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Integral Mean Estimates for Polynomials with Restricted Zeros *

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Abstract

In this paper, we prove some integral inequalities concerning polynomials and there by investigate the dependence of $|P(Rz) - P(z)|$ on $|P(z)|$ for $|z| = 1$. These results not only generalize some well-known L^q $(q > 1)$ inequalities, but also establish the validity of many in $(0, 1)$ as well.

Keywords and Phrases: *Polynomials, Integral mean estimates, Zeros..*

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1. Introduction

Let $P(z)$ be a polynomial of degree n and $P^{'}(z)$ its derivative, then for each $q \geq 1$,

$$\left\{\frac{1}{2\pi}\int\limits_{0}^{2\pi}|P^{'}(e^{i\theta})|^{q}d\theta\right\}^{\frac{1}{q}} \leq n\left\{\frac{1}{2\pi}\int\limits_{0}^{2\pi}|P(e^{i\theta})|^{q}d\theta\right\}^{\frac{1}{q}} \tag{1}$$

and for every $q > 0$,

$$\left\{\frac{1}{2\pi}\int\limits_{0}^{2\pi}|P(Re^{i\theta})|^{q}d\theta\right\}^{\frac{1}{q}} \leq R^{n}\left\{\frac{1}{2\pi}\int\limits_{0}^{2\pi}|P(e^{i\theta})|^{q}d\theta\right\}^{\frac{1}{q}}. \tag{2}$$

Inequality (1) is due to Zygmund [21], whereas inequality (2) is a simple consequence of a result due to Hardy [13]. Arestove [1] verified that (1) remains true for $0 < q < 1$ as well.

Inequalities (1) and (2) can be sharpened, if we restrict ourselves to the class of polynomials having no zeros in $|z| < k$ where $k \geq 1$. In case $k = 1$, inequality (1) can be replaced [8,18] by

$$\left\{\frac{1}{2\pi}\int\limits_{0}^{2\pi}|P^{'}(e^{i\theta})|^{q}d\theta\right\}^{\frac{1}{q}} \leq n\ A_{q}\left\{\frac{1}{2\pi}\int\limits_{0}^{2\pi}|P(e^{i\theta})|^{q}d\theta\right\}^{\frac{1}{q}}, \quad q > 0 \tag{3}$$

where

$$A_{q} = \left\{\frac{1}{2\pi}\int\limits_{0}^{2\pi}|1+e^{i\alpha}|^{q}d\alpha\right\}^{\frac{-1}{q}}.$$

Whereas inequality (2) can be replaced [7,17] by

$$\left\{\frac{1}{2\pi}\int\limits_{0}^{2\pi}|P(Re^{i\theta})|^{q}d\theta\right\}^{\frac{1}{q}} \leq B_{q}\left\{\frac{1}{2\pi}\int\limits_{0}^{2\pi}|P(e^{i\theta})|^{q}d\theta\right\}^{\frac{1}{q}}, \tag{4}$$

where

$$B_{q} = \frac{\left\{\frac{1}{2\pi}\int\limits_{0}^{2\pi}|1+R^{n}e^{i\alpha}|^{q}d\alpha\right\}^{\frac{1}{q}}}{\left\{\frac{1}{2\pi}\int\limits_{0}^{2\pi}|1+e^{i\alpha}|^{q}d\alpha\right\}^{\frac{1}{q}}}.$$

For $k \geq 1$, Govil and Rahman [10] have shown that, if $P(z)$ does not vanish in $|z| < k$, then for every $q \geq 1$,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}|P'(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \leq n\ C_q\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}, \tag{5}$$

where

$$C_q = \left\{\frac{1}{2\pi}\int_0^{2\pi}|k + e^{i\alpha}|^q d\alpha\right\}^{\frac{-1}{q}}.$$

The validity of (5) for $0 < q < 1$ is verified in [4,12]. On the other hand, the extension of (4) for $k \geq 1$ was proved by Aziz and Shah [5] to read as

