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Generalized Vector Valued Sequence Spaces Defined by Orlicz Functions*

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Abstract

In this paper we introduce the vector valued sequence spaces $w_0(M,\theta,\Delta^m,Q,p,u)$, $w_1(M,\theta,\Delta^m,Q,p,u)$, $w_\infty(M,\theta,\Delta^m,Q,p,u)$ and $S_\theta(\Delta^m_{uq})$ using an Orlicz function, the generalized difference operator Δ^m and the multiplier sequence $u=(u_k)$ of non-zero complex numbers. We give some relations related to these sequence spaces. It is also shown that if a sequence is strongly lacunary $\Delta^m_{uq}-$ Cesàro summable with respect to the Orlicz function M then it is $\Delta^m_{uq}-$ statistically convergent.

Keywords and Phrases: *Orlicz function, Seminorm, Sequence spaces.*

1. Introduction

Let w be the set of *all* sequences of real (or complex) numbers and ℓ_∞, c and c_0 be the linear spaces of *bounded*, *convergent* and *null* sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N} = \{1, 2, ...\}$, the set of positive integers. Throughout the article $w(X)$,

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$c(X)$, $c_0(X)$ and $\ell_\infty(X)$ will represent the spaces of *all*, *convergent*, *null* and *bounded* X valued sequence spaces. For $X = \mathbb{C}$, the field of complex numbers, these represent the corresponding scalar valued sequence spaces. The zero sequence is denoted by $\bar{\Theta} = (\Theta, \Theta, ...)$, where Θ is the zero element of X.

The studies on vector valued sequence spaces are done by Rath and Srivastava [19] , Das and Choudhary [4] , Leonard [13], Srivastava and Srivastava [22] , Tripathy and Sen [23] , Et *et al.* [6] and many others.

Let $u = (u_k)$ be a sequences of non-zero scalar. Then for a sequence space E, the multiplier sequnce space $E(u)$, associated with the multiplier sequnce u is defined as

$$E(u) = \{(x_k) \in w : (u_k x_k) \in E\}.$$

The studies on the multiplier sequnce spaces are done by Çolak [3], Srivastava and Srivastava [22] and many others.

The notion of difference sequence space was introduced by Kızmaz [11]. It was generalized by Et and Çolak [5] as follows:

$$X(\Delta^m) = \{(x_k) \in w : (\Delta^m x_k) \in X\},$$

for $X = c, c_0$ and ℓ_∞, where $m \in \mathbb{N}$, $\Delta^m x_k = \Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}$ and $\Delta^m x_k = \sum\limits_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$ for all $k \in \mathbb{N}$.

By a lacunary sequence $\theta = (k_r)$; $r = 0, 1, 2, ...$, where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. We denote $I_r = (k_{r-1}, k_r]$ and $\eta_r = \frac{k_r}{k_{r-1}}$ for $r = 0, 1, 2, ...$ The space of lacunary strongly convergent sequnces N_θ was defined by Freedman *et al.* [8] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_{r\to\infty} h_r^{-1} \sum_{k\in I_r} |x_k - \ell| = 0, \text{ for some } \ell \right\}.$$

The space N_θ is a BK - space with norm

$$\|(x)\|_\theta = \sup_r \left(h_r^{-1} \sum_{k\in I_r} |(x_k)| \right).$$

There is relation (see for instance [8]) between N_θ and the space $|\sigma_1|$ of strongly Cesàro summable sequences, which is defined by

$$|\sigma_1| = \left\{ x = (x_k) : \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} |x_k - \ell| = 0, \text{ for some } l \right\}.$$

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ and for all sequence (α_k) of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

A sequence space E is said to be symmetric if $(x_n) \in E$ implies $\left(x_{\pi(n)}\right) \in E$, where $\pi(n)$ is a permutation of elements of $\mathbb{N}$.

A sequence space E is said to be monotone if E contains preimages of all its step spaces.

A sequence space E is said to be sequence algebra if $x.y \in E$ whenever $x, y \in E$.

Lemma 1. *A sequence space E is solid implies E is monotone.*

An Orlicz function is a function on M :$[0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [14] used the notion of Orlicz function to construct the sequence space

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1, \text{ for some } \rho > 0 \right\}$$

become a Banach space, which is called an Orlicz sequence space. The space ℓ_M is closely related to the ℓ_p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$.

