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A New Approach for Ranking of L-R Type Generalized Fuzzy Numbers [*]

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Abstract

Ranking of fuzzy numbers play an important role in decision making, optimization, forecasting etc. Fuzzy numbers must be ranked before an action is taken by a decision maker. Cheng (A new approach for ranking fuzzy numbers by distance method. Fuzzy Sets and Systems 95 (1998) 307-317) pointed out that the proof of the statement "Ranking of generalized fuzzy numbers does not depend upon the height of fuzzy numbers" stated by Liou and Wang (Ranking fuzzy numbers with integral value. Fuzzy Sets and Systems 50 (1992) 247-255) is not ture. In this paper, by giving an alternative proof it is proved that the above statement is correct. Also with the help of several counter examples it is proved that the results proposed by Chen and Chen (Fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and

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different spreads. Expert Systems with Applications 36 (2009) 6833-6842) are not accurate. The main aim of this paper is to modify the Liou and Wang approach for the ranking of L-R type generalized fuzzy numbers. The main advantage of the proposed approach is that the proposed approach provide the correct ordering of generalized and normal fuzzy numbers and also the proposed approach is very simple and easy to apply in the real life problems. It is shown that proposed ranking function satisfy all the reasonable properties of fuzzy quantities.

Keywords and Phrases: *Ranking function, L-R type generalized fuzzy number.*

1. Introduction

Fuzzy set theory [19] is a powerful tool to deal with real life situations. Real numbers can be linearly ordered by $\geq$ or $\leq$, however this type of inequality does not exist in fuzzy numbers. Since fuzzy numbers are represented by possibility distribution, they can overlap with each other and it is difficult to determine clearly whether one fuzzy number is larger or smaller than other. An efficient approach for ordering the fuzzy numbers is by the use of a ranking function $\Re : F(R) \to R$, where $F(R)$ is a set of fuzzy numbers defined on real line, which maps each fuzzy number into the real line, where a natural order exists. Thus, specific ranking of fuzzy numbers is an important procedure for decision-making in a fuzzy environment and generally has become one of the main problems in fuzzy set theory.

The method for ranking was first proposed by Jain [9]. Yager [18] proposed four indices which may be employed for the purpose of ordering fuzzy quantities in [0,1]. In Kaufmann and Gupta [10], an approach is presented for the ranking of fuzzy numbers. Campos and Gonzalez [2] proposed a subjective approach for ranking fuzzy numbers. Liou and Wang [13] developed a ranking method based on integral value index. Cheng [5] presented a method for ranking fuzzy numbers by using the distance method. Kwang and Lee [11] considered the overall possibility distributions of fuzzy numbers in their evaluations and proposed a ranking method. Modarres and Nezhad [14] proposed a ranking method based on preference function which measures the fuzzy numbers point by point and at each point the most preferred number is identified.

Chu and Tsao [6] proposed a method for ranking fuzzy numbers with the area between the centroid point and original point. Deng and Liu [7] presented a centroid-index method for ranking fuzzy numbers. Liang et al. [12] and Wang and Lee [17] also used the centroid concept in developing their ranking index. Chen and Chen [3] presented a method for ranking generalized trapezoidal fuzzy numbers. Abbasbandy and Hajjari [1] introduced a new approach for ranking of trapezoidal fuzzy numbers based on the left and right spreads at some α-levels of trapezoidal fuzzy numbers. Chen and Chen [4] presented a method for fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads. A comprehensive survey of different ranking methods of fuzzy numbers can be seen in [15]

In this paper, with the help of several counter examples it is shown that the results proposed by Chen and Chen [4] are not accurate. The main aim of this paper is to propose a new approach for the ranking of L-R type generalized fuzzy numbers. It is shown that the ranking of L-R type generalized fuzzy numbers does not depend upon the height of fuzzy number. The main advantage of the proposed approach is that the proposed approach provide the correct ordering of generalized and normal fuzzy numbers and also the proposed approach is very simple and easy to apply in the real life problems.

This paper is organized as follows: In section 2, some basic definitions, arithmetic operations and ranking function are reviewed. In section 3, the shortcomings of Chen and Chen approach [4] is discussed. In section 4, a new approach is proposed for the ranking of L-R type generalized fuzzy numbers. In section 5, the correct ordering of fuzzy numbers are obtained and also it is shown that the proposed approach satisfies all the reasonable properties of fuzzy quantities. The conclusion is discussed in section 6.

