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A Note on the Generalization of Some New Čebyšev Type Inequalities *

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Abstract

In this paper, we present a generalized Čebyšev type inequality for absolutely continuous functions whose derivatives belong to $L_p[a,b]$, $p>1$. Applications for probabilty density functions are also given.

Keywords and Phrases: *Čebyšev type inequalities, Absolutely continuous function, Trapezoid like rule, Probability density function.*

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1. Introduction

For two measurable functions $f,\ g : [a,b] \to \mathbb{R}$, define the functional,

$$T(f,g;a,b) := \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1.1}$$

which is in literature called the Čebyšev functional, provided the integrals in (1.1) exists.

Moreover, in 1882 P. L. Čebyšev (see [5], p. 297) proved that,
if $f',\ g' \in L_\infty[a,b]$, then

$$|T(f,g;a,b)| \le \frac{1}{12}(b-a)^2 \left\|f'\right\|_\infty \left\|g'\right\|_\infty. \tag{1.2}$$

In the recent past, Čebyšev functional has remained an area of special interest for many researchers and has yielded many variants and generalizations in the field of inequalities. It has also played a key role in obtaining some new inequalities of Ostrowski type, for example, Ostrowski-Grüss type, Ostrowski-Čebyšev type, etc. The research papers [1, 2, 6] cover a comprehensive literature on the generalizations of Čebyšev functional and its associated bounds.

In [7], B. G. Pachpatte presented the following Čebyšev type inequality in L_p norm:

Theorem 1. *Let $f,g : [a,b] \to \mathbb{R}$ be absolutely continuous functions whose derivatives $f',\ g' \in L_p[a,b]$, $p > 1$ then*

$$|P(C,D,f,g)| \le \frac{1}{(b-a)^2} M^{\frac{2}{q}} \|f'\|_p \|g'\|_p, \tag{1.3}$$

where

$$\begin{aligned} C &= \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right], \\ D &= \frac{1}{3}\left[\frac{g(a)+g(b)}{2} + 2g\left(\frac{a+b}{2}\right)\right], \end{aligned}$$

$$M = \frac{\left(2^{q+1}+1\right)(b-a)^{q+1}}{3\,(q+1)\,6^{q}}$$

with $\frac{1}{p}+\frac{1}{q}=1,$ *and*

$$\|f\|_p = \left(\int_a^b |f(t)|^p\, dt\right)^{\frac{1}{p}} < \infty.$$

$$\begin{aligned} P(\alpha,\beta,f,g) &= \alpha\beta - \frac{1}{b-a}\left(\alpha\int_a^b g(t)\,dt + \beta\int_a^b f(t)\,dt\right) \\ &\quad + \left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right), \qquad (1.4) \end{aligned}$$

α and β are real constants.

Recently, in [4], Zheng Liu presented the following generalization of (1.3):

Theorem 2. *Let the assumptions of Theorem 1 hold, then for any $\theta \in [0,1]$,*

$$|P(\Gamma_\theta,\Delta_\theta,f,g)| \le \frac{1}{(b-a)^2} M_\theta^{\frac{2}{q}} \|f'\|_p \|g'\|_p\,, \qquad (1.5)$$

where

$$M_\theta = \frac{\theta^{q+1}+(1-\theta)^{q+1}}{(q+1)\,2^{q}}(b-a)^{q+1}\,,$$

and

$$\begin{aligned} \Gamma_\theta &= \frac{\theta}{2}[f(a)+f(b)] + (1-\theta) f\left(\frac{a+b}{2}\right), \\ \Delta_\theta &= \frac{\theta}{2}[g(a)+g(b)] + (1-\theta) g\left(\frac{a+b}{2}\right). \end{aligned}$$

In this paper, we obtain a generalization of the inequalities (1.3) and (1.5) and apply them to probability density functions.

2. Main Results

For suitable functions $f, g : [a, b] \to \mathbb{R}$ and $h \in [0, 1]$, we present the following notations:

$$\Gamma_{h,x} = (1-h) f(x) + h\left(\frac{(x-a) f(a) + (b-x) f(b)}{b-a}\right),$$

$$\Delta_{h,x} = (1-h) g(x) + h\left(\frac{(x-a) g(a) + (b-x) g(b)}{b-a}\right). \tag{2.1}$$

and $P(\alpha, \beta, f, g)$ is as defined above in (1.4).

The following result holds:

Theorem 3. *Let the assumptions of Theorem 1 hold, then for any $h \in [0, 1]$ and $x \in [a, b]$, we have:*

$$\left|P\left(\Gamma_{h,x}, \Delta_{h,x}, f, g\right)\right| \leq \frac{1}{(b-a)^2} M_{h,x}^{\frac{2}{q}} \left\|f'\right\|_p \left\|g'\right\|_p \tag{2.2}$$

where $\Gamma_{h,x}$ and $\Delta_{h,x}$ are as defined by (2.1) and

$$M_{h,x} = \frac{1}{q+1}\left[h^{q+1} + (1-h)^{q+1}\right]\left[(x-a)^{q+1} + (b-x)^{q+1}\right]. \tag{2.3}$$

Proof. We define the function

$$k(x,t;h) = \begin{cases} t-(1-h)a-hx, & t \in [a,x], \\ t-(1-h)b-hx, & t \in (x,b]. \end{cases}$$

Then, we obtain the following identities:

$$\Gamma_{h,x} - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b k(x,t;h) f^{'}(t)\,dt, \tag{2.4}$$

$$\Delta_{h,x} - \frac{1}{b-a}\int_a^b g(t)\,dt = \frac{1}{b-a}\int_a^b k(x,t;h) g^{'}(t)\,dt. \tag{2.5}$$