The Cesàro summable sequence space defined by Orlicz functions were studied by Parashar and Choudhary [18], Bhardwaj and Singh [1], Mursaleen *et al* [17], Et *et al* [6] and many others.

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Let q_1 and q_2 be seminorms on a vector space X. Then q_1 is said to be stronger than q_2 if whenever (x_n) is a sequence such that $q_1(x_n) \to 0$, than also $q_2(x_n) \to 0$. If each is stronger than the others q_1 and q_2 are said to be equivalent (one may refer to Wilansky [25]).

2. Main Results

In this section we prove some results involving the sequence spaces $w_\infty(M,\theta,\Delta^m,Q,p,u)$, $w_1(M,\theta,\Delta^m,Q,p,u)$ and $w_0(M,\theta,\Delta^m,Q,p,u)$.

Definition 1. Let $p=(p_k)$ be a sequence of strictly positive real numbers, M be an Orlicz function, X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q_k for each $k\in\mathbb{N}$ and $u=(u_k)$ be any fixed sequence of non-zero complex numbers u_k. We define the following sequence spaces:

$$w_0(M,\theta,\Delta^m,Q,p,u)= \left\{ \begin{array}{c} x\in w(X): \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(q_k\left(\frac{u_k\Delta^m x_k}{\rho}\right)\right)\right]^{p_k}\to 0, \text{ as } r\to\infty \\ \text{for some } \rho>0 \end{array}\right\},$$

$$w_1(M,\theta,\Delta^m,Q,p,u,)= \left\{ \begin{array}{c} x\in w(X): \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(q_k\left(\dfrac{u_k\Delta^m x_k-\ell}{\rho}\right)\right)\right]^{p_k}\to 0, \\ \text{as } r\to\infty, \text{for some } \rho>0 \quad \text{and } \ell\in X \end{array}\right\},$$

$$w_\infty(M,\theta,\Delta^m,Q,p,u)= \left\{ \begin{array}{c} x\in w(X): \sup_r \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(q_k\left(\frac{u_k\Delta^m x_k}{\rho}\right)\right)\right]^{p_k}<\infty, \\ \text{for some } \rho>0 \end{array}\right\}.$$

Throughout the paper Z will denote any one of the notation 0,1,∞. In the case $q_k=q$ for all $k\in\mathbb{N}$, we shall write $w_Z(M,\theta,\Delta^m,q,p,u)$ instead of $w_Z(M,\theta,\Delta^m,Q,p,u)$. If $x\in w_1(M,\theta,\Delta^m,q,p,u)$, we say that x is strongly lacunary $\Delta_p^m u_q$-Cesàro summable with respect to the Orlicz function M and we will write $x_k\to\ell\,(w_1(M,\theta,\Delta^m,q,p,u))$ and ℓ will be called $\Delta_p^m\ u_q-$limit of x with respect to the Orlicz function M. If $u_k=1$ and $p_k=1$ for all $k\in\mathbb{N}$, we obtain $w_Z(M,\theta,\Delta^m,Q)$ instead of $w_Z(M,\theta,\Delta^m,Q,p,u)$.

The proofs of the following theorems are obtained by using the known standard techniques, therefore we give them without proofs.

Theorem 2.1. *Let the sequence (p_k) be bounded. Then the spaces $w_Z(M,\theta,\Delta^m,q,p,u)$ are linear spaces.*

Theorem 2.2. *Let M be an Orlicz function and the sequence (p_k) be bounded, then*

$$w_0\left(M,\theta,\Delta^m,q,p,u\right)\subset w_1\left(M,\theta,\Delta^m,q,p,u\right)\subset w_\infty\left(M,\theta,\Delta^m,q,p,u\right)$$

and the inclusions are strict.

Theorem 2.3. *$w_0\left(M,\theta,\Delta^m,q,p,u\right)$ is a paranormed (need not total paranorm) space with*

$$g_\Delta(x)=\sum_{i=1}^m q(x_i)+\inf_{\rho>0}\inf_{n\geq 1}\{\rho^{\frac{p_n}{H}}:\sup_k\{[M(q(\frac{u_k\Delta^m x_k}{\rho}))]\}\leq 1,\rho>0,n\in\mathbb{N}\},$$

where $H=\max(1,\sup_k p_k)$.