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(Re^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \leq D_q\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}, \tag{6}$$

where

$$D_q = \frac{\left\{\frac{1}{2\pi}\int_0^{2\pi}|1 + R^n e^{i\alpha}|^q d\alpha\right\}^{\frac{1}{q}}}{\left\{\frac{1}{2\pi}\int_0^{2\pi}|1 + t_k e^{i\alpha}|^q d\alpha\right\}^{\frac{1}{q}}}, \quad with \ \ t_k = \left(\frac{1 + Rk}{R + k}\right)^n.$$

As a generalization of inequality (3), Aziz [2] obtained the following interesting result:

Theorem A. *If $P(z)$ is a polynomial of degree n with $\min_{|z|=1}|P(z)| = m$ and $P(z)$ has no zeros in $|z| < 1$, then for every given complex number β with $|\beta| \leq 1$ and for $q \geq 1$,*

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}|P'(e^{i\theta}) + mn\beta e^{i(n-1)\theta}|^q d\theta\right\}^{\frac{1}{q}} \leq n\ A_q\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta}) + m\beta e^{in\theta}|^q d\theta\right\}^{\frac{1}{q}}, \tag{7}$$

where A_q is defined above.

Recently Aziz and Rather [3] investigated the dependence of $|P(Re^{i\theta}) - P(e^{i\theta})|$ on $|P(e^{i\theta})|$ and proved the following:

Theorem B. *If $P(z)$ is a polynomial of degree n, then for every $q > 0$ and $R \geq 1$,*

$$\left\{\frac{1}{2\pi}\int_0^{2\pi} |P(Re^{i\theta}) - P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \leq (R^n - 1)\left\{\frac{1}{2\pi}\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}. \qquad (8)$$

In this paper, we first consider a class of polynomials $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu \leq n$ and prove the following more general result analogous to Theorem B, which among other things provide generalizations for some well-known polynomial inequalities in L^q spaces.

Theorem 1. *Let $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ \ 1 \leq \mu \leq n$ be a polynomial of degree at most n, having no zeros in $|z| < k$ where $k \geq 1$ and $\min_{|z|=k} |P(z)| = m$. Then for every complex number β with $|\beta| \leq 1,\ \ q > 0,\ \ R > 1$ and $0 \leq \theta < 2\pi$, $0 \leq \alpha < 2\pi$,*

$$\left\{\int_0^{2\pi} \left|\frac{P(Re^{i\theta}) - P(e^{i\theta})}{R^n - 1} + m\beta k^{-n} e^{in\theta}\right|^q d\theta\right\}^{\frac{1}{q}}$$

$$\leq \left\{\frac{1}{2\pi}\int_0^{2\pi} |C_{\mu k} + e^{i\alpha}|^q d\alpha\right\}^{-\frac{1}{q}} \left\{\int_0^{2\pi} |P(e^{i\theta}) + m\beta k^{-n} e^{in\theta}|^q d\theta\right\}^{\frac{1}{q}}, \qquad (9)$$

where

$$C_{\mu k} = k^{\mu+1}\left\{\frac{\frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| k^{\mu-1} + 1}{1 + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}}\right\}. \qquad (10)$$

The result is sharp in case $k = 1$ and equality holds for $P(z) = z^n + 1$.

The following corollary immediately follows from Theorem 1 by making $R \to 1$.

Corollary 1. If $P(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j$ is a polynomial of degree n with $\min\limits_{|z|=k} |P(z)| = m$ and having no zeros in the disk $|z| < k, \quad k \geq 1$, then for every given complex number β with $|\beta| \leq 1, \quad q > 0$ and for each θ with $0 \leq \theta < 2\pi$,$0 \leq \alpha < 2\pi$,

$$\left\{ \frac{1}{2\pi} \int\limits_0^{2\pi} |P^{'}(e^{i\theta}) + mn\beta k^{-n} e^{i(n-1)\theta}|^q d\theta \right\}^{\frac{1}{q}}$$

$$\leq \frac{n}{\left\{ \frac{1}{2\pi} \int\limits_0^{2\pi} |C^{'}_{\mu k} + e^{i\alpha}|^q d\alpha \right\}^{\frac{1}{q}}} \left\{ \frac{1}{2\pi} \int\limits_0^{2\pi} |P(e^{i\theta}) + m\beta k^{-n} e^{in\theta}|^q d\theta \right\}^{\frac{1}{q}}, \tag{11}$$

where

$$C^{'}_{\mu k} = k^{\mu+1} \left\{ \frac{\frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu-1} + 1}{1 + \frac{\mu}{n} \left| \frac{a_\mu}{a_0} \right| k^{\mu+1}} \right\}. \tag{12}$$

