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Basic Subgroups in Semi-simple Abelian Group Rings *

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Abstract

We consider the question of finding in an explicit form a basic subgroup of the group $S(KG)$ of normalized invertible p-elements in a group ring KG where K is a field of $\mathrm{char}(K) \neq p$ and G is an abelian p-group. In some partial cases the problem is completely resolved.

Keywords and Phrases: *Basic subgroups, Group rings, Units, Pure subgroups, Divisible groups, Direct sums of cyclic groups.*

1. Introduction

Throughout the text of the present paper, let K be a field of characteristic distinct from a prime p and let G be an abelian p-group with a basic subgroup B. As usual, KG designates the group algebra of G over K with group $S(KG)$ of all normalized p-units. All other notations and terminology used are standard and follow essentially those from [F] and [K], [Ka].

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Inspired by [D] and [Da], the purpose here is to establish in an explicit form, which is convenient for further applications, a proper basic subgroup $B_{S(KG)}$ of $S(KG)$ which eventually depends on B.

In [Dan] we have proved that $S(KB) \subseteq B_{S(KG)}$ and that $B_{S(KG)} \cong S(KB)$, provided that K is the first kind field with respect to p.

We shall demonstrate in the sequel that $B_{S(KG)}$ coincides with $(1+I(KG; B))\cap S(KG)$ where $I(KG; B)$ is the relative augmentation ideal of KG with respect to B, provided K is a field of the first kind with respect to p and G is a direct sum of cyclic groups (in particular, a countable separable group).

Before doing that, we first need the following preparatory technicalities.

2. Preliminaries

Lemma 1. *Suppose R is a commutative unitary ring and A is a multiplicative abelian group with a subgroup C. Then $T = [1+I(RA; C)]\cap S(RA)$ is a group.*

Proof. Choose $a, b \in T$. Clearly $ab \in S(RA)$. Moreover, $a = 1 + a_1$ and $b = 1 + b_1$ where $a_1, b_1 \in I(RA; C)$. Thus $ab = (1 + a_1)(1 + b_1) = 1 + a_1 + b_1 + a_1b_1 \in 1 + I(RA; C)$. Hence $ab \in T$.

On the other hand, there exists a positive integer m such that $a^{p^m} = 1$. Therefore, $a^{-1} = a^{p^m-1} \in T$ applying inductively the first step. □

Moreover, motivated by the preceding lemma, we obtain:

Proposition 2. *Suppose R is a commutative unitary ring and A is a multiplicative abelian p-group with a subgroup C. Then the following two implications hold:*

(i) If C is separable and pure in A, then $(1 + I(RA; C)) \cap S(RA)$ is pure in $S(RA)$.

(ii) If A/C is divisible, then $S(RA)/[(1 + I(RA; C)) \cap S(RA)]$ is divisible, provided that R is the first kind field with respect to p.

Proof. (i) We shall show that $(1 + I(RA; C)) \cap S^{p^n}(RA) = [(1 + I(RA; C)) \cap S(RA)]^{p^n}$ for each natural $n \geq 1$. In order to do this, given x in the left hand-side. Hence $x \in (1 + I(RA; F)) \cap S(RA)$ where $F \leq C$ is a finite subgroup. Because of pureness of C in A, we observe that F can be regarded as a direct factor of A (see [F]). Consequently, referring to ([Da], Lemma 6), $(1+I(RA;F))\cap S(RA)$ is a direct factor of $S(RA)$, hence it is its pure subgroup. Finally, $x \in [(1+I(RA;F))\cap S(RA)]^{p^n} \subseteq [(1+I(RA;C))\cap S(RA)]^{p^n}$, which is equivalent to the promised equality as expected.

(ii) According to [Dan] or [Danc], $S(RA)/[(1+I(RA;C))\cap S(RA)] \cong S(R(A/C))$. However, it was proved in [Dan] that $S(R(A/C))$ is divisible and so the claim follows. □

Proposition 3. *Suppose G is an infinite group with a subgroup C and K is the first kind field with respect to p. Then*

$$|(1+I(KG;C))\cap S(KG)| = |G|.$$

Proof. First of all, observe that $|G| = |\mathcal{F}|$ where $\mathcal{F} = \{F : F \leq G$ is finite $\}$. Choose an idempotent $e \in KF$ and fix an element $b \in C$ with $b \notin F$; actually because $|G/\langle b\rangle| = |G|$ we may take $F \cap \langle b\rangle = \{1\}$ for almost all finite subgroups F of G.

Construct the element $1 \neq 1 - e + eb = 1 - e(1-b)$ which clearly belongs to $(1+I(KG;C))\cap S(KG)$. If u is another idempotent from KF, $u \neq e$, we deduce that $1 - e + eb \neq 1 - u + ub$. Otherwise $u - e = (u-e)b$ and since $u - e \in KF$ it follows immediately that $b \in F$, a contradiction. By the same token, if $e' \in KF'$ is an idempotent for some finite $F' \leq G$ such that $(FF')\cap\langle b\rangle = 1$, we see as above that $1-e+eb = 1-e'+e'b$ only when $e = e'$. This substantiates our equality. □

The following statement which is of an independent interest strengthens a lemma due to Mollov proved in [Mo] when G is separable. Here we omit this restriction on G.