Theorem 2.4. *Let M_1 and M_2 any two Orlicz functions. For any bounded sequences $p=(p_k)$ and $t=(t_k)$ of strictly positive real numbers and for any two sequences of seminorms $q=(q_k)$, $r=(r_k)$ we have*

i) *$w_Z\left(M_1,\theta,\Delta^m,Q,p,u\right)\cap w_Z\left(M_1,\theta,\Delta^m,R,p,u\right)\subset w_Z\left(M_1,\theta,\Delta^m,Q+R,p,u\right)$,*

ii) *$w_Z\left(M_1,\theta,\Delta^m,Q,p,u\right)\cap w_Z\left(M_2,\theta,\Delta^m,Q,p,u\right)\subset w_Z\left(M_1+M_2,\theta,\Delta^m,Q,p,u\right)$,*

iii) *If q is stronger than r then $w_Z\left(M_1,\theta,\Delta^m,q,p,u\right)\subset\left(M_1,\theta,\Delta^m,r,p,u\right)$,*

iv) *If q equivalent to r then $w_Z\left(M_1,\theta,\Delta^m,q,p,u\right)=\left(M_1,\theta,\Delta^m,r,p,u\right)$,*

v) *$w_Z\left(M_1,\theta,\Delta^m,Q,p,u\right)\cap w_Z\left(M_1,\theta,\Delta^m,R,p,u\right)\neq\emptyset$.*

Theorem 2.5. *If $m\geq 1$, then $w_Z\left(M,\theta,\Delta^{m-1},Q\right)\subset w_Z\left(M,\theta,\Delta^m,Q\right)$. In general, $w_Z\left(M,\theta,\Delta^i,Q\right)\subset w_Z\left(M,\theta,\Delta^m,Q\right)$ for $i=1,2,...,m-1$.*

Proof. We prove it for $Z=0$. Let $(x_k)\in w_0\left(M,\theta,\Delta^{m-1},Q\right)$. Then

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(q_k\left(\frac{\Delta^{m-1}x_k}{\rho}\right)\right)\right]=0,\ \text{for some }\rho>0$$

Since M is non-decrasing, convex and q_k is semi-norm, for all $k\in\mathbb{N}$ we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in Ir}\left[M\left(q_k\left(\frac{\Delta^m x_k}{2\rho}\right)\right)\right] &= \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(q_k\left(\frac{(\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1})}{2\rho}\right)\right)\right]\\ \leq &\left\{\frac{1}{h_r}\sum\left[\frac{1}{2}M\left(q_k\left(\frac{\Delta^{m-1}x_k}{\rho}\right)\right)\right]+\frac{1}{h_r}\sum\left[\frac{1}{2}M\left(q_k\left(\frac{\Delta^{m-1}x_{k+1}}{\rho}\right)\right)\right]\right\}\\ &< \left\{\frac{1}{h_r}\sum M\left(q_k\left(\frac{\Delta^{m-1}x_k}{\rho}\right)\right)+\frac{1}{h_r}\sum M\left(q_k\left(\frac{\Delta^{m-1}x_{k+1}}{\rho}\right)\right)\right\}.\end{aligned}$$

Hence $x \in w_0(M,\theta,\Delta^m,Q)$. Proceeding inductively we have $w_Z(M,\theta,\Delta^i,Q) \subset w_Z(M,\theta,\Delta^m,Q)$ for $i = 1,2,...,m-1$.

The above inclusion is strict. For this consider the following example.

Example 1. Let $X = \mathbb{C}$, $M(x) = x$, $q_k(x) = |x|$, $u_k = 1$ and $p_k = 1$ for all $k \in \mathbb{N}$. We consider the lacunary sequence $\theta = (2^r)$ and the sequence $x_k = k^{m-1}$. Then $\Delta^m(x_k) = 0$, $\Delta^{m-1}(x_k) = (-1)^{m-1}(m-1)!$, therefore $x \in w_0(M,\theta,\Delta^m,Q)$ but $x \notin w_0(M,\theta,\Delta^{m-1},Q)$.

Theorem 2.6. *Let $0 < p_k \leq t_k$ and $\left(\frac{t_k}{p_k}\right)$ be bounded. Then $w_Z(M,\theta,\Delta^m,Q,t,u) \subset w_Z(M,\theta,\Delta^m,Q,p,u)$.*

Proof. If we take $w_k = \left[M\left(q_k\left(\frac{u_k\Delta^m x_k}{\rho}\right)\right)\right]^{t_k}$ for all $k \in \mathbb{N}$, then following technique applied for proving Theorem 5 by Maddox [16], the theorem can be easly proved.