2. Preliminaries

In this section some basic definitions, arithmetic operations and ranking function are reviewed.

2.1. Basic Definitions

Definition 1. *[10] The characteristic function μ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X. This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range i.e. $\mu_{\tilde{A}} : X \to [0,1]$. The assigned value indicate the membership grade of the element in the set A.*
The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)); x \in X\}$ defined by $\mu_{\tilde{A}}(x)$ for each $x \in X$ is called a fuzzy set.

Definition 2. *[10] A fuzzy set $\tilde{A}$, defined on the universal set of real numbers R, is said to be a fuzzy number if its membership function has the following characteristics:*
1. $\mu_{\tilde{A}} : R \longrightarrow [0,1]$ is continuous.
2. $\mu_{\tilde{A}}(x) = 0$ for all $x \in (-\infty, a] \bigcup [d, \infty)$.
3. $\mu_{\tilde{A}}(x)$ strictly increasing on $[a, b]$ and strictly decreasing on $[c, d]$.
4. $\mu_{\tilde{A}}(x) = 1$ for all $x \in [b, c]$, where $a < b < c < d$.

Definition 3. *[10] A fuzzy number $\tilde{A} = (a, b, c, d)$ is said to be a trapezoidal fuzzy number if its membership function is given by*

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}, & a \le x \le b \\ 1 & b \le x \le c \\ \frac{(x-d)}{(c-d)} & c \le x \le d \end{cases}$$

Definition 4. *[4] A fuzzy set $\tilde{A}$, defined on the universal set of real numbers R, is said to be generalized fuzzy number if its membership function has the following characteristics:*
1. $\mu_{\tilde{A}} : R \longrightarrow [0, w]$ is continuous.
2. $\mu_{\tilde{A}}(x) = 0$ for all $x \in (-\infty, a] \bigcup [d, \infty)$.
3. $\mu_{\tilde{A}}(x)$ strictly increasing on $[a, b]$ and strictly decreasing on $[c, d]$.
4. $\mu_{\tilde{A}}(x) = w$, for all $x \in [b, c]$, where $0 < w \le 1$.

Definition 5. *[8] A fuzzy number $\tilde{A} = (a, b, c, d; w)_{LR}$ is said to be a L-R type generalized fuzzy number if its membership function is given by*

$$\mu_{\tilde{A}}(x)=\begin{cases} wL(\frac{b-x}{b-a}) & , \quad a \le x \le b \\ w & , \quad b \le x \le c \\ wR(\frac{x-c}{d-c}) & , \quad c \le x \le d. \end{cases}$$

where L and R are reference functions.

Definition 6. *[8] A L-R type generalized fuzzy number $\tilde{A} = (a, b, c, d; w)_{LR}$ is said to be a generalized trapezoidal fuzzy number if its membership function is given by*

$$\mu_{\tilde{A}}(x)=\begin{cases} w\frac{(x-a)}{(b-a)} & , \quad a \le x \le b \\ w & , \quad b \le x \le c \\ w\frac{(x-d)}{(c-d)} & , \quad c \le x \le d \end{cases}$$

2.2. Arithmetic Operations

In this subsection, arithmetic operations between two L-R type generalized fuzzy numbers, defined on universal set of real numbers R, are reviewed [8].

Let $\tilde{A}_1 = (a_1, b_1, c_1, d_1; w_1)_{LR}$, $\tilde{A}_2 = (a_2, b_2, c_2, d_2; w_2)_{LR}$ and
$\tilde{A}_3 = (a_3, b_3, c_3, d_3; w_3)_{RL}$ be any L-R type generalized fuzzy numbers and then

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; minimum\ (w_1, w_2))_{LR}$
(ii) $\tilde{A}_1 \ominus \tilde{A}_3 = (a_1 - d_3, b_1 - c_3, c_1 - b_3, d_1 - a_3; minimum\ (w_1, w_3))_{LR}$
(iii) $\lambda\tilde{A}_1 = \begin{cases} (\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1; w_1)_{LR} & \lambda > 0 \\ (\lambda d_1, \lambda c_1, \lambda b_1, \lambda a_1; w_1)_{RL} & \lambda < 0. \end{cases}$

2.3. Ranking Function

An efficient approach for comparing the fuzzy numbers is by the use of a ranking function [9] $\Re : F(R) \to R$, where $F(R)$ is a set of fuzzy numbers defined on set of real numbers, which maps each fuzzy number into the real line, where a natural order exists i.e.,