Multiplying the left and right hand side of (2.4) and (2.5), we get,

$$P(\Gamma_{h,x}, \Delta_{h,x}, f, g) = \frac{1}{(b-a)^2} \left(\int_a^b k(x,t;h) f'(t)\, dt \right) \left(\int_a^b k(x,t;h) g'(t)\, dt \right),$$

implies

$$|P(\Gamma_{h,x}, \Delta_{h,x}, f, g)| \leq \frac{1}{(b-a)^2} \left(\int_a^b |k(x,t;h)| \left| f'(t) \right| dt \right) \left(\int_a^b |k(x,t;h)| \left| g'(t) \right| dt \right). \tag{2.6}$$

Thus, by using the Hölder's integral inequality:

$$\begin{aligned} & |P(\Gamma_{h,x}, \Delta_{h,x}, f, g)| \\ \leq\ & \frac{1}{(b-a)^2} \left[\left(\int_a^b |k(x,t;h)|^q dt \right)^{\frac{1}{q}} \left(\int_a^b \left| f'(t) \right|^p dt \right)^{\frac{1}{p}} \right] \\ & \times \left[\left(\int_a^b |k(x,t;h)|^q dt \right)^{\frac{1}{q}} \left(\int_a^b \left| g'(t) \right|^p dt \right)^{\frac{1}{p}} \right] \\ =\ & \frac{1}{(b-a)^2} \left(\int_a^b |k(x,t;h)|^q dt \right)^{\frac{2}{q}} \left\| f' \right\|_p \left\| g' \right\|_p . \end{aligned} \tag{2.7}$$

From the definition of $k(x,t;h)$, it follows that

$$\int_a^b |k(x,t;h)|^q dt = \frac{1}{(q+1)} \left[h^{q+1} + (1-h)^{q+1} \right] \left[(x-a)^{q+1} + (b-x)^{q+1} \right]. \tag{2.8}$$

By using (2.7) – (2.8), (2.2) follows. □

Remark 1. *For $x = \frac{a+b}{2}$, $h = \frac{1}{3}$ in (2.2), (1.3) is recaptured.*

Remark 2. *For $x = \frac{a+b}{2}$ in (2.2), (1.5) is recaptured.*

We, now, state a special case of Theorem 3 in the form of the following corollary:

Corollary 1. *Let the assumptions of Theorem 1 hold, then*

$$\left|P\left(\Gamma_{1,\frac{a+b}{2}}, \Delta_{1,\frac{a+b}{2}}, f, g\right)\right|$$
$$\leq \frac{1}{(b-a)^2} M_{1,\frac{a+b}{2}}^{\frac{2}{q}} \|f'\|_p \|g'\|_p \tag{2.9}$$

where

$$M_{1,\frac{a+b}{2}} = \frac{1}{2^q (q+1)} (b-a)^{q+1}, \tag{2.10}$$

and

$$\Gamma_{1,\frac{a+b}{2}} = \frac{f(a)+f(b)}{2},$$
$$\Delta_{1,\frac{a+b}{2}} = \frac{g(a)+g(b)}{2}. \tag{2.11}$$

We, now apply (2.9) to probability density functions as follows:

3. Applications for PDF's

Let X be a continuous random variable with the probability density function $f : [a,b] \to \mathbb{R}_+$ and the expectation of X is given by

$$E(X) = \int_a^b t\ f(t)\,dt. \tag{3.1}$$

The cumulative distribution function F is given as:

$$F(x) = \int_a^x f(t)\,dt, \tag{3.2}$$

for $x \in [a,b]$.

Moreover, let Y be another continuous variable with the probability density function $h : [a,b] \to \mathbb{R}_+$ and the expectation of Y is given by

$$E(Y) = \int_a^b t\ h(t)\,dt. \tag{3.3}$$

The cumulative distribution function H is given as:

$$H(y) = \int_a^y h(t)\,dt, \tag{3.4}$$

for $y \in [a,b]$. Then,

$$\begin{aligned} \int_a^b F(x)\,dx &= b - E(X), \\ F(a) &= 0,\ F(b) = 1, \\ \frac{F(a)+F(b)}{2} &= \frac{1}{2} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \int_a^b H(y)\,dy &= b - E(Y), \\ H(a) &= 0,\ H(b) = 1, \\ \frac{H(a)+H(b)}{2} &= \frac{1}{2}. \end{aligned} \tag{3.6}$$

The following proposition holds:

Proposition 1. *Let X, Y, F and H be defined as above. Then, the following holds:*

$$\left|\frac{1}{4}\left(1-\left(\frac{E(Y)-E(X)}{b-a}\right)\right)-\frac{1}{b-a}\left(b-\frac{E(X)+E(Y)}{2}\right)\left(1-\frac{b-\frac{E(X)+E(Y)}{2}}{b-a}\right)\right|$$

$$\leq \frac{1}{4}\left(\frac{b-a}{q+1}\right)^{\frac{2}{q}}\|f\|_p\,\|h\|_p\,. \tag{3.7}$$

Proof. By choosing $f = F$ and $g = H$ in (2.9)-(2.11) and simplifying with the help of (3.1)-(3.6), we get the required inequality. □

Remark 3. *If in (3.7), we choose $F = H$, then we have:*

$$\left|\frac{1}{2} - \frac{1}{b-a}\left(b - E\left(X\right)\right)\right| \leq \frac{1}{2}\left(\frac{b-a}{q+1}\right)^{\frac{1}{q}} \left\|h\right\|_p , \tag{3.8}$$

which is known in literature as "trapezoid inequality" for cumulative distribution functions (see [3], p. 34 for $f = H$).

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