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Topologies Generated by Probabilistic Quasi-pseudo-metrics*

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Abstract

In this paper, we give the properties of some topological spaces generated by *Pqp*-metrics and we study properties (separation axiom) of a bitopological space (X, T_P, T_Q) generated by a *Pqp*-metrics.

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Introduction

The advances in the art of measurement in the 19th century stimulated a corresponded concern with the accompanying errors. Motivated by the idea that in practice and as, e.g., quantum mechanics implies even in theory, some measurements are necessarily inexact, in his note Statistical Metrics [8] Karl Menger proposed transferring the probabilistic notions of quantum mechanics from the physics to the underlying geometry and showed how one could replace a numerical distance between points p and q by a distribution function F_{pq}. Subsequently, numerous authors studied such spaces, as did e.g., the excellent books Probabilistic Metric Spaces[17] by Berthold Schweizer and Abe Sklar. The contribution of Menger to resolving the interpretative issue of quantum mechanics turned out to be of fundamental importance in probabilistic functional analysis and nonlinear analysis; see e.g. [1]. Probabilistic metric spaces notation is useful in modelling some phenomena where it is necessary to study the relationship between two probability functions as will observe in [14]; for instance, it has a direct physic motivation in the context of the two-slit experiment as the foundation of E-infinity of high energy physics, recently studied by El Naschie in England [11, 12, 15], Giordano et al. [13], and by Zmeskal et. al. in Czech Republic [19] and etc. For instance, the process in the analysis of the probability involved in the two-slit experiment can be modelled by means of a probabilistic metric.

In this paper, we give the properties of some topological spaces generated by Pqp-metrics.

In section 1. we define the concept of t_I-norms and give its basic properties.

In section 2. some properties of t_{Δ^+}-norms are given and studical.

In section 3. we give the basic concepts, some notation, definitions related to the subject of $PqpM$-spaces.

In section 4. we introduce a one-parameter family of some neighbourhoods $\{N_x^P(t)\}_{x\in X}$ for any $t > 0$ in a $PqpM$-space $(X, P, *)$ and give some conditions for the t_{Δ^+}-norm $*$ that the family is a complete system of neighbourhoods in X.

A motivation for such a definition of a topology on X is provided by the fact we prove, namely, that a topology T_{G_p} of a P-simple space (X, G_p) induced by the Pqp-metric G_p is equivalent to the topology T_p generated by the quasi-pseudo-metric p of the space (X, p). In particular, if $G = u_1$, then these topologies are identical. Thus a study of the properties of topological spaces generated by quasi-pseudo-metrics can be reduced to the study of topologies of simple P-spaces. However, the topological spaces formed by probabilistic metrics lead to a supremum topology $T_P \vee T_Q$, where P and Q are mutually conjugate Pqp-metrics.

In section 5. we study some properties (separation axioms) of a bitopological space (X, T_P, T_Q) generated by a Pqp-metric P.

1. t_I-Norms and Their Properties

Now, we shall give some definitions and properties of t_I-norms (K. Menger [8]) defined on the unit interval $I = [0, 1]$. A t_I-norm $T : I^2 \to I$ is in interval I an abelian semigroup with unit, and the t_I-norm T is nondecreasing with respect to each variable.

Definition 1.1. Let T be a t_I-norm.

(1) T is called a *continuous t_I-norm* if the function T is continuous with respect to the product topology on the set $I \times I$.

(2) The function T is said to be *left-continuous* if, for every $x, y \in (0, 1]$, the following condition holds:

$$T(x, y) = \sup\{T(u, v) : 0 < u < x,\ 0 < v < y\}.$$

(3) The function T is said to be *right-continuous* if, for every $x, y \in [0, 1)$, the following condition holds:

$$T(x, y) = \inf\{T(u, v) : x < u < 1,\ y < v < 1\}.$$

Note that the continuity of a t_I-norm T implies both left and right-continuity of it.

Definition 1.2. Let T be a t_I-norm. For each $n \in \mathbb{N}$ and each $x \in I$, let

$$x^0 = 1 \quad \text{and} \quad x^{n+1} = T(x^n, x).$$

Then the function T is called an *Archimedean t_I-norm* if, for every $x, y \in (0,1)$, there is an $n \in \mathbb{N}$ such that

$$x^n < y, \quad \text{that is,} \quad x^n \leq y \quad \text{and} \quad x^n \neq y. \tag{TA}$$

From an immediate consequence of the above definition, we have the following:

Lemma 1.3. *If T is an archimedean t_I-norm, then, for all $x \in (0,1)$, the following inequality holds:*

$$T(x,x) < x.$$

Proof. Indeed, by (TA), there exists an $n \in \mathbb{N}$ such that $x^n < x$. Let $m \in \mathbb{N}$ be the smallest number with $x^m = x$. Then we have

$$x^{m+1} = T(x^m, x) = T(x,x) < x.$$

This completes the proof.