Remark 1. For $k = 1$, it follows from Corollary 1, that Theorem A holds true for $0 < q < 1$ and for $\beta = 0, \quad k = 1$, Corollary 1 reduces to de Bruijn's theorem [8] for every $q > 0$.

Remark 2. Since $\frac{R^\mu - 1}{R^n - 1} \leq \frac{\mu}{n}$ for all $R > 1, \quad 1 \leq \mu \leq n$ (for refrence see [6]) and $\left\{ \frac{1}{2\pi} \int\limits_0^{2\pi} |k + e^{i\alpha}|^q d\alpha \right\}^{\frac{1}{q}} \leq \left\{ \frac{1}{2\pi} \int\limits_0^{2\pi} |C_{\mu k} + e^{i\alpha}|^q d\alpha \right\}^{\frac{1}{q}}$ (for refrence see [9]), the following improvement as well as generalization of a result of Govil and Rahman follows from Theorem 1 by taking $\beta = 0$.

Corollary 2. If $P(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j$ is a polynomial of degree n, having no zeros in $|z| < k, \quad k \geq 1$, then for every $q > 0, \quad R > 1$ and $0 \leq \theta < 2\pi$,

$$\left\{ \frac{1}{2\pi} \int\limits_0^{2\pi} \left| \frac{P(Re^{i\theta}) - P(e^{i\theta})}{R^n - 1} \right|^q d\theta \right\}^{\frac{1}{q}}$$

$$\leq \frac{1}{\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}|C_{\mu k}+e^{i\alpha}|^q d\alpha\right\}^{\frac{1}{q}}}\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}|P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}, \tag{13}$$

where $C_{\mu k}$ is given by (10).

Remark 3. By making $q \to \infty$ and noting that

$$\lim_{q\to\infty}\left\{\frac{1}{2\pi}\int\limits_0^{2\pi}|P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} = \max_{|z|=1}|P(z)|,$$

we have from inequality (9)

$$\max_{|z|=1}\left|\frac{P(Rz)-P(z)}{R^n-1}+\frac{m\beta z^n}{k^n}\right|$$

$$\leq \left\{\frac{\frac{R^\mu-1}{R^n-1}\left|\frac{a_\mu}{a_0}\right|k^{2\mu}+k^{\mu+1}}{1+\frac{R^\mu-1}{R^n-1}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}+1\right\}^{-1}\max_{|z|=1}|P(z)+m\beta k^{-n}z^n|.$$

Equivalently

$$\left|\frac{P(Rz)-P(z)}{R^n-1}+\frac{m\beta z^n}{k^n}\right|$$

$$\leq \left\{\frac{1+\frac{R^\mu-1}{R^n-1}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}{(1+k^{\mu+1})+\frac{R^\mu-1}{R^n-1}\left|\frac{a_\mu}{a_0}\right|(k^{2\mu}+k^{\mu+1})}\right\}(\max_{|z|=1}|P(z)|+m|\beta|k^{-n}). \tag{14}$$

Choosing argument of β suitably, so that for $|z|=1$,

$$\left|\frac{P(Rz)-P(z)}{R^n-1}+\frac{m\beta z^n}{k^n}\right| = \left|\frac{P(Rz)-P(z)}{R^n-1}\right|+\frac{m|\beta|}{k^n}$$

and then making $|\beta|\to 1$, we get from inequality (14) the following:

Corollary 3. If $P(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j, \quad 1 \leq \mu \leq n$ is a polynomial of degree at most n, having no zeros in $|z| < k, \quad k \geq 1$, then for $R > 1$,

$$\left|\frac{P(Rz) - P(z)}{R^n - 1}\right| \leq \left\{\frac{1 + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}}{(1 + k^{\mu+1}) + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right|(k^{2\mu} + k^{\mu+1})}\right\} \max_{|z|=1} |P(z)|$$

$$-\frac{1}{k^n}\left\{1 - \frac{1 + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}}{(1 + k^{\mu+1}) + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right|(k^{2\mu} + k^{\mu+1})}\right\} \min_{|z|=k} |P(z)|. \tag{15}$$

The following result which is an improvement as well as a generalization of a result due to Govil, Rahman and Schmeisser [11] (see also Qazi [16]) follows from Corollary 3 by making $R \to 1$.

