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The Hahn Sequence Space of Fuzzy Numbers *

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Abstract

In this article we introduce a new sequence space $h(F)$ called the Hahn sequence space of fuzzy numbers. It is proved that the β-dual and γ-dual of $h(F)$ is the Cesaro space of the set of all Fuzzy bounded sequences.

1. Introduction

In recent years there has been an increasing interest in mathematical aspects of operations defined on fuzzy sets. The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh [1] and subsequently several authors have discussed various aspects of theory and applications of fuzzy sets, such

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as topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events and fuzzy mathematical programming. The theory of fuzzy numbers is not only the foundation of fuzzy analysis, but it also has important applications in fuzzy optimization, fuzzy decision making etc. [2, 3]. Many authors have found interest in the study of theory of fuzzy numbers [4, 5]. Matloka [6] introduced bounded and convergent sequences of fuzzy numbers. In addition sequences of fuzzy numbers have been discussed by Aytar and Pehlian [7], Basarir and Mursaleen [8] Nanda [9] and many others.

The idea of difference sequence space of fuzzy numbers was introduced by Savas [10] and further generalized by Rifat Colak [11] and many others. Recently Talo and Basar [12] introduced and studied the space $b_p(F)$ of sequences of p-bounded variation of fuzzy numbers. The study of Hahn-sequence space was initiated by Chandrasekhara Rao [13] with certain specific purpose in Banach space theory. Indeed, he got interested in finding a semi Hahn space and proved that the intersection of all semi Hahn spaces is the Hahn space [14]. This idea motivates us to study fuzzy Hahn sequence space. Talo and Basar [15] gave the idea of determining the dual of sequence space of fuzzy numbers by using the concept of convergence of a series of fuzzy numbers [16]. The present paper is devoted to the study of Hahn sequence space of fuzzy numbers. In Section 2 we recall some basic definitions and results about fuzzy numbers. In Section 3 we proved the completeness of the space $h(F)$ and showed that the β-dual and γ-dual of $h(F)$ is the Cesaro space of the set of all Fuzzy bounded sequences.

2. Definitions and Preliminaries

We begin with giving some required definitions and statements of theorems, propositions and lemmas. A fuzzy number is a fuzzy set on the real axis i.e. a mapping $u : R \to [0, 1]$ which satisfies the following four conditions.

(i) u is normal i.e. there exists an $x_0 \in R$ such that $u(x_0) = 1$.

(ii) u is fuzzy convex i.e. $u[\lambda x + (1 - \lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in R$ and for all $\lambda \in [0, 1]$.

(iii) u is upper semi continuous

(iv) The set $[u]_0 = \overline{\{x \in R : u(x) > 0\}}$ is compact [1] where $\overline{\{x \in R : u(x) > 0\}}$ denotes the closure of the set $\{x \in R : u(x) > 0\}$ in the usual topology

of R. We denote the set of all fuzzy numbers on R by E' and called it as the space of fuzzy numbers. The λ-level set $[u]_\lambda$ of $u \in E'$ is defined by $[u]_\lambda = \begin{cases} \{t \in R : u(t) \geq \lambda\}, & (0 < \lambda \leq 1) \\ \overline{\{t \in R : u(t) > \lambda\}}, & (\lambda = 0). \end{cases}$
The set $[u]_\lambda$ is a closed bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$. $\mathbb{R}$ can be embedded in E'. Since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by
$\overline{r} = \begin{cases} 1, & (x = r) \\ 0, & (x \neq r). \end{cases}$

Let $u, w \in E'$ and $k \in \mathbb{R}$. The operations addition, scalar multiplication and product defined on E' by

$$\begin{aligned} u + v = w &\Leftrightarrow [w]_\lambda = [u]_\lambda + [v]_\lambda \quad \text{for all } \lambda \in [0,1] \\ &\Leftrightarrow [w]^-(\lambda) = [u^-(\lambda), v^-(\lambda)] \text{ and } [w]^+(\lambda) = [u^+(\lambda), v^+(\lambda)] \\ &\quad \text{for all } \lambda \in [0,1] \end{aligned}$$

$[ku]_\lambda = k[u]_\lambda$ for all $\lambda \in [0,1]$ and $uv = w \Leftrightarrow [w]_\lambda = [u]_\lambda [v]_\lambda$ for all $\lambda \in [0,1]$ where it is immediate that

$$\begin{aligned} &[w]^-(\lambda) = \min\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\} \\ \text{and} \quad &[w]^+(\lambda) = \max\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\} \end{aligned}$$

for all $\lambda \in [0,1]$.