Lemma 1.4. *If T is a continuous t_I-norm, strictly increasing in $(0,1]^2$, then it is Archimedean.*

Proof. By the strict monotonicity of T, for every $x \in (0,1)$, we have $T(x,x) < x$ and so $x^{n+1} < x^n$ for all $n \in \mathbb{N}$. Since T is continuous, it follows that $\lim_{n\to\infty} x^n = 0$. This completes the proof.

Definition 1.5. Let T be a t_I-norm. Then T is said to be *positive* if $T(x,y) > 0$ for all $x, y \in (0,1]$.

Note that every t_I-norm satisfying the assumption of Lemma 0.2.4 is positive.

We shall now establish the notation related to a few most important t_I-norms defined by:

$$M(x,y) = \text{Min}(x,y) = x \wedge y \tag{T-M}$$

for all $x, y \in I$. The function M is continuous and positive, but is not Archimedean (in fact, it fails to satisfy the strict monotonicity condition).

$$\Pi(x,y) = x \cdot y \tag{T-Π}$$

for all $x, y \in I$. The function Π is strictly increasing and continuous and hence it is a positive archimedean t_I-norm.

$$W(x,y) = \mathrm{Max}(x+y-1,0) \tag{T-W}$$

for all $x, y \in I$. The function W is continuous and Archimedean, but it is not positive and hence it fails to be a strictly increasing t_I-norm.

$$Z(x,y) = \begin{cases} x & \text{if } x \in I \text{ and } y = 1, \\ y & \text{if } x = 1 \text{ and } y \in I, \\ 0 & \text{if } x, y \in [0,1). \end{cases} \tag{T-Z}$$

The function Z is Archimedean and right-continuous, but it fails to be left-continuous.

We give the following relations among the t_I-norms defined above:

$$M \geq \Pi \geq W \geq Z, \tag{T,$\geq$}$$
$$M \gg \Pi \gg W \gg Z. \tag{T,$\gg$}$$

Remark 1.6. For any number $p \in (-\infty, +\infty)$, one can define a t_I-norm T_p as follows:

$$T_p(x,y) = \begin{cases} (\mathrm{Max}(x^p + y^p - 1, 0))^{1/p} & \text{if } p \neq 0, \\ \Pi(x,y) = x \cdot y & \text{if } p = 0. \end{cases}$$

The function T_p is continuous and strictly monotone if and only if $p \leq 0$. Also, notice that $p = -1$ yields

$$T_{-1}(x,y) = \frac{xy}{x+y-xy}$$

for any $x, y \in (0,1]$ and, for $p = 1$, $T_1 = W$,

$$\lim_{p \to -\infty} T_p(x,y) = M(x,y)$$

and

$$\lim_{p \to +\infty} T_p(x,y) = Z(x,y)$$

for any $x, y \in (0,1]$

2. t_{Δ^+}-Norms and Their Properties

In this section, we shall now present some properties of the t_S-norms defined on Δ^+ (Šerstnev [18]).

The ordered pair $(\Delta^+, *)$ is an abelian semigroup with the unit $u_0 \in \Delta^+$ and the operation $* : \Delta^+ \times \Delta^+ \to \Delta^+$ is a nondecreasing function. We note that $u_\infty \in \Delta^+$ is a zero of Δ^+. Indeed, by (L-R) we obtain

$$u_\infty \le u_\infty * F \le u_\infty * u_0 = u_\infty \text{ for all } F \in \Delta^+.$$

Let $T(\Delta^+, *)$ denote the family of all t_{Δ^+}-norms $*$ on the set Δ^+.

Then the relation $\le$ defined by:

$(\Delta^+, \le)$ $\quad *_1 \le *_2$ iff $F *_1 G \le F *_2 G$ for all $F, G \in \Delta^+$ partially orders the family $T(\Delta^+, *)$.

Now, we are going to define the next relation in the $T(\Delta^+, *)$. It will be denoted by $\gg$ and is defined as follows:

$(\Delta^+, \gg)$ $\quad *_1 \gg *_2$ iff for all $F, G, P, Q \in \Delta^+[(F *_2 P) *_1 (G *_2 R)] \ge [(F *_G) *_2 (P * R)]$. By putting $G = P = u_0$ we obtain $F *_1 R \ge F *_2 R$ for $F, R \in \Delta^+$ and hence $*_1 \ge *_2$. Then follows that $*_1 \gg *_2 \Rightarrow *_1 \ge *_2$.

Theorem 2.1. *Let T be a left-continuous t_I-norm. Then the function $T : \Delta^+ \times \Delta^+ \to \Delta^+$ defined by*

$$T(F, G)(t) = T(F(t), G(t)) \tag{2.1.1}$$

for any $t \in [0, +\infty]$ is a t_{Δ^+}-norm on the set Δ^+.

Theorem 2.2. *For every t_{Δ^+}-norm $*$, the following inequality holds:*

$$* \le M,$$

where M is the t_I-norm of Definiton 1.15.