(i) $\tilde{A} \succ \tilde{B}$ iff $\Re(\tilde{A}) > \Re(\tilde{B})$
(ii) $\tilde{A} \prec \tilde{B}$ iff $\Re(\tilde{A}) < \Re(\tilde{B})$
(iii) $\tilde{A} \sim \tilde{B}$ iff $\Re(\tilde{A}) = \Re(\tilde{B})$

Remark 1. [16] For all fuzzy numbers $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ we have

(i) $\tilde{A} \succ \tilde{B} \Rightarrow \tilde{A} \oplus \tilde{C} \succ \tilde{B} \oplus \tilde{C}$
(ii) $\tilde{A} \succ \tilde{B} \Rightarrow \tilde{A} \ominus \tilde{C} \succ \tilde{B} \ominus \tilde{C}$
(iii) $\tilde{A} \sim \tilde{B} \Rightarrow \tilde{A} \oplus \tilde{C} \sim \tilde{B} \oplus \tilde{C}$
(iv) $\tilde{A} \succ \tilde{B}, \tilde{C} \succ \tilde{D} \Rightarrow \tilde{A} \oplus \tilde{C} \succ \tilde{B} \oplus \tilde{D}$

3. Shortcomings of Chen and Chen Approach [4]

In this section, the shortcomings of Chen and Chen approach on the basis of reasonable properties of fuzzy quantities and on the basis of height of fuzzy numbers, are pointed out .

3.1. On the Basis of Reasonable Properties of Fuzzy Quantities [16]

Let $\tilde{A}$ and $\tilde{B}$ be any two fuzzy numbers then

$\tilde{A} \succ \tilde{B} \Rightarrow \tilde{A} \ominus \tilde{B} \succ \tilde{B} \ominus \tilde{B}$ (Using remark 1)
i.e., $\Re(\tilde{A}) > \Re(\tilde{B}) \Rightarrow \Re(\tilde{A} \ominus \tilde{B}) > \Re(\tilde{B} \ominus \tilde{B})$

In this subsection, several examples are chosen to prove that the ranking function, proposed by Chen and Chen, does not satisfy the reasonable property, $\tilde{A} \succ \tilde{B} \Rightarrow \tilde{A} \ominus \tilde{B} \succ \tilde{B} \ominus \tilde{B}$, for the ordering of fuzzy quantities i.e., according to Chen Chen approach $\tilde{A} \succ \tilde{B} \not\Rightarrow \tilde{A} \ominus \tilde{B} \succ \tilde{B} \ominus \tilde{B}$, which is contradiction according to Wang and Kerre [16].

Example 1. Let $\tilde{A} = (0.1, 0.3, 0.3, 0.5; 1)$ and $\tilde{B} = (0.2, 0.3, 0.3, 0.4; 1)$ be two generalized trapezoidal fuzzy numbers then according to Chen and Chen approach $\tilde{B} \succ \tilde{A}$ but $\tilde{B} \ominus \tilde{A} \prec \tilde{A} \ominus \tilde{A}$ i.e., $\tilde{B} \succ \tilde{A} \not\Rightarrow \tilde{B} \ominus \tilde{A} \succ \tilde{A} \ominus \tilde{A}$.

Example 2. Let $\tilde{A} = (0.1, 0.3, 0.3, 0.5; 0.8)$ and $\tilde{B} = (0.1, 0.3, 0.3, 0.5; 1)$ be two generalized trapezoidal fuzzy numbers then according to Chen and Chen approach $\tilde{B} \succ \tilde{A}$ but $\tilde{B} \ominus \tilde{A} \prec \tilde{A} \ominus \tilde{A}$ i.e., $\tilde{B} \succ \tilde{A} \not\Rightarrow \tilde{B} \ominus \tilde{A} \succ \tilde{A} \ominus \tilde{A}$.

Example 3. Let $\tilde{A} = (-0.8, -0.6, -0.4, -0.2; 0.35)$ and $\tilde{B} = (-0.4, -0.3, -0.2, -0.1; 0.7)$ be two generalized trapezoidal fuzzy numbers then according to Chen and Chen approach $\tilde{A} \succ \tilde{B}$ but $\tilde{A} \ominus \tilde{B} \prec \tilde{B} \ominus \tilde{B}$ i.e., $\tilde{A} \succ \tilde{B} \not\Rightarrow \tilde{A} \ominus \tilde{B} \succ \tilde{B} \ominus \tilde{B}$.