Theorem 2.7. *The spaces $w_Z(M,\theta,\Delta^m,Q,p,u)$ are not solid, in general.*

For showing that the spaces $w_Z(M,\theta,\Delta^m,Q,p,u)$ are not solid, consider the following example.

Example 2. Let $X = \mathbb{C}$, $M(x) = x$, $m \geq 2$, $q_k(x) = |x|$, $u_k = 1$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence $x_k = k$, for all $k \in \mathbb{N}$ and the lacunary sequence $\theta = (2^r)$. Then $(x_k) \in w_Z(M,\theta,\Delta^m,Q,p,u)$. Let $\alpha_k = (-1)^k$ for all $k \in \mathbb{N}$, then $(\alpha_k x_k) \notin w_Z(M,\theta,\Delta^m,Q,p,u)$.

Theorem 2.8. *The spaces $w_Z(M,\theta,\Delta^m,Q,p,u)$ are not symmetric in general.*

Proof. To show that the spaces are not symmetric in general, consider the following example.

Example 3. Let $X = \mathbb{C}$, $M(x) = x$, $m \geq 2$, $q_k(x) = |x|$, $u_k = 1$ and $p_k = 1$ for all $k \in \mathbb{N}$ and consider the lacunary sequence $\theta = (2^r)$. Then the sequence (x_k) defined by $x_k = k$ for all $k \in \mathbb{N}$ is in $w_Z(M,\theta,\Delta^m,Q,p,u)$. Consider the following sequence (y_k) which is the rearrangement of the sequnce (x_k) defined as

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10,}...\}$$

Then (y_k) does not belong to $w_Z(M,\theta,\Delta^m,Q,p,u)$.

The proofs of the following two results are obtained by using the known standard techniques, therefore we give them without proofs.

Proposition 2.9. Let $\theta = (k_r)$ be a lacunary sequence with lim $\inf_r \eta_r > 1$, then for any Orlicz function M and a sequence of a seminorms $q = (q_k)$, $w_0(M, \Delta^m, Q, p, u) \subset w_0(M, \theta, \Delta^m, Q, p, u)$,
where

$$w_0(M, \Delta^m, Q, p, u) = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[M\left(q_k \left(\frac{u_k \Delta^m x_k}{\rho} \right) \right) \right]^{p_k} = 0, \text{ for some } \rho > 0 \right\}.$$

Proposition 2.10. Let $\theta = (k_r)$ be a lacunary sequence with $\text{limsup}_r \eta_r < \infty$, then for any Orlicz function M, $w_Z(M, \theta, \Delta^m, Q, p, u) \subset w_Z(M, \Delta^m, Q, p, u)$.

Theorem 2.11. *Let $\theta = (k_r)$ be a lacunary sequence with 1 < lim $\inf_r \eta_r \leq$ $\text{limsup}_r \eta_r < \infty$, then for any Orlicz M, $w_Z(M, \theta, \Delta^m, q, p, u) = w_Z(M, \Delta^m, q, p, u)$.*

Proof. The proof follows from Proposition 2.9 and Proposition 2.10.

Theorem 2.12. *The sequence spaces $w_0(M, \theta, \Delta^m, q, p, u)$, $w_1(M, \theta, \Delta^m, q, p, u)$ and $w_\infty(M, \theta, \Delta^m, q, p, u)$ are not sequence algebras for $m > 1$.*

Proof. To show that the spaces are not sequnce algebras, consider the following example.

Example 4. Let $X = \mathbb{C}$, $M(x) = x$, $\theta = (2^r)$, $m \geq 2$, $q_k(x) = |x|$, $u_k = 1$ and $p_k = 1$ for all $k \in \mathbb{N}$. Consider the sequence $x = (k)$, $y = (k^{m-1})$, then $x, y \in w_0(M, \theta, \Delta^m, Q, p, u)$ but $x.y \notin w_0(M, \theta, \Delta^m, Q, p, u)$. For the others consider the sequences $x = (k)$, $y = (k^m)$.

3. Lacunary $\Delta^m u_q-$ Statistical Convergence

The notion of statistical convergence were introduced by Fast [7] and Schoenberg [21], independently. Over the years and under differrent names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory. Later on it was further investigated from sequence spaces point of view and linked summability theory by Fridy and Orhan [9], Connor [2], Sálát [20], Mursaleen *et al.*[17], Işık [10], Kolk [12], Maddox [15], Tripathy and Sen [24] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and structure of ideals of bounded countinous functions on locally compact spaces.