Proof. For every F, $G \in \Delta^+$, we have by definition of $(\Delta^+, *)$, $F * G \le F * u_0 = F$ and, by symmetry, also $F * G \le G$. Thus, for every $t \in [0, +\infty]$, we have

$$(F * G)(t) \le M(F(t), G(t)) = M(F, G)(t). \tag{2.2.1}$$

Theorem 2.3. *If T is a left-continuous t_I-norm, then the function $*_T$: $\Delta^+ \times \Delta^+ \to \Delta^+$ defined by*

$$F *_T G(t) = \sup\{T(F(u), G(s)) : u + s = t,\ u, s > 0\} \tag{2.3.1}$$

is a t_{Δ^+}-norm on Δ^+.

Proof. The function $F *_T G \in \Delta^+$ is nondereasing and satisfies the condition $F *_T G(+\infty) = 1$ for all $F, G \in \Delta^+$. Thus it suffices to check that $F *_T G$ is left-continuous, i.e., for every $t \in (0, +\infty)$ and $h > 0$, there exists $0 < t_1 < t$ such that

$$F *_T G(t_1) > F *_T G(t) - h.$$

Let $t \in (0, +\infty)$. Then there exist $u, s > 0$ such that $u + s = t$ and

$$T(F(u), G(s)) > F *_T G(t) - \frac{h}{2}. \tag{2.3.2}$$

By the left-continuity of F, G and the t_I-norm T, it follows that there are numbers $0 \leq u_1 < u$ and $0 \leq s_1 \leq s$ such that

$$T(F(u_1), G(s_1)) > T(F(u), G(s)) - \frac{h}{2}. \tag{2.3.3}$$

Now, put $t_1 = u_1 + s_1$. Then $t_1 < t$ and, by (2.5.3), we obtain

$$F *_T G(t) \geq T(F(u_1), G(s_1)). \tag{2.3.4}$$

This completes the proof.

Theorem 2.4. *Let T be a continuous t_I-norm. Then the t_{Δ^+}-norms $*_T$ and T are uniformly continuous on (Δ^+, d_L).*

Proof. Let us observe that the continuity of the t_I-norm T implies its uniform continuity on $I \times I$ with the product topology. Take an $h \in (0, 1)$. Then there exists $s > 0$ such that

$$T(\mathrm{Min}(z + s, 1), w) < T(z, w) + \frac{h}{4}$$

and

$$T(z, \mathrm{Min}(w + s, 1)) < T(z, w) + \frac{h}{4} \tag{2.4.1}$$

for all $z, w \in I$. Let $u < 1/s$ and $v < 1/s$ be such that $u + v < 2/h$. Next, by (2.3.1), for every $F, G \in \Delta^+$ and $t \in (0, 2/h)$, there exist $u, v > 0$ such that $u + v = t$ and

$$F *_T G(t) < T(F(u), G(v)) + \frac{h}{4}.$$

Now, let $F_1 \in \Delta^+$ be such that $d_L(F, F_1) < s$, which means that

$$F(u) \leq F_1(u + s) + s$$

for all $u \in (0, \frac{1}{s})$. Since $u + v = t < 2/h$, we have $u < 2/h$. Therefore, we obtain

$$\begin{aligned} F *_T G(t) &< T(\mathrm{Min}(F_1(u+s) + s, 1), G(v)) + \frac{h}{2} \\ &< T(F_1(u+s), G(v)) + \frac{h}{2} \end{aligned}$$

and

$$\begin{aligned} F *_T G(t) &< F_1 *_T G(u + s + v) + \frac{h}{2} \\ &\leq F_1 *_T G(u + v + \frac{h}{2}) + \frac{h}{2} \\ &= F_1 *_T G(t + \frac{h}{2}) + \frac{h}{2}. \end{aligned}$$

Thus, we have

$$p_L(F_1 *_T G, G) \leq \frac{h}{2}, \quad q_L(F *_T G, F_1 *_T G) \leq \frac{h}{2}$$

and so we have

$$d_L(F_1 *_T G, F *_T G) \leq \frac{h}{2}.$$

If $d_L(G, G_1) < s$, then we have

$$d_L(F_1 *_T G_1, F_1 *_T G) \leq \frac{h}{2}$$

and so let $F, F_1, G, G_1 \in \Delta^+$ satisfy the conditions $d_L(F, F_1) < s$ and $d_L(G, G_1) < s$. Then we have

$$\begin{aligned} &d_L(F_1 *_T G_1, F *_T G) \\ &\leq d_L(F_1 *_T G_1, F_1 *_T G) + d_L(F_1 *_T G, F *_T G) \\ &\leq \frac{h}{2} + \frac{h}{2} = h. \end{aligned}$$

It follows that the t_{Δ^+}-norm $*_T$ is uniformly continuous in the space (Δ^+, d_L). The second part is a simple restatement of the first one. This completes the proof.

Remark 2.5. There exist t_{Δ^+}-norms which are not continuous on (Δ^+, d_L). Among them, there is the function $*_Z$ of (2.3.1) and (T-Z). Indeed, this can be seen by the following example.