Example 4. Let $\tilde{A} = (0.2, 0.4, 0.6, 0.8; 0.35)$ and $\tilde{B} = (0.1, 0.2, 0.3, 0.4; 0.7)$ be two generalized trapezoidal fuzzy numbers then according to Chen and Chen approach $\tilde{B} \succ \tilde{A}$ but $\tilde{B} \ominus \tilde{A} \prec \tilde{A} \ominus \tilde{A}$ i.e., $\tilde{B} \succ \tilde{A} \not\Rightarrow \tilde{B} \ominus \tilde{A} \succ \tilde{A} \ominus \tilde{A}$.

3.2. On the Basis of Height of Fuzzy Numbers

In this subsection, it is proved that, in some cases, Chen and Chen approach [4] states that the ranking of fuzzy numbers depends upon height of fuzzy numbers, while in several cases the ranking does not depend upon the height of fuzzy numbers.

Let $\tilde{A} = (a_1, a_2, a_3, a_4; w_1)$ and $\tilde{B} = (a_1, a_2, a_3, a_4; w_2)$ be two generalized trapezoidal fuzzy numbers then according to Chen and Chen [4]

Case (i) If $(a_1 + a_2 + a_3 + a_4) \neq 0$ then $\begin{cases} \tilde{A} \prec \tilde{B}, & \text{if } w_1 < w_2 \\ \tilde{A} \succ \tilde{B}, & \text{if } w_1 > w_2 \\ \tilde{A} \sim \tilde{B}, & \text{if } w_1 = w_2. \end{cases}$

Case (ii) If $(a_1 + a_2 + a_3 + a_4) = 0$ then $\tilde{A} \sim \tilde{B}$ for all values of w_1 and w_2.

Example 5. Let $\tilde{A} = (1, 1, 1, 1; w_1)$ and $\tilde{B} = (1, 1, 1, 1; w_2)$ be two generalized trapezoidal fuzzy numbers then according to Chen and Chen approach

$\tilde{A} \prec \tilde{B}$ if $w_1 < w_2$, $\tilde{A} \succ \tilde{B}$ if $w_1 > w_2$ and $\tilde{A} \sim \tilde{B}$ if $w_1 = w_2$.

Example 6. Let $\tilde{A} = (-.4, -.2, -.1, .7; w_1)$ and $\tilde{B} = (-.4, -.2, -.1, .7; w_2)$, be two generalized trapezoidal fuzzy numbers then $\tilde{A} \sim \tilde{B}$ for all values of w_1 and w_2.

According to Chen and Chen [4] in first case ranking of fuzzy numbers depends upon height and in second case ranking does not depend upon the height which is contradiction.

4. Proposed Approach

Cheng [5] pointed out that the proof of the statement "Ranking of generalized fuzzy numbers does not depend upon the height of fuzzy numbers" stated by Liou and Wang [13], is incorrect. In this section, by modifying the Liou and Wang approach [13] a new approach is proposed for the ranking of generalized fuzzy numbers and using the proposed approach, in the proposition 1, it is proved that the above statement is correct.

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)_{L_1 R_1}$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)_{L_2 R_2}$ be two L-R generalized fuzzy numbers then

(i) $\tilde{A} \succ \tilde{B}$ if $\Re(\tilde{A}) > \Re(\tilde{B})$
(ii) $\tilde{A} \prec \tilde{B}$ if $\Re(\tilde{A}) < \Re(\tilde{B})$
(iii) $\tilde{A} \sim \tilde{B}$ if $\Re(\tilde{A}) = \Re(\tilde{B})$ (1)

4.1. Method to Find the Values of $\Re(\tilde{A})$ and $\Re(\tilde{B})$

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)_{L_1 R_1}$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)_{L_2 R_2}$ be two L-R type generalized fuzzy numbers then use the following steps to find the values of $\Re(\tilde{A})$ and $\Re(\tilde{B})$