Let W be the set of all closed and bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$ i.e., $A = [\underline{A}, \overline{A}\,]$. Define the relation d on W by

$$d(A,B) = \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

Then it can be observed that d is a metric on W [10] and (W, d) is a complete metric space [11]. Now we can define the metric D on E' by means of a Hausdroff metric d as

$$D(u,v) = \sup_{\lambda \in [0,1]} d([u]_\lambda, [v]_\lambda) = \sup_{\lambda \in [0,1]} \{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}.$$

(E', D) is a complete metric space [17] one can extend the natural order relation on the real line to intervals as follows.

$$A \leq B \quad \text{if and only if} \quad \underline{A} \leq \underline{B} \quad \text{and} \quad \overline{A} \leq \overline{B}.$$

The partial order relation on E' is defined as follows.

$u \leq v \Leftrightarrow [u]_\lambda \leq [v]_\lambda \Leftrightarrow u^-(\lambda) \leq v^-(\lambda)$ and $u^+(\lambda) \leq v^+(\lambda)$ for all $\lambda \in [0,1]$.

An absolute value $|u|$ of a fuzzy number u is defined by

$$|u|(t) = \begin{cases} \max\{u(t), u(-t)\}, & (t \geq 0) \\ 0, & (t < 0) \end{cases}$$

λ-level set $[|u|]_\lambda$ of the absolute value of $u \in E'$ is in the form $[|u|]_\lambda$ where $|u|^-(\lambda) = \max\{0, u^-(\lambda), u^+(\lambda)\}$ and $|u|^+(\lambda) = \max\{|u^-(\lambda)|, |u^+(\lambda)|\}$. The absolute value $|uv|$ of $u, v \in E'$ satisfies the inequalities [15]

$$\begin{aligned} &|uv|^-(\lambda) \leq |uv|^+(\lambda) \\ &\quad \leq \max\left\{|u|^-(\lambda)|\,|v|^-(\lambda), |u|^-(\lambda)|\,|v|^+(\lambda), |u|^+(\lambda)|\,|v|^-(\lambda), |u|^+(\lambda)|\,|v|^+(\lambda), \right\} \end{aligned}$$

In the sequel, we require the following definitions and lemmas.

Definition 2.1. A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set N into the set E'. The fuzzy number u_k denotes the value of the function at $k \in \mathbb{N}$ and is called the kth term of the sequence. Let $w(F)$ denote the set of all sequences.

Lemma 2.2. *The following statements hold*

1. $D(uv, \overline{0}) \leq D(u, \overline{0})D(v, \overline{0})$ *for all* $u, v \in E'$.
2. *If* $u_k \to u$ *as* $k \to \infty$ *then* $D(u_k, \overline{0}) \to D(u, 0)$ *as* $k \to \infty$ *where* $(u_k) \in w(F)$.

Definition 2.3. A sequence $(u_k) \in w(F)$ is called convergent with limit $u \in E'$ if and only if for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon) \in N$ such that

$$D(u_k, u) < \varepsilon \quad \text{for all } k \geq n_0.$$

If the sequence $(u_k) \in w(F)$ converges to a fuzzy number u then by the definition of D the sequences of functions $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are uniformly convergent to $u^-(\lambda)$ and $u^+(\lambda)$ in $[0,1]$ respectively.

Definition 2.4. A sequence $(u_k) \in w(F)$ is called bounded if and only if the set of all fuzzy numbers consisting of the terms of the sequence (u_k) is a bounded set.

That is to say that a sequences $(u_k) \in w(F)$ is said to be bounded if and only if there exist two fuzzy numbers m and M such that $m \leq u_k \leq M$ for all $k \in N$.