Let $F_n(t) = 1 - e^{-\frac{t}{n}}$, where $n \in \mathbb{N}$. Then

$$F_n \xrightarrow{\text{w}} u_0$$

while the sequence $\{F_n *_Z F_n\}$ fails to be weakly convergent to $u_0 *_Z u_0$ because $F_n *_Z F_n = u_\infty$ for all $n \in \mathbb{N}$. We note that this example actually shows much more: the t_{Δ^+}-norm $*_Z$ is not continuous on (D^+, d_L). In particular, it is not continuous at the point (u_0, u_0).

We finish this section by showing a few properties of the relation defined in $(\Delta^+, \gg)$ in the context of t_{Δ^+}-norms.

Lemma 2.6. *If T_1 and T_2 are continuous t_I-norms, then*

$$T_1 \gg T_2 \quad \textit{if and only if} \quad *_{T_1} \gg *_{T_2}.$$

Lemma 2.7. *If T is a continuous t_I-norm and T is the t_{Δ^+}-norm of* (2.1.1), *then:*

$$T \gg *_T, \tag{i}$$

$$M \gg * \quad \textit{for all} \;\; t_{\Delta^+}\textit{-norms} \;\, * \,. \tag{ii}$$

3. Properties of *PqpM*-Spaces

First, we give the definition of *PqpM*-spaces and some properties of *PqpM*-spaces and others.

Definition 3.1 ([3]). By a *PqpM*-space we mean an *ordered triple* $(X, P, *)$, where X is a nonempty set, the operation $*$ is a t_{Δ^+}-norm and $P : X^2 \to \Delta^+$

satisfies the following conditions (by P_{xy} we denote the value of P at $(x,y) \in X^2$): for all $x,y,z \in X$,

$$P_{xx} = u_0, \tag{3.1.1}$$
$$P_{xy} * P_{yz} \le P_{xz}. \tag{3.1.2}$$

If P satisfies also the additional condition:

$$P_{xy} \ne u_0 \quad \text{if} \quad x \ne y, \tag{3.1.3}$$

then $(X,P,*)$ is called a *probabilistic quasi-metric space* (denoted by PqM-space).

Moreover, if P satisfies the condition of symmetry:

$$P_{xy} = P_{yx}, \tag{3.1.4}$$

then $(X,P,*)$ is called a *probabilistic metric space* (denoted by PM-space).

Definition 3.2. [3] Let $(X,P,*)$ be a $PqpM$-space and let $Q : X^2 \to \Delta^+$ be defined by the following condition:

$$Q_{xy} = P_{yx}$$

for all $x,y \in X$. Then the ordered triple $(X,Q,*)$ is also a $PqpM$-space. We say that the function P is called a *conjugate* Pqp*-metric* of the function Q. By $(X,P,Q,*)$ we denote the structure generated by the Pqp-metric P on X.

Now, we shall characterize the relationships between Pqp-metrics and probabilistic pseudo-metrics.

Lemma 3.3. *Let $(X,P,Q,*)$ be a structure defined by a Pqp-metric P and let $*_1 \gg *_2 (\Delta^+, \gg)$. Then the ordered triple $(X,F^{*_1},*)$ is a probabilistic pseudo-metric space (denoted by PPM-space) whenever the function $F^{*_1} : X^2 \to \Delta^+$ is defined in the following way:*

$$F^{*_1}_{xy} = P_{xy} *_1 Q_{xy} \tag{3.3.1}$$

for all $x,y \in X$. If, additionally, P satisfies the condition:

$$P_{xy} \ne u_0 \quad \text{or} \quad Q_{xy} \ne u_0 \tag{3.3.2}$$

for $x \ne y$, then $(X,F^{*_1},*)$ is a PM-space.

Proof. For any $x, y \in X$, we have

$$F^{*_1}_{xy} \in \Delta^+, \quad F^{*_1}_{xy} = F^{*_1}_{yx}.$$

By (3.1.1), we obtain

$$F^{*_1}_{xx} = P_{xx} *_1 Q_{xx} = u_0 *_1 u_0 = u_0.$$

Next, by (3.1.2), $(\Delta^+, \gg)$ and the monotonicity of t_{Δ^+}-norm, we obtain

$$\begin{aligned} F^{*_1}_{xy} &= P_{xy} *_1 Q_{xy} \\ &\geq (P_{xz} * P_{xz}) *_1 (Q_{xz} * Q_{zy}) \\ &\geq (P_{xz} *_1 Q_{xz}) * (P_{zy} *_1 Q_{zy}) \\ &= F^{*_1}_{xz} * F^{*_1}_{zy}. \end{aligned}$$

The proof of the second part of the theorem is a direct consequence of the fact that the conditions (3.3.2) and (3.3.1) both imply the statement that

$$F^{*_1}_{xz} = P_{xy} *_1 Q_{xy} = u_0 \quad \text{if and only if} \quad P_{xy} = Q_{xy} = u_0.$$

It follows that, whenever $x \neq y, P_{xy} \neq u_0$ or $Q_{xy} \neq u_0$ and hence $P_{xy} *_1 Q_{xy} \neq u_0$. This completes the proof.