Step 1. Find $w = \mathit{minimum}\ (w_1, w_2)$

Step 2. Let the membership function of $\tilde{A}$ and $\tilde{B}$ be $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ respectively, where, $\mu_{\tilde{A}}(x)=\begin{cases} w_1 L_1(\frac{b_1-x}{b_1-a_1}) & , \ a_1 \le x \le b_1 \\ w_1 & , \ b_1 \le x \le c_1 \\ w_1 R_1(\frac{x-c_1}{d_1-c_1}) & , \ c_1 \le x \le d_1. \end{cases}$

and

$$\mu_{\tilde{B}}(x)=\begin{cases} w_2 L_2(\frac{b_2-x}{b_2-a_2}) & , \ a_2 < x < b_2 \\ w_2 & , \ b_2 \le x \le c_2 \\ w_2 R_2(\frac{x-c_2}{d_2-c_2}) & , \ c_2 < x < d_2. \end{cases}$$

$\mu_{\tilde{A}}(x) \in [0, w_1]$ and $\mu_{\tilde{B}}(x) \in [0, w_2]$

Let λ represent the membership value, which is common for both $\tilde{A}$ and $\tilde{B}$, then $\lambda \in (0, w]$, where $w = minimum\ (w_1, w_2)$

$$\Rightarrow \mu_{\tilde{A}}(x)=\begin{cases} w L_1(\frac{b_1-x}{b_1-a_1}) & , \ a_1 \le x \le b_1 \\ w & , \ b_1 \le x \le c_1 \\ w R_1(\frac{x-c_1}{d_1-c_1}) & , \ c_1 \le x \le d_1. \end{cases}$$

and

$$\mu_{\tilde{B}}(x)=\begin{cases} w L_2(\frac{b_2-x}{b_2-a_2}) & , \ a_2 \le x \le b_2 \\ w & , \ b_2 \le x \le c_2 \\ w R_2(\frac{x-c_2}{d_2-c_2}) & , \ c_2 \le x \le d_2. \end{cases}$$

Let $wL_1(\frac{b_1-x}{b_1-a_1}) = \lambda$ and $wR_1(\frac{x-c_1}{d_1-c_1}) = \lambda \Rightarrow \frac{b_1-x}{b_1-a_1} = L_1^{-1}(\frac{\lambda}{w}) \Rightarrow x = b_1 - (b_1 - a_1)L_1^{-1}(\frac{\lambda}{w})$ and $\Rightarrow \frac{x-c_1}{d_1-c_1} = R_1^{-1}(\frac{\lambda}{w}) \Rightarrow x = c_1 + (d_1 - c_1)R_1^{-1}(\frac{\lambda}{w})$

Now $\Re(\tilde{A}) = \alpha \int_0^w (b_1 - (b_1 - a_1)L_1^{-1}(\frac{\lambda}{w}))d\lambda + (1-\alpha) \int_0^w (c_1 + (d_1 - c_1)R_1^{-1}(\frac{\lambda}{w}))d\lambda$

Let $\frac{\lambda}{w} = h \Rightarrow d\lambda = wdh$
$\Re(\tilde{A}) = w\alpha \int_0^1 (b_1 - (b_1 - a_1)L_1^{-1}(h))dh + w(1-\alpha) \int_0^1 (c_1 + (d_1 - c_1)R_1^{-1}(h))dh$

Similarly $\Re(\tilde{B}) = w\alpha \int_0^1 (b_2 - (b_2 - a_2)L_2^{-1}(h))dh + w(1-\alpha) \int_0^1 (c_2 + (d_2 - c_2)R_2^{-1}(h))dh$

Remark 2. [13] The index of optimism (α) is representing the degree of optimism of a decision maker. A larger value of α indicates a higher degree of optimism. For $\alpha = 0$ and $\alpha = 1$ value of $\Re(\tilde{A})$ and $\Re(\tilde{B})$ represents the view points of a pessimistic and optimistic decision maker respectively while for $\alpha = 0.5$ values of $\Re(\tilde{A})$ and $\Re(\tilde{B})$ represents the view points of a moderate decision maker.

Remark 3. The arithmetic operations between two fuzzy numbers is obtained using the α- cut method [4] and the maximum value of λ, that will be common for both fuzzy numbers, will be obtained by finding the minimum value of the height of the fuzzy numbers, due to which in *minimum* $(w_1, w_2) = w$ is considered.