Lemma 4. *Let P be a pure separable subgroup of the group G and let R be a commutative unitary ring. Then $S(RP)$ is pure in $S(RG)$.*

Proof. Take $x \in S(RP) \cap S^{p^n}(RG)$ where n is an arbitrary natural, whence $x \in S(RE)$ for some finite subgroup E of P. Because of the separability of P, we may assume that E is a direct factor of P. Thus E must be a direct factor of G (see, e.g., [F]). Furthermore, one easily checks that $S(RE)$ is a direct factor of $S(RG)$ and hence $x \in S^{p^n}(RE) \subseteq S^{p^n}(RP)$ as required. □

3. Basic Subgroups in Commutative Semi-Simple Group Algebras

We are now ready to establish a proper basic subgroup of $S(KG)$ (it is worthwhile noticing that in this situation when G is a direct sum of cyclic groups $S(KG)$ is a basic subgroup of itself).

Theorem 5. *Suppose G is an infinite direct sum of cyclic p-groups (in particular, a countable separable abelian p-group) and K is a field of the first kind with respect to p of char$(K) = p \neq 0$. Then*

$$B_{S(KG)} = (1 + I(KG; B)) \cap S(KG).$$

Proof. It is well known that each countable separable abelian p-group G is a direct sum of cyclic groups (see [F]). Appealing to [M] we deduce that $S(KG)$ is a direct sum of cyclic groups, whence so is its subgroup $(1 + I(KG; B)) \cap S(KG)$. Furthermore, we apply Proposition 2 to conclude that $(1 + I(KG; B)) \cap S(KG)$ is a proper basic subgroup of $S(KG)$, indeed. □

Remark. There is another method for attacking the proof. In fact, in accordance with Proposition 2 we need to show only that $(1 + I(KG; B)) \cap S(KG)$ is a direct sum of cyclic groups. Indeed, write $B = \cup_{n<\omega} B_n$ where $B_n \subseteq B_{n+1}$ are finite and pure in B, whence pure in G. Thus $(1 + I(KG; B)) \cap S(KG) = \cup_{n<\omega}[(1 + I(KG; B_n)) \cap S(KG)]$. It is only a technical exercise to check that the members of the union have only a finite number of finite heights in $S(KG)$. On the other hand, by Proposition 2(i) they are also pure in $S(KG)$, and hence in the former group $(1 + I(KG; B)) \cap S(KG)$. Finally, we wish apply [H] to deduce the wanted claim, whence we obtain the stated equality. □

Note. In the case when G is countable, it follows from [M] or [Dan] that $S(KG)$ is countable, whence so is its subgroup $(1 + I(KG; B)) \cap S(KG)$. Furthermore, to show once again that this subgroup is a direct sum of cyclic groups, it is strongly enough to prove that it is separable. However, the idea of proof is analogous to that in Theorem 5.

Standardly, A_d and A_r denote the maximal divisible subgroup and the reduced part of an abelian p-group A, respectively.

Theorem 6. *Suppose G is an abelian p-group and K is a field of the second kind with respect to p of char$(K) \neq p$. Then*

$$B_{S(KG)} = S(KG)_r.$$

Proof. Employing [M], we write $S(KG) = S(KG)_d \times S(KG)_r$ where $S(KG)_r$ is a direct sum of cyclic groups. Moreover, $S(KG)_r$ is pure in $S(KG)$ and $S(KG)/S(KG)_r \cong S(KG)_d$ is divisible. So, the equality follows. □

We finish with the following crucial

Remark. Proposition 3 illustrates that Theorem 5 is not longer true when G is not starred, i.e., $|G| > |B|$. In fact, if $|G| = \aleph_1 > \aleph_0 = |B|$, then $B_{S(KG)} \neq (1 + I(KG; B)) \cap S(KG)$ since by virtue of [Dan] we have $B_{S(KG)} \cong S(KB)$, whence in view of [M] we have $|B_{S(KG)}| = |B| = \aleph_0$ whereas Proposition 3 tells us that $|(1 + I(KG; B)) \cap S(KG)| = |G| = \aleph_1$.

References

[1] P. V. Danchev, Sylow p-subgroups of commutative modular and semisimple group rings, *Compt. rend. Acad. bulg. Sci.*, **54** no.6 (2001), 5-6.

[2] P. V. Danchev, Basic subgroups in abelian group rings, *Czechoslovak Math. J.*, **52** no.1 (2002), 129-140.

[3] P. V. Danchev, Sylow p-subgroups of abelian group rings, *Serdica Math. J.*, **29** no.1 (2003), 33-44.

[4] P. V. Danchev, Ulm-Kaplansky invariants of $S(KG)/G$, *Bull. Polish Acad. Sci. -Math.*, **53** no.3 (2005), 147-156.

[5] P. V. Danchev, On the coproducts of cyclics in commutative modular and semisimple group rings, *Bull. Acad. Sci. Moldova - Math.*, **51** no.2 (2006), 26-33.

[6] L. Fuchs, *Infinite Abelian Groups*, volume I, Mir, Moscow, 1974. (In Russian.)

[7] P. D. Hill, Primary groups whose subgroups of smaller cardinality are direct sums of cyclic groups, *Pac. J. Math.*, **42** no.1 (1972), 63-67.

[8] G. Karpilovsky, *Commutative Group Algebras*, Marcel Dekker, New York, 1983.

[9] G. Karpilovsky, *Unit Groups of Group Rings*, John Wiley and Sons, New York, 1989.

[10] T. Zh. Mollov, Sylow p-subgroups of the normalized unit groups of semisimple group algebras of uncountable abelian p-groups, *Pliska Stud. Math. Bulg.*, **8** (1986), 34-46. (In Russian.)

[11] T. Zh. Mollov, Ulm-Kaplansky invariants of the Sylow p-subgroups of the normed unit groups of semisimple group algebras of infinite separable abelian p-groups, *Pliska Stud. Math. Bulg.*, **8** (1986), 101-106. (In Russian.)