Remark 3.4. For an arbitrary t_{Δ^+}-norm $*_1$, by Lemma 3.4, $M \gg *_1$ holds. By (3.1.3), we have

$$F^M(x,y) \geq F^{*_1}(x,y) \tag{3.4.1}$$

for all $x, y \in X$.

The function F^M will be called the *natural probabilistic pseudo-metric* generated by the *Pqp*-metric P. It is the "greatest" among all the probabilistic pseudo-metrics generated by P.

Definition 3.5. Let (X, p) be a quasi-pseudo-metric-space and $G \in \Delta^+$ be distinct from u_0 and u_∞. Then (X, G_p) is called a P-simple space genrated by (G, p) and G. Define of function $G_p : X^0 \to \Delta^+$ by

$$G_p(x,y) = G\left(\frac{t}{p(x,y)}\right) \quad \text{for all } t \in R^+ \tag{3.5.1}$$

and $G(\frac{t}{0}) = G(\infty) = 1$, for $t > 0, G(\frac{0}{0} = G(0) = 0$.

Theorem 3.6. *Every P-simple space (X, G_p) is a quasi-pseudo-Menger space respect to the t_I-norm M.*

Proof. For all $x, y, z \in X$, by the triangle condition for the quasi-pseudo-metric p, we have

$$p(x, y) \geq p(x, y) + p(y, z).$$

Assume,that all at $p(x, z), p(x, y)$ and $p(y, z)$ are distinct from zero. For any $t_1, t_2 > 0$, we obtain

$$\frac{t_1 + t_2}{p(x, z)} \geq \frac{t_1 + t_2}{p(x, y) + p(y, z)} \tag{3.6.1}$$

and hence we infer that

$$\text{Max}\left\{\frac{t_1}{p(x, y)}, \frac{t_2}{p(y, z)}\right\} \geq \frac{t_1 + t_2}{p(x, y) + p(y, z)} \geq \text{Min}\left\{\frac{t_1}{p(x, y)}, \frac{t_2}{p(y, z)}\right\}.$$

This inequality and the monotonicity of G imply that

$$G_p(x, z)(t_1 + t_2) \geq \text{Min}\,(G_p(x, y)(t_1), G_p(y, z)(t_2)),$$

for $t_1, t_2 \geq 0$.

4. Topologies in $PqpM$-Spaces

Definition 4.1. Let $(X, P, *)$ be a $PqpM$-space. For all $x \in X$ and $t > 0$, a P-neighbourhood of the point x is the set

$$N_x^P(t) = \{y \in X : d_L(P_{xy}, u_0) < t\} = \{y \in X : P_{xy}(t) > 1 - t\}. \tag{4.1.1}$$

Theorem 4.2. *Let $(X, P, *)$ be a $PqpM$-space. If the function $*$ is a continuous t_{Δ^+}-norm, then the family $\{N_x^P(t) : t \in \mathbb{R}^+\}$, where $x \in X$ forms a complete system of neighbourhoods in X.*

Proof. We note that, for every $t > 1, N_x^P(t) = X$. This is a consequence of the property (2.5.5) of the metric d_L. Since $d_L(P_{xx}, u_0) = d_L(u_0, u_0) = 0$, we have $x \in N_X^P(t)$ for $t > 0$. If $t_1 < t_2$, then

$$N_x^P(t_1) \subset N_x^P(t_2)$$

by (4.1.1). Thus it follows that

$$N_x^P(\min(t_1, t_2)) \subset N_x^P(t_1) \cap N_x^P(t_2).$$

Let $y \in N_x^P(t)$ and let $d_L(P_{xy}, u_0) = k$. Then $t - k > 0$. By Theorem 2.4, the t_{Δ^+}-norm $*$ is uniformly continuous. Thus there exists $t_1 > 0$ such that

$$d_L(P_{xy} * G, P_{xy}) < t - k$$

whenever $d_L(G, u_0) < t_1$.

Now, let $z \in N_y^P(t_1)$. Then $d_L(P_{yz}, u_0) < t$ and, by (3.1.2) and the Lemma 2.3.2, we have

$$\begin{aligned} d_L(P_{xz}, u_0) &\leq d_L(P_{xy} * P_{yz}, u_0) \\ &\leq d_L(P_{xy} * P_{yz}, P_{xy}) + d_L(P_{xy}, u_0) \\ &< t - k + k \\ &= t. \end{aligned}$$

It follows that $z \in N_x^P(t)$. Consequently, we have

$$N_y^P(t_1) \subset N_x^P(t).$$

This completes the proof.

As an immediate consequence of the above theorem, we have the fact that a *Pqp*-metric P does generate a topology on X. Let us denote it by T_p. Also, the *Pqp*-metric Q which is a conjugate of P generates a topology T_Q. Thus the natural topological structure associated with a *Pqp*-metric is a bitopological space (X, T_P, T_Q) (cf., Kelly [7]).