Remark 4. For generalized trapezoidal fuzzy numbers $\tilde{A} = (a, b, c, d; w)$, $\Re(\tilde{A}) = \frac{w}{2}\alpha(a + b) + \frac{w}{2}(1 - \alpha)(c + d)$ and for a generalized triangular fuzzy number $\tilde{A} = (a, b, c; w)$, $\Re(\tilde{A}) = \frac{w}{2}\alpha(a+b)+\frac{w}{2}(1-\alpha)(b+c)$ and for a L-R type normal fuzzy number $\tilde{A} = (a, b, c, d)_{LR}$ $\Re(\tilde{A}) = \alpha \int_0^1 (b - (b - a)L^{-1}(h))dh + (1 - \alpha) \int_0^1 (c + (d - c)R^{-1}(h))dh$

Proposition 1. Prove that ranking of L-R type generalized fuzzy numbers does not depend upon the height of fuzzy numbers i.e., if $\tilde{A}'$, $\tilde{B}'$ are two L-R type generalized fuzzy numbers and $\tilde{A}$, $\tilde{B}$ are normalize fuzzy numbers, obtained from $\tilde{A}'$ and $\tilde{B}'$ respectively, then (i) $\tilde{A}' \succ \tilde{B}'$ iff $\tilde{A} \succ \tilde{B}$, (ii) $\tilde{A}' \prec \tilde{B}'$ iff $\tilde{A} \prec \tilde{B}$, (iii) $\tilde{A}' \sim \tilde{B}'$ iff $\tilde{A} \sim \tilde{B}$.

Proof. Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)_{L_1R_1}$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)_{L_2R_2}$ be two L-R type generalized fuzzy numbers and Let $\tilde{A}' = (a_1, b_1, c_1, d_1; 1)_{L_1R_1}$ and $\tilde{B}' = (a_2, b_2, c_2, d_2; 1)_{L_2R_2}$ be the corresponding normal fuzzy numbers respectively and $w = min(w_1, w_2)$ then using the proposed approach

$\tilde{A} \succ \tilde{B} \Leftrightarrow \Re(\tilde{A}) > \Re(\tilde{B}) \Leftrightarrow w\alpha \int_0^1 (b_1 - (b_1 - a_1)L_1^{-1}(h))dh + w(1 - \alpha) \int_0^1 (c_1 + (d_1 - c_1)R_1^{-1}(h))dh > w\alpha \int_0^1 (b_2 - (b_2 - a_2)L_2^{-1}(h))dh + w(1 - \alpha) \int_0^1 (c_2 + (d_2 - c_2)R_2^{-1}(h))dh \Leftrightarrow \alpha \int_0^1 (b_1 - (b_1 - a_1)L_1^{-1}(h))dh + (1 - \alpha) \int_0^1 (c_1 + (d_1 - c_1)R_1^{-1}(h))dh > \alpha \int_0^1 (b_2 - (b_2 - a_2)L_2^{-1}(h))dh + (1 - \alpha) \int_0^1 (c_2 + (d_2 - c_2)R_2^{-1}(h))dh$ $\Leftrightarrow \Re(\tilde{A}') > \Re(\tilde{B}') \Leftrightarrow \tilde{A}' \succ \tilde{B}'$.

Silmilarly it can easily proved that $\tilde{A}' \prec \tilde{B}'$ iff $\tilde{A} \prec \tilde{B}$ and $\tilde{A}' \sim \tilde{B}'$ iff $\tilde{A} \sim \tilde{B}$.

5. Results and Discussion

In this section the correct ordering of fuzzy numbers, discussed in section 3, are obtained. Also, in the Table 1, it is shown that proposed ranking function satisfies the all reasonable properties of fuzzy quantities proposed by Wang and Kerre [16].

Example 7. Let $\tilde{A} = (0.1, 0.3, 0.3, 0.5; 1)$ and $\tilde{B} = (0.2, 0.3, 0.3, 0.4; 1)$ be two generalized trapezoidal fuzzy numbers.
Step 1. *minimum* $(1, 1) = 1$
Step 2. $\Re(\tilde{A}) = \frac{1}{2}(\alpha(0.4) + (1-\alpha)(0.8)) = 0.4 - 0.2\alpha$ and $\Re(\tilde{B}) = \frac{1}{2}(\alpha(0.5) + (1-\alpha)(0.7)) = \frac{1}{2}(0.7 - 0.2\alpha)$
For a pessimistic decision maker, with $\alpha = 0$, $\Re(\tilde{A}) = 0.4$ and $\Re(\tilde{B}) = 0.35$. Since $\Re(\tilde{A}) > \Re(\tilde{B})$ so $\tilde{A} \succ \tilde{B}$.
For optimistic decision maker, with $\alpha = 1$, $\Re(\tilde{A}) = 0.2$ and $\Re(\tilde{B}) = 0.25$. Since $\Re(\tilde{A}) < \Re(\tilde{B})$ so $\tilde{A} \prec \tilde{B}$.
For moderate decision maker, with $\alpha = 0.5$, $\Re(\tilde{A}) = 0.3$ and $\Re(\tilde{B}) = 0.3$. Since $\Re(\tilde{A}) = \Re(\tilde{B})$ so $\tilde{A} \sim \tilde{B}$.

