \left[R_i^L\left(\frac{\alpha}{w}\right), R_i^U\left(\frac{\alpha}{w}\right)\right]\right\}$ and
$R_s^U\left(\alpha\right) = \max\left\{\prod_{i=1}^{n}\left(1 - \left(1 - R_i\right)^{x_i}\right) : R_i \in \left[R_i^L\left(\frac{\alpha}{w}\right), R_i^U\left(\frac{\alpha}{w}\right)\right]\right\}$

Here $\frac{\partial}{\partial R_k}\prod_{i=1}^{n}\left(1 - \left(1 - R_i\right)^{x_i}\right) = \prod_{\substack{i=1\\ i\neq k}}^{n}\left(1 - \left(1 - R_i\right)^{x_i}\right)\ x_k\ \left(1 - R_k\right)^{x_k} > 0$ for $0 < R_i < 1$, $i = 1, 2, ..., n$.

Therefore $R_s\left(\alpha\right) = \left[R_s^L\left(\alpha\right), R_s^U\left(\alpha\right)\right]$ where $R_s^L\left(\alpha\right) = \prod_{i=1}^{n}\left(1 - \left(1 - R_i^L\left(\frac{\alpha}{w}\right)\right)^{x_i}\right)$ and $R_s^U\left(\alpha\right) = \prod_{i=1}^{n}\left(1 - \left(1 - R_i^U\left(\frac{\alpha}{w}\right)\right)^{x_i}\right)$ and $w = \min_{\forall i}\left\{w_i\right\}$ for i=1,2,. . . ,n.

So $\tilde{R}_s$ is an approximate GTFN as

$$\tilde{R}_s = \left(\prod_{i=1}^{n} (1-(1-R_{i1})^{x_i}), \prod_{i=1}^{n} (1-(1-R_{i2})^{x_i}), \prod_{i=1}^{n} (1-(1-R_{i3})^{x_i}) ; w \right)$$

$w = \min_{\forall i} \{w_i\}$ for i=1,2,...,n. $\log \tilde{R}_s$ is approximated to a GTFN (Kaufmann and Gupta [7] page-61) as

$$\left(\log\left(\prod_{i=1}^{n} (1-(1-R_{i1})^{x_i}) \right), \log\left(\prod_{i=1}^{n} (1-(1-R_{i2})^{x_i}) \right), \log\left(\prod_{i=1}^{n} (1-(1-R_{i3})^{x_i}) \right) ; w \right)$$

$w = \min_{\forall i} \{w_i\}$ for i=1,2,...,n.

$$\begin{aligned} \text{Again} \qquad & \log\left(\prod_{i=1}^{n} (1-(1-R_{ij})^{x_i}) \right) \\ &= \sum_{i=1}^{n} (1-(1-R_{ij})^{x_i}) \;\; for \;\; j = 1,2,3. \\ &= -\sum_{i=1}^{n} \left((1-R_{ij})^{x_i} + \frac{1}{2}(1-R_{ij})^{2x_i} + \frac{1}{3}(1-R_{ij})^{3x_i} \ldots \right) \\ &\simeq -\sum_{i=1}^{n} (1-R_{ij})^{x_i} \;\; for \;\; j = 1,2,3. \end{aligned}$$

[In general $0.5 << R_{ij} < 1$ so that $0 < 1 - R_{ij} << 0.5$ therefore higher power of $(1-R_{ij})^{x_i}$ are neglected for j=1,2,3]

Therefore $\log \tilde{R}_s$ is approximate GTFN as

$$\left(-\sum_{i=1}^{n} (1-R_{i1})^{x_i}, -\sum_{i=1}^{n} (1-R_{i2})^{x_i}, -\sum_{i=1}^{n} (1-R_{i3})^{x_i} \right)$$

So $\log \tilde{R}_s \tilde{=} -\sum_{i=1}^{n} (1-R_i)^{x_i}$

Hence the approximation has significance to reduce the problem in to the standard form of primal GP.

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