We note that, if $P = Q$, which is equivalent to the statement that the $PqpM$-space satisfies the symmetry condition (3.1.4), then the topology T_P is identical with T_Q, and the bitopological space (X, T_P, T_Q) reduces itself to the topological space (X, T_P) generated by P.

Let (X, p) be a quasi-pseudo-metri space. Let (X, G_p) be a P-simple space generated by p and G defined in (3.5.1). A relationship between the topologies generated by p and G_p is provided by the following:

Theorem 4.3. *Let (X, p) be a quasi-pseudo-metric space and let (X, G_p) be a P-simple space generated by p and G. Then the topology T_G is equivalent to the topology T_p genrated by the quasi-pseudo-metric p.*

Proof. Let $N_x^G(t)$ be a G-neighbourhood of $x \in X$. Let $k > 0$ be such that $G(\frac{t}{k}) > 1 - t$ and suppose that $y \in U_x^p(k)$, where

$$U_x^p(k) = \{y \in X : p(x, y) < k\}.$$

Then $0 \le p(x,y) < k$ and hence

$$P_{xy}(t) = G\Big(\frac{t}{p(x,y)}\Big) \ge G\Big(\frac{t}{k}\Big) > 1 - t.$$

It follows that $y \in N_x^p(t)$ and so

$$U_x^p(k) \subset N_x^G(t).$$

On the other hand, let $U_x^p(k)$ be a p-neighbourhood of x. Since $G \neq u_0$, there exists a number $a \in [0,1)$ such that $G^\wedge(a) > 0$, where $G^\wedge(s) = \sup\{t : G(t) < s\}$ denotes the left-continuous quasi-inverse of G. Let $t > 0$ be such that $\frac{t}{G(1-t)} < k$ and let $y \in N_x^G(t)$. Then we have

$$P_{xy}(t) > 1 - t, \quad P_{xy}(1-t) = p(x,y) \cdot G^\wedge(1-t) < t,$$

and

$$p(x,y) < \frac{t}{G(1-t)} < k.$$

Hence it follows that $p(x,y) < k$, i.e., $y \in U_x^P(x)$ and so we have

$$N_x^G(t) \subset U_x^P(t).$$

This completes the proof.

Corollary 4.4. *Let (X, G_p) be a P-simple space and let $G = u_1$. Then the topologies T_G and T_P generated by functions G_P and p, respectively, are identical.*

Proof. We will show that $N_x^G(t) = U_x^p(t)$ for all $t > 0$. Indeed, we have

$$\begin{aligned} N_x^G(t) &= \{y \in X : P_{xy}(t) > 1 - t\} \\ &= \Big\{y \in X : G\Big(\frac{t}{p(x,y)}\Big) > 1 - t\Big\} \\ &= \Big\{y \in X : u_1\Big(\frac{t}{p(x,y)}\Big) > 1 - t\Big\} \\ &= \{y \in X : p(x,y) < t\} \\ &= U_x^p(t). \end{aligned}$$

This completes the proof.

Definition 4.5. If $(X, P, *_M)$ is a quasi-pseudo-Menger space with respect to the t_I-norm T =Min, then, for all $a \in [0,1]$, the function $r_a : X^2 \to \mathbb{R}^+$ defined by

$$r_a(x,y) = P^{\wedge}_{xy}(a) = \sup\{t : R_{xy}(t) < a\} \tag{4.5.1}$$

for all $x, y \in X$ is a quasi-pseudo-metric on X.

The following example shows that the topology T_p generated by P fails to be equivalent to any topology $T_r(a)$ generated by the quasi-pseudo-metric r_a.

Example 4.6. Let $X = [0,1]$ and let $P : X^2 \to \Delta^+$ be defined by

$$P_{xy}(a) = \begin{cases} 0 & \text{if } x = y \text{ or } a \le y \le x, \\ 1 & \text{if } y < \text{Min } (a,b) \text{ or } y > x \end{cases}$$

for all $x, y \in [0,1]$ and $a \in [0,1]$. We note that

$$P^{\wedge}_{xy}(a) \le P^{\wedge}_{xz}(a) + P^{\wedge}_{zy}(a).$$

It follows that $(X, P, *_{\text{Min}})$ is a quasi-pseudo-Menger space. Then, for each $t \in (0,1)$, a P-neighbourhood of $x = 1 \in [0,1]$ is given by

$$N^P_1(t) = \{y : P_{1y}(t) > 1 - t\} = \{y : P^{\wedge}_{1y}(1-t) < t\} = [1-t, 1], \tag{4.6.1}$$

where the r_a-neighbourhoods of $x = 1$ are given by

$$U^r_1(t) = \{y : P^{\wedge}_{1y}(a) < t\} = [a, 1]. \tag{4.6.2}$$

Suppose that the topology T_p is equivalent to the topology T generated by a finite family of quasi-metrics $r_{a_1}, \ldots, r_{a_n}$. Take a number k such that

$$0 < k < \text{Min } (1 - a_1, 1 - a_2, \ldots, 1 - a_n)$$

and consider the P-neighbourhood $N^P_1(k)$. By our assumption, there exists $t > 0$ such that

$$U^{r(a_i)}_1(t) \subset N^p_1(k)$$

for some $1 \le i \le n$. By (4.6.1) and (4.6.2), we have $[a_1, 1] \subset [1-k, 1]$. It follows that $k \ge 1 - a_i$, which is a contradiction.