Corollary 4. If $P(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j, \quad 1 \leq \mu \leq n$ is a polynomial of degree n, having no zeros in $|z| < k, \quad k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \leq n\left\{\frac{1 + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}}{(1 + k^{\mu+1}) + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{2\mu} + k^{\mu+1})}\right\} \max_{|z|=1} |P(z)|$$

$$-\frac{n}{k^n}\left\{1 - \frac{1 + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}}{(1 + k^{\mu+1}) + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right|(k^{2\mu} + k^{\mu+1})}\right\} \min_{|z|=k} |P(z)|. \tag{16}$$

It can be easily verified (see for refrence [6])that

$$\frac{1 + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}}{(1 + k^{\mu+1}) + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right|(k^{2\mu} + k^{\mu+1})} \leq \frac{1}{1 + k^\mu}. \tag{17}$$

Using (17) in (15),we get the following:

Corollary 5. If $P(z) = a_0 + \sum\limits_{j=\mu}^{n} a_j z^j$, is a polynomial of degree n, which does not vanish in the disk $|z| < k$ where $k \geq 1$, then for $R > 1$,

$$\left| \frac{P(Rz) - P(z)}{R^n - 1} \right| \leq \frac{1}{1 + k^\mu} \max_{|z|=1} |P(z)| - \frac{1}{k^{n-\mu}(1 + k^\mu)} \min_{|z|=k} |P(z)|. \tag{18}$$

The result is best possible either in case $\mu = n$ or $R \to 1$ and in both cases equality holds for polynomials $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$.

Remark 4. If we make $R \to 1$ in (18), then Corollary 5 not only gives a generalization of a result due to Malik [15], but also for $k = 1$ yields a refinement of Erdös conjecture proved by Lax [14].

Next, we consider a class of polynomials having a zero of order s at the origin and the rest of the zeros outside, or on the circle of radius k, $k \geq 1$ and prove the following result which generalizes some known L^q inequalities for polynomials. We prove:

Theorem 2. *If* $P(z) = z^s \left\{ a_0 + \sum\limits_{j=\mu}^{n-s} a_j z^j \right\}$, $1 \leq \mu \leq n - s$ *is a polynomial of degree n, having all its zeros in* $|z| \geq k$ *where* $k \geq 1$ *except s-fold zeros at the origin, then*

$$\left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(Re^{i\theta}) - P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}}$$

$$\leq \left\{ (R^s - 1) + \frac{R^n - R^s}{\left\{ \frac{1}{2\pi} \int_0^{2\pi} |C_{\mu k} + e^{i\alpha}|^q d\alpha \right\}^{\frac{1}{q}}} \right\} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}}, \tag{19}$$

where $C_{\mu k}$ *is given by (10).*

The following corollary immediately follows from Theorem 2, by dividing the two sides of (19) by $R - 1$ and making $R \to 1$.

Corollary 6. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \geq k$

where $k \geq 1$ except s-fold zeros at the origin, then for every $q > 1$,

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}|P'(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \leq \left\{s+\frac{n-s}{\left\{\frac{1}{2\pi}\int_0^{2\pi}|C_{\mu k}+e^{i\alpha}|^q d\alpha\right\}^{\frac{1}{q}}}\right\}\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}. \tag{20}$$

Remark 5. The result of Dewan, Bhat and Pukhta [9] is a special case of Corollary 6, when $s = 0$.