The density of a subset E of $\mathbb{N}$ is defined by

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_E(k) \text{ provided the limit exists,}$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta\left(E^c\right) = 1 - \delta\left(E\right)$.

In this section we introduce lacunary $\Delta^m u_q-$statistically convergent sequences and give some relations between lacunary $\Delta^m u_q-$statistically convergent sequences and $w_1\left(M, \theta, \Delta^m, q, p, u\right)-$
summable sequences.

Definition 3. A sequence $x = (x_k)$ is said to be lacunary $\Delta^m u_q-$statistically convergent to ℓ, if for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : q\left(u_k \Delta^m x_k - \ell\right) \geq \varepsilon\right\}\right| = 0.$$

In this case we write $x_k \to \ell\left(\left(S_\theta\left(\Delta^m_{uq}\right)\right)\right)$. The set of all lacunary $\Delta^m u_q-$statistically convergent sequences is denoted by $S_\theta\left(\Delta^m_{uq}\right)$.

Theorem 3.1. *Let θ be a lacunary sequence. Then*

i) *If $x_k \to \ell\left(w_1\left(\theta, \Delta^m, q, u\right)\right)$ then $x_k \to \ell\left(S_\theta\left(\Delta^m_{uq}\right)\right)$,*

ii) *If $x \in \ell_\infty\left(\Delta^m_{u_q}\right)$ and $x_k \to \ell\left(S_\theta\left(\Delta^m_{uq}\right)\right)$, then $x_k \to \ell\left(w_1\left(\theta, \Delta^m, q, u\right)\right)$,*

iii) *$S_\theta\left(\Delta^m_{uq}\right) \cap \ell_\infty\left(\Delta^m_{u_q}\right) = w_1\left(\theta, \Delta^m, q, u\right) \cap \ell_\infty\left(\Delta^m_{u_q}\right)$,*

where $\ell_\infty\left(\Delta^m_{u_q}\right) = \left\{x \in w\left(X\right) : \sup_k q\left(u_k \Delta^m x_k\right) < \infty\right\}$.

Proof. (i) Let $\varepsilon > 0$ and $x_k \to \ell\left(w_1\left(\theta, \Delta^m, q, u\right)\right)$. Then we can write

$$\sum_{k\in I_r} q\left(u_k \Delta^m x_k - \ell\right) \geq \sum_{\substack{k\in I_r \\ q(u_k\Delta^m x_k - \ell)\geq\varepsilon}} q\left(u_k \Delta^m x_k - \ell\right) \geq \varepsilon\left|\left\{k \in I_r : q\left(u_k \Delta^m x_k - \ell\right) \geq \varepsilon\right\}\right|.$$

Hence $x_k \to \ell\left(S_\theta\left(\Delta^m_{uq}\right)\right)$.

ii) Suppose that $x_k \to \ell\left(S_\theta\left(\Delta^m_{uq}\right)\right)$ and let $x \in \ell_\infty\left(\Delta^m_{u_q}\right)$. Let $\varepsilon > 0$ be given and select N_ε such that

$$\frac{1}{h_r}\left|\left\{k \in I_r : q\left(u_k \Delta^m x_k - \ell\right) \geq \left(\frac{\varepsilon}{2}\right)\right\}\right| \leq \frac{\varepsilon}{2K}$$

for all $r > N_\varepsilon$ and set $L_r = \left\{k \in I_r : q\left(u_k \Delta^m x_k - \ell\right) \geq \left(\frac{\varepsilon}{2}\right)\right\}$, where $K = \sup_k q\left(u_k \Delta^m x_k\right)$. Now for all $r > N_\varepsilon$ we have

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r} q\left(u_k\Delta^m x_k-\ell\right) &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ k\in L_r}} q\left(u_k\Delta^m x_k-\ell\right)+\frac{1}{h_r}\sum_{\substack{k\in I_r\\ k\notin L_r}} q\left(u_k\Delta^m x_k-\ell\right)\\
&\le \frac{1}{h_r}\left(\frac{h_r\varepsilon}{2K}\right)K+\frac{\varepsilon}{2h_r}h_r=\varepsilon.
\end{aligned}$$

Thus $x_k\to\ell\left(w_1\left(\theta,\Delta^m,q,u\right)\right)$.