Definition 2.5. Let $(u_k) \in w(F)$. Then the expression $\sum u_k$ is called a series of fuzzy numbers. Denote $S_n = \sum_{k=0}^{n} u_k$ for all $n \in N$, if the sequences (S_n) converges to a fuzzy number u then we say that the series $\sum u_k$ of fuzzy numbers converges to u and write $\sum_{k=0}^{n} u_k = u$ which implies as $n \to \infty$ that $\sum_{k=0}^{n} u_k^-(\lambda) \to u_k^-(\lambda)$ and $\sum_{k=0}^{n} u_k^+(\lambda) \to u_k^+(\lambda)$ uniformly in $\lambda \in [0,1]$. Conversely, if the fuzzy numbers $u_k = \{[u_k^-(\lambda), u_k^+(\lambda)] : \lambda \in [0,1]\}$, $\sum u_k^-(\lambda)$ and $\sum u_k^+(\lambda)$ converge uniformly in λ then $u = \big\{[u^-(\lambda),\, u^+(\lambda)] : \lambda \in [0,1]\big\}$ defines a fuzzy number such that $u = \sum u_k$.

We say otherwise the series of fuzzy numbers diverges. Additionally if the sequence (S_n) is bounded then we say that the series $\sum u_k$ of fuzzy numbers is bounded. By $\mathrm{cs}(F)$ and $\mathrm{bs}(F)$ we denote the sets of all convergent and bounded series of fuzzy numbers respectively.

Lemma 2.6. *Let for the series of functions $\sum_k u_k(x)$ and $\sum_k v_k(x)$ there exists an $n_0 \in N$ such that $|u_k(x)| \leq v_k(x)$ for all $k \geq n_0$ and for all $x \in [a,b]$ with $u_k : [a,b] \to \mathbb{R}$ and $v_k : [a,b] \to \mathbb{R}$. If the series converges uniformly in $[a,b]$ then the series $\sum_k |u_k(x)|$ and $\sum_k v_k(x)$ are uniformly convergent in $[a,b]$.*

Weierstrass M test

Let $u_k : [a,b] \to \mathbb{R}$ be given. If there exists an $M_k \geq 0$ such that $|u_k(x)| \leq M_k$ for all $k \in N$ and the series $\sum_k M_k$ converges then the series $\sum_k u_k(x)$ is uniformly and absolutely convergent in $[a,b]$.

Definition 2.7. A mapping T from X_1 into X_2 is said to be fuzzy isometric if $d_2(Tx,Ty) = d_1(x,y)$ for all $x,y \in X_1$. The space X_1 is said to be fuzzy isometric with the space X_2 if there exists a bijective fuzzy isometry from X_1 onto X_2 and write $X_1 \cong X_2$ .The spaces X_1 and X_2 are then called fuzzy isometric spaces.

Definition 2.8. Let S_1 and S_2 are two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk} where $n,k \in N$. Then the matrix A defines a transformation from S_1 into S_2, if for every sequence $x = (x_k) \in S_1$ the sequence $Ax = \big((Ax)_n\big)$, the A-transform of x, exists and is in S_2 where $(Ax)_n = \sum_k a_{nk}x_k$.

For a sequence space S, the matrix domain S_A of an infinite matrix A is defined by $S_A = \{x = (x_k) \in w : Ax \in S\}$.

The Hahn sequence space is the space of all sequences $x = (x_k)$ such that $\sum_{k=1}^{\infty} k|x_k - x_{k-1}|$ converges and $\lim_{k\to\infty} x_k = 0$.