2. Lemmas

For the proofs of these theorems, we need the following lemmas:

Lemma 1. *Let* $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ *be a polynomial of degree* n*, having no zeros in* $|z| < k$ *where* $k \geq 1$*. If* $m = \min_{|z|=k}|P(z)|$*, then for every given complex number* β *with* $|\beta| \leq 1$ *and* $R \geq 1$*,*

$$\frac{|P(Rz)-P(z)+(R^n-1)m\beta k^{-n}z^n|}{\left|R^nP\left(\frac{z}{R}\right)-P(z)\right|} \leq \frac{1}{k^{\mu+1}}\left\{\frac{\frac{R^\mu-1}{R^n-1}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}+1}{1+\frac{R^\mu-1}{R^n-1}\left|\frac{a_\mu}{a_0}\right|k^{\mu-1}}\right\} \quad for \ \ |z|=1. \tag{21}$$

Proof of Lemma 1. The result is trivial if $R = 1$, so we suppose that $R > 1$. Since $P(z)$ has all zeros in $|z| \geq k$ where $k \geq 1$, therefore $P(kz)$ has all zeros in $|z| \geq 1$. Also, $m \leq |P(z)|$ for $|z| = k$, so that $m \leq |P(kz)|$ for $|z| = 1$. This gives for any β with $|\beta| < 1$, $|m\beta z^n| < |P(kz)|$ for $|z| = 1$. By Rouche's theorem the polynomial $F(z) = P(kz) + \beta m z^n$ has also all zeros in $|z| \geq 1$. Therefore, the polynomial $G(z) = z^n \overline{F\left(\frac{1}{\bar{z}}\right)}$ has all its zeros in $|z| \leq 1$ and $|F(z)| = |G(z)|$ for $|z| = 1$. Hence the function $\frac{G(z)}{F(z)}$ is analytic in $|z| < 1$ and

$\left|\frac{G(z)}{F(z)}\right| = \frac{|G(z)|}{|F(z)|} = 1$. A direct application of the maximum modulus principle shows that

$$|G(z)| \leq |F(z)| \quad for \quad |z| \leq 1. \tag{22}$$

We now show that all the zeros of $f(z) = F(z) - \alpha G(z)$ lie in $|z| \leq 1$ for every α with $|\alpha| > 1$. First suppose that $F(z)$ has all its zeros on $|z| = 1$. If $z_1, z_2, \cdots, z_n$ are zeros of $F(z)$, then $|z_j| = 1$ for all $j = 1, 2, \cdots, n$ and we have

$$F(z) = c\prod_{j=1}^{n}(z - z_j),$$

so that

$$G(z) = z^n \overline{F\left(\frac{1}{\bar{z}}\right)} = \bar{c}\prod_{j=1}^{n}(1 - z\bar{z}_j) = uF(z),$$

$$where \quad |u| = \left|\frac{\bar{c}}{c}(-1)^n \prod_{j=1}^{n}\frac{1}{z_j}\right| = 1.$$

This shows that all the zeros of $f(z) = F(z) - \alpha G(z) = (1 - \alpha u)F(z)$ also lie on $|z| = 1$ and inparticular in $|z| \leq 1$. Next, suppose that $F(z)$ has atleast one zero in $|z| < 1$, then obviously $\frac{G(z)}{F(z)}$ is not a constant and hence from (22), it follows that

$$|G(z)| < |F(z)| \quad for \quad |z| < 1. \tag{23}$$

Replacing z by $\frac{1}{\bar{z}}$ in (23), we get $|F(z)| < |G(z)|$, for $|z| > 1$. By Rouche's theorem, we conclude that the polynomial $f(z) = F(z) - \alpha G(z)$ has all its zeros in $|z| \leq 1$. Thus in any case the polynomial $f(z)$ has all its zeros in $|z| \leq 1$ for every α with $|\alpha| > 1$. Since $|f(z)| < |f(Rz)| \quad for \quad |z| = 1$ and $R > 1$, and all the zeros of $f(Rz)$ lie in $|z| \leq \frac{1}{R} < 1$, again Rouche's theorem shows that the polynomial

$$g(z) = f(Rz) - f(z) \tag{24}$$

$$= \{F(Rz) - F(z)\} - \alpha\{G(Rz) - G(z)\}$$

has all its zeros in $|z| < 1$, for every complex number α with $|\alpha| > 1$ and $R > 1$. This implies

$$|F(Rz) - F(z)| \leq |G(Rz) - G(z)| \quad for \quad |z| \geq 1 \quad and \quad R > 1. \tag{25}$$