The proof of (iii) follows from (i) and (ii).

In the following two theorems we shall assume that the sequence $p=(p_k)$ is bounded and $0<h=\inf_k\le p_k\le\sup_k=H<\infty$.

Theorem 3.2. *Let M be a Orlicz function. Then $w_1\left(M,\theta,\Delta^m,q,p,u\right)\subset S_\theta\left(\Delta^m_{uq}\right)$.*

Proof. Let $x\in w_1\left(M,\theta,\Delta^m,q,p,u\right)$ and let $\varepsilon>0$ be given. Let $\sum_1$ and $\sum_2$ denote the sums over $k\in I_r$ with $q\left(u_k\Delta^m x_k-\ell\right)\ge\varepsilon$ and $q\left(u_k\Delta^m x_k-\ell\right)<\varepsilon$, respectively. Then

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(q\left(\frac{u_k\Delta^m x_k-\ell}{\rho}\right)\right)\right]^{p_k} &= \frac{1}{h_r}\sum_1\left[M\left(q\left(\frac{u_k\Delta^m x_k-\ell}{\rho}\right)\right)\right]^{p_k}\\
&\quad+\frac{1}{h_r}\sum_2\left[M\left(q\left(\frac{u_k\Delta^m x_k-\ell}{\rho}\right)\right)\right]^{p_k}\\
&\ge \frac{1}{h_r}\sum_1\left[M\left(\varepsilon_1\right)\right]^{p_k},\text{where } \varepsilon_1=\frac{\varepsilon}{\rho}\\
&\ge \frac{1}{h_r}\sum_1 \min\left(\left[M\left(\varepsilon_1\right)\right]^h,\left[M\left(\varepsilon_1\right)\right]^H\right)\\
&\ge \frac{1}{h_r}\left|\left\{k\in I_r: q\left(u_k\Delta^m x_k-\ell\right)\ge\varepsilon\right\}\right|\\
&\quad \min\left(\left[M\left(\varepsilon_1\right)\right]^h,\left[M\left(\varepsilon_1\right)\right]^H\right).
\end{aligned}$$

Hence $x\in S_\theta\left(\Delta^m_{uq}\right)$.

Theorem 3.3. Let $\theta=(k_r)$ be a lacunary sequence, M be an Orlicz function and $x=(x_k)\in\ell_\infty\left(\Delta^m_{u_q}\right)$ then $S_\theta(\Delta^m_{uq})\subset w_1\left(M,\theta,\Delta^m,q,p,u\right)$.

Proof. Let $x \in S_\theta(\Delta^m_{uq})$. Since $x \in \ell_\infty\left(\Delta^m_{u_q}\right)$, there exists $T > 0$ such that $\sup_k q\left(u_k \Delta^m x_k\right) \leq T$. Let $\varepsilon > 0$, $\sum_1$and $\sum_2$ denote in previous theorem, then we have

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(q\left(\frac{u_k\Delta^m x_k-\ell}{\rho}\right)\right)\right]^{p_k} &= \frac{1}{h_r}\sum_1\left[M\left(q\left(\frac{u_k\Delta^m x_k-\ell}{\rho}\right)\right)\right]^{p_k} \\
&\quad +\frac{1}{h_r}\sum_2\left[M\left(q\left(\frac{u_k\Delta^m x_k-\ell}{\rho}\right)\right)\right]^{p_k} \\
&\leq \frac{1}{h_r}\sum_1 \max\left\{\left[M\left(\frac{T}{\rho}\right)\right]^h, \left[M\left(\frac{T}{\rho}\right)\right]^H\right\} \\
&\quad +\frac{1}{h_r}\sum_2\left[M\left(\frac{\varepsilon}{\rho}\right)\right]^{p_k} \\
&\leq \max\left\{[M(K)]^h, [M(K)]^H\right\}.\frac{1}{h_r}\left|\{k\in I_r : q\left(u_k\Delta^m x_k-\ell\right)\geq\varepsilon\}\right| \\
&\quad +\max\left\{[M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H\right\}, \quad \frac{T}{\rho}=K, \quad \frac{\varepsilon}{\rho}=\varepsilon_1
\end{aligned}$$

Hence $x \in w_1\left(M, \theta, \Delta^m, q, p, u\right)$.

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