The following spaces are needed for our work

$$\begin{aligned}
\ell_\infty(F) &= \big\{(u_k) \in w(F) : \sup_{k\in N} D(u_k, \overline{0}) < \infty)\big\},\\
c(F) &= \big\{(u_k) \in w(F) : \exists \ell \in E' \lim_{k\to\infty} D(u_k, \ell) = 0\big\},\\
c_0(F) &= \big\{(u_k) \in w(F) : \lim_{k\to\infty} D(u_k, 0) = 0\big\},\\
\ell_p(F) &= \big\{(u_k) \in w(F) : \sum_k D(u_k, \overline{0}) < \infty\big\}.
\end{aligned}$$

3. Main Results

Let A denote the matrix $A = (a_{nk})$ defined by

$$a_{n_k} = \begin{cases} n(-1)^{n-k}, & n-1 \le k \le n \\ 0, & 1 \le k \le n-1 \text{ or } k > n \end{cases} \tag{1}$$

Define the sequence $y = (y_k)$ which will be frequently used as the A-transform of a sequence $x = (x_k)$,

$$\text{i.e.,} \quad y_k = (Ax)_k = k(x_k - x_{k-1})k \ge 1. \tag{2}$$

We introduce the sequence spaces $h(F)$ as the set of all sequences such that the A-transforms of them are in $\ell(F)$ that is

$$h(F) = \big\{u = (u_k) \in w(F) : \sum_k D[(Au)_k, \overline{0}\,] < \infty \text{ and } \lim_{k\to\infty} D[u_k, \overline{0}\,] = 0\big\}$$

and $\quad h_\infty(F) = \big\{u = (u_k) \in w(F) : \sup_k D[(Au)_k, \overline{0}\,] < \infty\big\}.$

Example 3.1. Consider the sequence $u = \{u_k\}$ defined by

$$u_k = \begin{cases} \overline{1}, & 1 \le k \le n \\ \overline{0}, & k > n \end{cases}$$

$$\begin{aligned} \sum D\lfloor (Au)_k, \overline{0} \rfloor &= \sum D\lfloor k(u_k - u_{k-1}), \overline{0} \rfloor \\ &= 0 \quad \text{which is convergent} \end{aligned}$$

Also $\lim_{k\to\infty} D\left(u_k, \overline{0}\right) = 0$. Hence $u \in h(F)$.

Now we proceed to prove the completeness of $h(F)$ and $h_\infty(F)$.

Theorem 3.2. *$h(F)$ and $h_\infty(F)$ are complete metric spaces with the metrics dh and dh_∞ defined by*

$$dh(u, v) = \sum_k D[(Au)_k, (Av)_k]$$
$$\textit{and} \quad dh_\infty(u, v) = \sup_{k\in N} D[(Au)_k, (Av)_k]$$

respectively, where $u = (u_k)$ and $v = (v_k)$ are the elements of the spaces $h(F)$ or $h_\infty(F)$.

Proof. Let $\{u^i\}$ be any Cauchy sequence in the space $h(F)$, where $u^i = \{u_0^{(i)}, u_1^{(i)}, u_2^{(i)} \ldots\}$. Then for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$dh(u^i, u^j) = \sum_n D[(Au)^i_n, (Au)^j_n] < \varepsilon \tag{3}$$

for $i, j \ge n_0(\varepsilon)$. We obtain for each fixed $n \in N$ from (3) that

$$D[(Au)^i_n, (Au)^j_n] < \varepsilon \tag{4}$$

for every $i, j \ge n_0(\varepsilon)$. We obtain for each fixed $n \in \mathbb{N}$ from (3) that

$$\sum_{k=0}^{m} D[(Au)^i_n, (Au)^j_n] \le dh(u^i, u^j) < \varepsilon. \tag{5}$$

Take any $i \ge n_0(\varepsilon)$ and taking limit as $j \to \infty$ first and next $m \to \infty$ in (3) we obtain

$$dh(u^i, u) < \varepsilon. \tag{6}$$

Finally we proceed to prove $u \in h(F)$. Since $\{u^i\}$ is a Cauchy sequence in $h(F)$, we have

$$\sum_k D[(Au)^i_k, \overline{0}\,] \leq \varepsilon \quad \text{and} \quad \lim_{k\to\infty}[(Au)^i_k, \overline{0}\,] = 0.$$

Now

$$D[(Au)_k, \overline{0}\,] \leq D[(Au)_k, (Au)^i_k] + D[(Au)^i_k, (Au)^j_k] + D[(Au)^j_k, \overline{0}\,]. \qquad (7)$$

Hence

$$D[(Au)_k, \overline{0}\,] \leq \sum_k D[(Au)_k, (Au)^i_k] + \sum_k D[(Au)^i_k, (Au)^j_k] + \sum_k D[(Au)^j_k, \overline{0}\,] < \varepsilon.$$

Also from (5) $\lim_{k\to\infty} D[(Au)_k, \overline{0}\,] = 0$. Hence $u \in h(F)$. Since $\{u^i\}$ is an arbitrary Cauchy sequence, the space $h(F)$ is complete.