Remark 4.7. In Theorem 3.6, we proved that every P-simple space is a quasi-pseudo-Menger space with respect to Min. An examination of the Theorem

4.3 and the Example 4.6 leads to the conclusion that there are quasi-pseudo-Menger spaces with respect to Min which are not P-simple.

Let P and Q be two conjugates of Pqp-metrics and let $F_{P\vee Q}$ be the natural probabilistic pseudo-metric associated with P and Q according to (3.1.1.). The following gives a relationship that holds between topologies generated by these functions.

Lemma 4.9. *The topology T_F generated by the probabilistic pseudo-metric $F_{P\vee Q}$ is the smallest topology containing the topologies T_P and T_Q generated by the Pqp-metrics P and Q. Thus the topology T_F is the supremum of T_P and T_Q.*

Proof. It suffices to observe that every F-neighbourhood is of the form

$$N_x^F(t) = N_x^P(t) \cap N_x^Q(t).$$

Indeed, by (3.1.1) and (3.4.1), we have

$$\begin{aligned} F^M(x,y)(t) &= F_{P\vee Q}(x,y)(t) = \text{Min } (P_{xy}(t), Q_{xy}(t)), \\ N_x^F(t) &= \{y \in X : F_{P\vee Q}(x,y)(t) > 1-t\} \\ &= \{y \in X : \text{Min } (P_{xy}(t), Q_{xy}(t)) > 1-t\} \\ &= \{y \in X : P_{xy}(t) > 1-t\} \cap \{y \in X : Q_{xy}(t) > 1-t\} \\ &= N_x^P(t) \cap N_x^P(t). \end{aligned}$$

This completes the proof.

Example 4.10. Let $(\mathbb{R}, G_p)$ be a P-simple space as defined in (3.5.1), where the quasi-pseudo-metric $p : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$p(x,y) = \begin{cases} y-x, & \text{if } x \le y, \\ 0, & \text{if } x > y. \end{cases}$$

Then, by Theorem 4.3, the topologies T_{G_p} and T_{G_q} consist, respectively, of the sets of the forms $(-\infty, a)$ and $(b, +\infty)$ for any $a, b \in \mathbb{R}$. The supremum $T_{G_p} \vee T_{G_q} = T_{G_{p\vee q}}$ is the natural topology $T_{\mathbb{R}}$ on the real line. However, the infimum $T_{G_p} \wedge T_{G_q} = T_{G_{p\wedge q}}$ is the indiscrete topology $T_{\emptyset}$.

Example 4.11. Let $(\mathbb{R}, G_{p_1})$ be the P-simple space given in Definition 3.5 and let the quasi-pseudo-metric $p_1 : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$p_1(x,y) = \begin{cases} \text{Min } (y-x, 1), & \text{if } x \le y, \\ 1, & \text{if } x > y. \end{cases}$$

Then, by Theorem 4.10, $(\mathbb{R}, T_{G_{p_1}})$ and $(\mathbb{R}, T_{G_{q_1}})$ are the Sorgenfrey lines. The supremum $T_{G_{p_1}} \vee T_{G_{q_1}}$ is the discrete topology on $\mathbb{R}$.

The above examples provide a sequence of the relations between these topologies:

$$\begin{array}{ccccccccc} & & T_{G_p} & & \longrightarrow & & T_{G_{p_1}} & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & \\ T_0 & & \longrightarrow & & T_{\mathbb{R}} & & \longrightarrow & & T_X \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & T_{G_q} & & \longrightarrow & & T_{G_{q_1}} & & \end{array}$$

Remark 4.12. Schwizer, Sklar and Thorp [16] introduced a two-parameter family of neighbourhoods of elements of X for the theory of probabilistic metric spaces. Let $(X, P, *)$ be a $PqpM$-space. Then an (ε, λ)-neighbourhood of $x \in X$ is a set defined as follows[1]:

$$N_x^P(\varepsilon, \lambda) = \{y \in X : P_{xy}(\varepsilon) > 1 - \lambda\}, \tag{4.11.1}$$

where $\varepsilon > 0$ and $\lambda > 0$.

We note that topologies generated by the complete systems of neighbourhoods defined in Definition 4.1 and (4.11.1)[2] are equivalent. Indeed, for every $t > 0$, we have

$$N_x^P(t, t) = N_x^P(t)$$

and, for all $\varepsilon, \lambda > 0$, we have

$$N_x^P(\text{Min } (\varepsilon, \lambda) \subset N_x(\varepsilon, \lambda).$$

The following papers mentioned here in a chronological order are devoted to topologies in PM-spaces, i.e., Schweizer, Sklar and Throp [16], Fritsche [2], Höhle [5]. The last two papers characterize the topologies in PM-spaces in a manner different from that accepted in this work.