If inequality (25) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$ such that $|F(Rz_0) - F(z_0)| > |G(Rz_0) - G(z_0)|$. Since $G(z)$ has all its zeros in $|z| \leq 1$, it follows that all the zeros of $G(Rz) - G(z)$ lie in $|z| < 1$, for every $R > 1$. Hence $G(Rz_0) - G(z_0) \neq 0$ for $|z_0| \geq 1$. We take $\alpha = \frac{F(Rz_0)-F(z_0)}{G(Rz_0)-G(z_0)}$, so that $|\alpha| > 1$ and with this choice of α, from (24), we get $g(z_0) = 0$, where $|z_0| \geq 1$. This contradicts the fact that all the zeros of $g(z)$ lie in $|z| < 1$. Thus $|F(Rz) - F(z)| \leq |G(Rz) - G(z)|\ for\ \ |z| \geq 1\ \ and\ \ R > 1$. Replacing $F(z)$ by $P(kz) + \beta m z^n$ and $G(z)$ by $z^n \overline{P\left(\frac{k}{\bar{z}}\right)} + \bar{\beta} m$, we get

$$|P(Rkz) - P(kz) + (R^n - 1)m\beta z^n| \leq \left|R^n z^n \overline{P\left(\frac{k}{R\bar{z}}\right)} - z^n \overline{P\left(\frac{k}{\bar{z}}\right)}\right|$$

$$= \left|R^n P\left(\frac{kz}{R}\right) - P(kz)\right| \quad for \quad |z| = 1 \quad and \quad R > 1.$$

Since the polynomial $R^n P\left(\frac{kz}{R}\right) - P(kz)$ does not vanish in $|z| \leq 1$, therefore $H(z) = \frac{P(Rkz)-P(kz)+(R^n-1)m\beta z^n}{R^n P\left(\frac{kz}{R}\right)-P(kz)}$ is analytic in $|z| \leq 1$ and by the maximum modulus principle, we have $|H(z)| \leq 1$, for $|z| \leq 1$. Also, it can be easily seen that $H(0) = H^{'}(0) = \cdots = H^{\mu-1}(0) = 0$ and $H^{\mu}(0) = \frac{R^\mu-1}{R^n-1}\left|\frac{a_\mu}{a_0}\right|k^\mu$. By a generalized form of Schwarz's lemma, we have

$$|H(z)| \leq |z|^\mu \; \frac{|z| + \frac{R^\mu-1}{R^n-1}\left|\frac{a_\mu}{a_0}\right|k^\mu}{\frac{R^\mu-1}{R^n-1}\left|\frac{a_\mu}{a_0}\right|k^\mu|z| + 1} \quad for \quad |z| \leq 1.$$

Equivalently

$$\left|\frac{P(Rkz) - P(kz) + (R^n - 1)m\beta z^n}{R^n P\left(\frac{kz}{R}\right) - P(kz)}\right| \leq |z|^\mu \frac{|z| + \frac{R^\mu-1}{R^n-1}\left|\frac{a_\mu}{a_0}\right|k^\mu}{\frac{R^\mu-1}{R^n-1}\left|\frac{a_\mu}{a_0}\right|k^\mu|z| + 1} \quad for \quad |z| \leq 1.$$

We take $z = \frac{e^{i\theta}}{k}$, $0 \leq \theta < 2\pi$, so that $|z| = \frac{1}{k}$ and we get

$$\left|\frac{P(Re^{i\theta}) - P(e^{i\theta}) + (R^n - 1)m\beta k^{-n}e^{in\theta}}{R^n P\left(\frac{e^{i\theta}}{R}\right) - P(e^{i\theta})}\right| \leq \frac{1}{k^{\mu+1}} \frac{1 + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}}{\frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| k^{\mu-1} + 1}.$$

This implies for $|z| = 1$,

$$\left|\frac{P(Rz) - P(z) + (R^n - 1)m\beta k^{-n}z^n}{R^n P\left(\frac{z}{R}\right) - P(z)}\right| \leq \frac{1}{k^{\mu+1}} \frac{1 + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}}{\frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| k^{\mu-1} + 1}.$$

This completes proof of Lemma 1.