Definition 3.3. The space $h(F)$ is isomorphic to the space $\ell(F)$.

Proof. Consider the transformation T defined from $h(F)$ to $\ell(F)$ by $x \to y = T(x)$. To prove the fact $h(F) \cong \ell(F)$, we should show the existence of a bijection between the spaces $h(F)$ and $\ell(F)$. We can find that only one $x \in h(F)$ with $Tx = y$. This means that T is injective.
Let $y \in \ell(F)$. Define the sequence $x = (x_k)$ such that $(Ax)_k = y_k$ for all $k \in N$.

Then $dh(x, 0) = \sum_k D[(Ax)_k, \overline{0}\,] = \sum_k D[y_k, \overline{0}\,] < \infty$. Thus $x \in h(F)$.

Consequently T is bijective and is isometric. Therefore $h(F)$ and $\ell(F)$ are isomorphic.

Theorem 3.4. *Let d denote the set of all sequences of fuzzy numbers defined as follows*

$$d = \{x = (x_k) \in w(F) : \sum_k k|x_k - x_{k-1}| < \infty \quad \textit{and} \quad x \in c_0(F)\}$$

Then the set d is identical with the set $h(F)$.

Proof. Let $x \in h(F)$. Then

$$\sum_k D((Ax)_k, \overline{0}) < \infty \quad \text{and} \quad \lim_{k\to\infty} D(x_k, \overline{0}) = 0. \qquad (8)$$

Using (2),

$$\sum_k D(y_k,\overline{0}) < \infty \quad \text{and} \quad \lim_{k\to\infty} D(x_k,\overline{0}) = 0. \tag{9}$$

We have $\sum_k D(y_k,\overline{0}) = \sup_{\lambda\in[0,1]} \max\{|y_k^-(\lambda)|, |y_k^+(\lambda)|\}$.

Now $\max\{|y_k^-(\lambda)|, |y_k^+(\lambda)|\} \leq \sum_k D(y_k,\overline{0}) < \infty$.

This implies that $\sum_k |y_k| < \infty$. That is $\sum_k k|x_k - x_{k-1}| < \infty$.

Also from (8), $x \in c_0(F)$. Thus $x \in d$. Conversely suppose $x \in d$. Then $\sum_k k|x_k - x_{k-1}| < \infty$.

That is $\sum_k |y_k| < \infty$. Therefore $\sum_k k \max\{|y_k^-(\lambda)|, |y_k^+(\lambda)|\}$. converges for $\lambda \in [0,1]$. This gives for $\lambda = 0$, $\sum_k D(y_k,\overline{0}) < \infty$.

Also $(x_k) \in c_0(F)$ implies $\lim_{k\to\infty} D(x_k,\overline{0}) = 0$. This completes the proof.

Now we define the duals of the sequence space of fuzzy numbers.

Definition 3.5. The α-dual, β-dual and γ-dual of a set $S(F) \subset w(F)$ which are respectively denoted and defined by

$$\{S(F)\}^\alpha = \{(u_k) \in w(F) : (u_k\, v_k) \in \ell_1(F) \quad \text{for all} \quad (v_k) \in S(F)\},$$
$$\{S(F)\}^\beta = \{(u_k) \in w(F) : (u_k v_k) \in \text{cs}(F) \quad \text{for all} \quad (v_k) \in S(F)\}$$

and

$$\{S(F)\}^\gamma = \{(u_k) \in w(F) : (u_k v_k) \in \text{bs}(F) \quad \text{for all} \quad (v_k) \in S(F)\}.$$

Definition 3.6. Let B denote the matrix $B = (b_{n_k})$ defined by

$$b_{n_k} = \begin{cases} 1/n, & \text{if } 1 \leq k \leq n \\ 0, & \text{otherwise} \end{cases}$$

Define the sequence $y = (y_k)$ which will be frequently used as the B-transform of a sequence $x = (x_k)$ i.e., $y_k = (Bx)_k = \frac{1}{k}\sum_{i=1}^{k} x_i$.