Example 8. Let $\tilde{A} = (0.1, 0.3, 0.3, 0.5; 0.8)$ and $\tilde{B} = (0.1, 0.3, 0.3, 0.5; 1)$ be two generalized trapezoidal fuzzy numbers.
Step 1. *minimum* $(0.8, 1) = 0.8$
Step 2. $\Re(\tilde{A}) = \frac{0.8}{2}(\alpha(0.4) + (1-\alpha)(0.8)) = 0.4(0.8 - 0.4\alpha)$ and $\Re(\tilde{B}) = \frac{0.8}{2}(\alpha(0.4) + (1-\alpha)(0.8)) = 0.4(0.8 - 0.4\alpha)$.
Since $\Re(\tilde{A}) = \Re(\tilde{B})\ \ \forall \alpha$ so $\tilde{A} \sim \tilde{B}$

Example 9. Let $\tilde{A} = (-0.8, -0.6, -0.4, -0.2; 0.35)$ and $\tilde{B} = (-0.4, -0.3, -0.2, -0.1; 0.7)$ be two generalized trapezoidal fuzzy numbers.
Step 1. *minimum* $(0.35, 0.7) = 0.35$
Step 2. $\Re(\tilde{A}) = 0.35(-0.3 - 0.4\alpha)$ and $\Re(\tilde{B}) = \frac{0.35}{2}(-0.3 - 0.4\alpha)$.
For a pessimistic decision maker, with $\alpha = 0$, $\Re(\tilde{A}) = -0.105$ and $\Re(\tilde{B}) = -0.0525$. Since $\Re(\tilde{A}) < \Re(\tilde{B})$ so $\tilde{A} \prec \tilde{B}$.
For optimistic decision maker, with $\alpha = 1$, $\Re(\tilde{A}) = -0.245$ and $\Re(\tilde{B}) = -0.1225$. Since $\Re(\tilde{A}) < \Re(\tilde{B})$ so $\tilde{A} \prec \tilde{B}$.
For moderate decision maker, with $\alpha = 0.5$, $\Re(\tilde{A}) = -0.175$ and $\Re(\tilde{B}) = -0.0875$. Since $\Re(\tilde{A}) < \Re(\tilde{B})$ so $\tilde{A} \prec \tilde{B}$.

Example 10. Let $\tilde{A} = (0.2, 0.4, 0.6, 0.8; 0.35)$ and $\tilde{B} = (0.1, 0.2, 0.3, 0.4; 0.7)$ be two generalized trapezoidal fuzzy numbers.
Step 1. *minimum* $(0.35, 0.7) = 0.35$
Step 2. $\Re(\tilde{A}) = 0.35(0.7 - 0.4\alpha)$ and $\Re(\tilde{B}) = \frac{0.35}{2}(0.7 - 0.4\alpha)$.
For a pessimistic decision maker, with $\alpha = 0$, $\Re(\tilde{A}) = 0.245$ and $\Re(\tilde{B}) = 0.1225$. Since $\Re(\tilde{A}) > \Re(\tilde{B})$ so $\tilde{A} \succ \tilde{B}$.
For optimistic decision maker, with $\alpha = 1$, $\Re(\tilde{A}) = 0.105$ and $\Re(\tilde{B}) = 0.0525$. Since $\Re(\tilde{A}) > \Re(\tilde{B})$ so $\tilde{A} \succ \tilde{B}$.
For moderate decision maker, with $\alpha = 0.5$, $\Re(\tilde{A}) = 0.175$ and $\Re(\tilde{B}) = 0.0875$. Since $\Re(\tilde{A}) > \Re(\tilde{B})$ so $\tilde{A} \succ \tilde{B}$.