The next lemma which we need is due to Aziz and Rather [3].

Lemma 2. *If $P(z)$ is a polynomial of degree n, then for each $q > 0$, $R \geq 1$, α real and $0 \leq \theta < 2\pi$,*

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|(P(Re^{i\theta}) - P(e^{i\theta})) + e^{i\alpha}\left(R^n P\left(\frac{e^{i\theta}}{R}\right) - P(e^{i\theta})\right)\right|^q d\theta\right\}^{\frac{1}{q}}$$

$$\leq (R^n - 1)\left\{\frac{1}{2\pi}\int_0^{2\pi}|P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}. \tag{26}$$

3. Proofs of the Theorems

Proof of Theorem 1. Applying Lemma 2 to the polynomial $P(z) + \frac{\beta m z^n}{k^n}$, which is of degree at most n, we get for every $q > 0$, $R \geq 1$, α real and $0 \leq \theta < 2\pi$,

$$\int_0^{2\pi}|F_1(\theta) + e^{i\alpha}G_1(\theta)|^q d\theta \leq (R^n - 1)^q \int_0^{2\pi}|P(e^{i\theta}) + \frac{\beta m e^{in\theta}}{k^n}|^q d\theta. \tag{27}$$

where

$$F_1(\theta) = P(Re^{i\theta}) - P(e^{i\theta}) + \beta m k^{-n}(R^n - 1)e^{in\theta}$$

and

$$G_1(\theta) = R^n P\left(\frac{e^{i\theta}}{R}\right) - P(e^{i\theta}).$$

Integrate both sides of (27) with respect to α from 0 to 2π, we have for $q > 0$,

$$\int_0^{2\pi}\int_0^{2\pi} |F_1(\theta) + e^{i\alpha}G_1(\theta)|^q d\alpha d\theta \leq 2\pi(R^n - 1)^q \int_0^{2\pi} |P(e^{i\theta}) + \beta m k^{-n} e^{in\theta}|^q d\theta. \quad (28)$$

Now since for every real α and $A \geq B \geq 1$, we have

$$|A + e^{i\alpha}| \geq |B + e^{i\alpha}|.$$

This gives

$$\int_0^{2\pi} |A + e^{i\alpha}|^q d\alpha \geq \int_0^{2\pi} |B + e^{i\alpha}|^q d\alpha, \quad q > 0. \quad (29)$$

If $F_1(\theta) \neq 0$, we take $A = \frac{|G_1(\theta)|}{|F_1(\theta)|}$ and $B = k^{\mu+1}\left\{\frac{\frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right|k^{\mu-1}+1}{1+\frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right|k^{\mu+1}}\right\}$, $1 \leq \mu \leq n$.

Since $P(z)$ is a polynomial of degree at most n, having no zeros in $|z| < k$, $k \geq 1$ then by Lemma 1, for $A \geq B \geq 1$, we get by using (29)

$$\int_0^{2\pi} |F_1(\theta) + e^{i\alpha}G_1(\theta)|^q d\alpha = |F_1(\theta)|^q \int_0^{2\pi} \left|1 + \frac{G_1(\theta)}{F_1(\theta)}e^{i\alpha}\right|^q d\alpha$$

$$= |F_1(\theta)|^q \int_0^{2\pi} \left|\frac{G_1(\theta)}{F_1(\theta)} + e^{i\alpha}\right|^q d\alpha$$

$$= |F_1(\theta)|^q \int_0^{2\pi} \left|\left|\frac{G_1(\theta)}{F_1(\theta)}\right| + e^{i\alpha}\right|^q d\alpha$$

$$\geq |F_1(\theta)|^q \int_0^{2\pi} \left|C_{\mu k} + e^{i\alpha}\right|^q d\alpha$$

$$= |P(Re^{i\theta}) - P(e^{i\theta}) + \beta m k^{-n}(R^n - 1)e^{in\theta}|^q \int_0^{2\pi} \left|C_{\mu k} + e^{i\alpha}\right|^q d\alpha. \tag{30}$$