The Cesaro space of $\ell_\infty(F)$ is the set of all sequences such that the B-transforms of them are in $\ell_\infty(F)$. That is

$$\sigma(\ell_\infty(F)) = \left\{ x = (x_k) : \sup_k D\left[(Bx)_k, \overline{0}\right] < \infty \right\}.$$

Theorem 3.7. *$\sigma(\ell_\infty(F))$ is a complete metric space with the metric.*

$$d_\sigma(u,v) = \sup_k D\left[(Bu)_k, (Bv)_k\right]$$

where $u = (u_k)$ and $v = (v_k)$ are the elements of the space $\sigma\left(\ell_\infty(F)\right)$

Theorem 3.8. *The β- and γ-dual of the set $h(F)$ of sequences of fuzzy number is the set $\sigma\left(\ell_\infty(F)\right)$.*

Proof. Let $(u_k) \in h(F)$ and $(v_k) \in \sigma\left(\ell_\infty(F)\right)$.
$(u_k) \in h(F) \Rightarrow \lim_{k\to\infty} D\left[u_k, \overline{0}\right] = 0$.
Therefore for given $\varepsilon > 0$ there exist n_0 such that $D(u_k, \overline{0}) < \varepsilon$.
$(v_k) \in \sigma(\ell_\infty(F)) \Rightarrow \sup_k D\left[(Bv)_k \overline{0}\right] < \infty$.
Then $D(v_k, \overline{0}) < \infty$ for all k and n.
Hence there exist a $M > 0$ such that $D(v_k, \overline{0}) < M$ for all k and n.
Now,

$$|(u_k)^-(\lambda)| \leq D(u_k, \overline{0}) \leq D(u_k, \overline{0}) D(v_k, \overline{0}) < \varepsilon M.$$

Weierstras Test yields that $\sum_k (u_k)^-(\lambda)$ and $\sum_k (u_k)^+(\lambda)$ converge uniformly and hence $\sum_k u_k$ converges.

Thus $\sigma\left(\ell_\infty(F)\right) \subset h^\beta(F)$.

Conversely suppose that $(v_k) \in h^\beta(F)$. Then the series $\sum_k u_k v_k$ converges for all $(u_k) \in h(F)$. This also holds for the sequence (u_k) of fuzzy numbers defined by $u_k = \chi[-1,1]$ for all $k \in N$. Then since $u_k^-(\lambda) = -1$ and $u_k^+(\lambda) = 1$ for all $\lambda \in [0,1]$ the series

$$\begin{aligned}\sum_k (u_k v_k)^+(\lambda) &= \sum_k \max\{u_k^-(\lambda)v_k^-(\lambda), u_k^-(\lambda)v_k^+(\lambda), u_k^+(\lambda)v_k^-(\lambda), u_k^+(\lambda)v_k^+(\lambda)\} \\ &= \sum_k \max\{-v_k^-(\lambda), -v_k^+(\lambda), v_k^-(\lambda), v_k^+(\lambda)\} \\ &= \sum_k \max\{|v_k^-(\lambda)|, |v_k^+(\lambda)|\}\end{aligned}$$

converges uniformly. Thus $\sup_k D\left[(Bv)_k, \overline{0}\right] < \infty$. Hence $(v_k) \in \sigma\big(l_\infty(F)\big)$ and $h^\beta(F) = \sigma\big(l_\infty(F)\big)$. This completes the proof.

4. Conclusions

In this paper we introduced Hahn sequence space of fuzzy numbers and we discussed some of its topological properties. We can further proceed the work to find the matrix transformations between this space and some of the known spaces of sequences of fuzzy numbers.

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