Example 11. Let $\tilde{A} = (1, 1, 1, 1; w_1)$ and $\tilde{B} = (1, 1, 1, 1; w_2)$ be two generalized trapezoidal fuzzy numbers then
Step 1. *minimum* $(w_1, w_2) = w$
Step 2. $\Re(\tilde{A}) = \frac{w}{2}(\alpha(2)+(1-\alpha)(2)) = w$ and $\Re(\tilde{B}) = \frac{w}{2}(\alpha(2)+(1-\alpha)(2)) = w$
Since $\Re(\tilde{A}) = \Re(\tilde{B})\ \ \forall\alpha$ so $\tilde{A} \sim \tilde{B}$

Example 12. Let $\tilde{A} = (-.4, -.2, -.1, .7; w_1)$ and $\tilde{B} = (-.4, -.2, -.1, .7; w_2)$, be two generalized trapezoidal fuzzy numbers then
Step 1. *minimum* $(w_1, w_2) = w$
Step 2. $\Re(\tilde{A}) = \frac{w}{2}(\alpha(-0.6) + (1-\alpha)(0.6)) = 0.3$ and $\Re(\tilde{B}) = \frac{w}{2}(\alpha(-0.6) + (1-\alpha)(0.6)) = 0.3$
Since $\Re(\tilde{A}) = \Re(\tilde{B})\ \ \forall\alpha$ so $\tilde{A} \sim \tilde{B}$

5.1. Validation of the Results

In the above examples it can be easily checked that, for a particular value of α

(i) $\tilde{A} \sim \tilde{B} \Rightarrow \tilde{A}\ominus\tilde{B} \sim \tilde{B}\ominus\tilde{B}$ i.e., $\Re((\tilde{A}\ominus\tilde{B})\ominus(\tilde{B}\ominus\tilde{B})) = \Re((\tilde{B}\ominus\tilde{B})\ominus(\tilde{B}\ominus\tilde{B}))$

(ii) $\tilde{A} \succ \tilde{B} \Rightarrow \tilde{A}\ominus\tilde{B} \succ \tilde{B}\ominus\tilde{B}$ i.e., $\Re((\tilde{A}\ominus\tilde{B})\ominus(\tilde{B}\ominus\tilde{B})) > \Re((\tilde{B}\ominus\tilde{B})\ominus(\tilde{B}\ominus\tilde{B}))$

(iii) $\tilde{A} \prec \tilde{B} \Rightarrow \tilde{A}\ominus\tilde{B} \prec \tilde{B}\ominus\tilde{B}$ i.e., $\Re((\tilde{A}\ominus\tilde{B})\ominus(\tilde{B}\ominus\tilde{B})) < \Re((\tilde{B}\ominus\tilde{B})\ominus(\tilde{B}\ominus\tilde{B}))$

5.2. Validation of Proposed Ranking Function

For the validation of the proposed ranking function, in Table 1, it is shown that proposed ranking function satisfies the all reasonable properties of fuzzy quantities proposed by Wang and Kerre [16]

Table 1.
Fulfilment of the axioms for the ordering in the first and second class [16]

Index	A_1	A_2	A_3	A_4	A'_4	A_5	A_6	A'_6	A_7
Y_1	Y	Y	Y	Y	Y	Y	N	N	N
Y_2	Y	Y	Y	Y	Y	Y	Y	Y	N
Y_3	Y	Y	Y	N	N	Y	N	N	N
Y_4	Y	Y	Y	Y	Y	Y	N	N	N
C	Y	Y	Y	N	N	Y	N	N	N
FR	Y	Y	Y	Y	Y	Y	Y	Y	N
CL	Y	Y	Y	Y	Y	Y	Y	Y	N
LW^{λ}	Y	Y	Y	Y	Y	Y	Y	Y	N
CM_1^{λ}	Y	Y	Y	Y	Y	Y	Y	Y	N
CM_2^{λ}	Y	Y	Y	Y	Y	Y	Y	Y	N
K	Y	Y	Y	N	N	N	N	N	N
W	Y	Y	Y	Y	N	N	N	N	N
J^k	Y	Y	Y	Y	Y	N	N	N	N
CH^k	Y	Y	Y	Y	Y	N	N	N	N
KP^k	Y	Y	Y	Y	Y	N	N	N	N
Proposed Approach	Y	Y	Y	Y	Y	Y	Y	Y	N

6. Conclusion

In this paper, the shortcomings of Chen and Chen [4] approach are pointed out and a new ranking approach is proposed for finding the correct ordering of L-R type generalized fuzzy numbers. It is shown that proposed ranking function satisfies all the reasonable properties of fuzzy quantities proposed by Wang and Kerre [16].

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