For $F_1(\theta) = 0$, this inequality is trivially true. Using (30) in (28), we conclude that for each $q > 0$, $R \geq 1$, α real and $0 \leq \theta < 2\pi$,

$$\int_0^{2\pi} \left|C_{\mu k} + e^{i\alpha}\right|^q d\alpha \int_0^{2\pi} |P(Re^{i\theta}) - P(e^{i\theta}) + \beta m k^{-n}(R^n - 1)e^{in\theta}|^q d\theta$$

$$\leq 2\pi(R^n - 1)^q \int_0^{2\pi} |P(e^{i\theta}) + m\beta k^{-n} e^{in\theta}|^q d\theta.$$

This implies

$$\left\{\int_0^{2\pi} \left|\frac{P(Re^{i\theta}) - P(e^{i\theta})}{R^n - 1} + m\beta k^{-n} e^{in\theta}\right|^q d\theta\right\}^{\frac{1}{q}} \leq \frac{\left\{\int_0^{2\pi} |P(e^{i\theta}) + m\beta k^{-n} e^{in\theta}|^q d\theta\right\}^{\frac{1}{q}}}{\left\{\frac{1}{2\pi}\int_0^{2\pi} |C_{\mu k} + e^{i\alpha}|^q d\alpha\right\}^{\frac{1}{q}}},$$

where

$$C_{\mu k} = k^{\mu+1}\left\{\frac{\frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| k^{\mu-1} + 1}{1 + \frac{R^\mu - 1}{R^n - 1}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}}\right\}.$$

This proves Theorem 1.

Proof of Theorem 2. Since $P(z)$ has s-zeros at the origin, we write $P(z) = z^s f(z)$, where $f(z)$ has all its zeros in $|z| \geq k$, $k \geq 1$. So that

$$P(Rz) - P(z) = z^s[R^s f(Rz) - f(z)]$$

$$= z^s[(R^s - 1)f(z) + R^s f(Rz) - f(z)],$$

which implies for $0 \leq \theta < 2\pi$,

$$|P(Re^{i\theta}) - P(e^{i\theta})| = |(R^s - 1)f(e^{i\theta}) + R^s f(Re^{i\theta}) - f(e^{i\theta})|. \tag{31}$$

By Minkowski's inequality, we get from (31), for every $q \geq 1$,

$$\begin{aligned}\left\{\int_0^{2\pi} |P(Re^{i\theta}) - P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} &= \left\{\int_0^{2\pi} |(R^s - 1)f(e^{i\theta}) + R^s f(Re^{i\theta}) - f(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \\ &\leq (R^s - 1)\left\{\int_0^{2\pi} |f(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \\ &\quad + R^s\left\{\int_0^{2\pi} |f(Re^{i\theta}) - f(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}. \end{aligned} \tag{32}$$

Using inequality (9) with $\beta = 0$ and noting that $|f(e^{i\theta})| = |e^{is\theta} f(e^{i\theta})| = |P(e^{i\theta})|$, $0 \leq \theta < 2\pi$, we get from inequality (32)

$$\begin{aligned}\left\{\int_0^{2\pi} |P(Re^{i\theta}) - P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} &\leq (R^s - 1)\left\{\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \\ &\quad + R^s\left\{\frac{R^{n-s} - 1}{\left\{\frac{1}{2\pi}\int_0^{2\pi} |C_{\mu k} + e^{i\alpha}|^q d\alpha\right\}^{\frac{1}{q}}}\right\}\left\{\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \\ = \left\{(R^s - 1) + \frac{R^n - R^s}{\left\{\frac{1}{2\pi}\int_0^{2\pi} |C_{\mu k} + e^{i\alpha}|^q d\alpha\right\}^{\frac{1}{q}}}\right\}&\left\{\int_0^{2\pi} |P(e^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}.\end{aligned}$$

This completes proof of Theorem 2.

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