[1] A neighbourhhod $N_x^P(\varepsilon, \lambda)$ can be interpreted as a set of those $y \in X$ for which the distance between x and y is smaller than ε with a probability greater than $1 - \lambda$.

[2] A proof that the family of all (ε, λ)-neighbourhoods given in (5.11.1) is a complete neighbourhood system in a $PqpM$-space is a simple restatement of the proof of Theorem 7.2 in [16].

5. Separation Axioms in $PqpM$-Spaces

Definition 5.1. ([9]) A bitopological space (X, T_1, T_2) is called a *pairwise semi-Hausdorff space* if, for all distinct $x, y \in X$, there exist a T_1-open subset U and a T_2-open subset V such that $x \in U, y \in V$ or $x \in V$ and $y \in U$ and $U \cap V = \emptyset$.

Definition 5.2. ([9]) A bitopological space (X, T_1, T_2) is called a *pairwise Hausdorff space* if, for all distinct $x, y \in X$, there exist a T_1-open subset U and a T_2-open subset V such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Theorem 5.3. *Let* $(X, P, *)$ *be a* $PqpM$*-space satisfying* (3.1.6)*. Then* (X, T_P, T_Q) *generated by* P *is a pairwise semi-Hausdorff space. If, moreover,* P *satisfies* (3.1.3)*, then* (X, T_P, T_Q) *is a pairwise Hausdorff space.*

Proof. By (3.1.6), it follows that, any $x, y \in X$ with $x \neq y$, we have $P_{xy} \neq u_0$ or $Q_{xy} \neq u_0$. Assume that $P_{xy} \neq u_0$. Then we have

$$0 < k = d_L(P_{xy}, u_0).$$

By the uniform continuity of $*$, there is $t > 0$ such that

$$d_L(G_1 * G_2, u_0) < k$$

whenever $d_L(G_1, u_0) < t$ and $d_L(G_2, u_0) < t$. Suppose that $z \in N_x^P(t) \cap N_x^Q(t)$. Then we have

$$d_L(P_{xz}, u_0) < t, \quad d_L(Q_{yz}, u_0) < t$$

and, by Lemma 2.7 and Definition 3.1.2, we obtain

$$d_L(P_{xy}, u_0) \leq d_L(F_{xz} * G_{yz}, u_0) < k,$$

which is a constradiction.

The second part of the proof follows immediately by (3.1.3). This completes the proof.

Definition 5.4. ([6]) Let (X, T_1, T_2) be a bitopological space. Then T_1 is said to be *regular* with respect to T_2 if, for all $x \in X$ and T_1-closed set P with $x \notin P$, there exist a T_1-open set U and T_2-open set V disjoint from U such that $x \in U$ and $P \subset V$.

Definition 5.5. ([6]) A bitopological space (X, T_1, T_2) is sais to be *pairwise normal* if, for each T_1-closed set A and T_2-closed set B disjoint from A, there exist a T_1-open set U and a T_2-open set V disjoint from U such that $A \subset U$ and $B \subset V$.

Lemma 5.6. *Let $(X, P, *)$ be a PqpM-space such that the t_{Δ^+}-norm is sup-continuous*[3]*, i.e., for all $F, G \in \Delta^+, \lambda \in \Lambda \neq \emptyset$, the following holds:*

$$\sup_{\lambda \in \Lambda}\{F_\lambda * G\} = (\{\sup_{\lambda \in \Lambda} F_\lambda\}) * G.$$

Then, for each $\emptyset \neq A \subset X$ and $P_{xA} = \sup\{P_{xy} : y \in A\}$, we have

$$P_{xA} \leq P_{xz} * P_{zA}$$

for all $x \in X$.

Proof. Let $x \in X$. Then, by (3.1.2), we have

$$P_{xA} = \sup\{P_{xy} : y \in A\} \geq P_{xy} \geq P_{xz} * P_{zy}$$

for all $y \in A$. Since the t_{Δ^*}-norm $*$ is sup-continuous, it follows that

$$P_{xA} \geq \sup\{y \in A : P_{xz} * P_{zy}\} = P_{xz} * P_{zA}.$$

This completes the proof.

Lemma 5.7. *Let $(X, P, *)$ be a PqpM-space. Let the t_{Δ^+}-norm $*$ be sup-continuous. Then, for any $A \subset X$, the function $f_A : X \to [0, 1]$ defined by $f_A(x) = d_L(P_{xA}, u_0)$ is upper Q-semicontinuous. If we additionally assume that $* \geq *_W$, then f_A is lower P-semicontinuous. Also, the function $g_A : X \to [0, 1]$ defined by $g_A(x) = d_L(Q_{xA}, u_0)$ is an upper P-semicontinuous function g_A, which is also lower Q-semicontinuous.*

Proof. It suffices to show that, for each $t \in \mathbb{R}^+$, the sets $V = \{y \in X : d_L(P_{yA}, u_0) < t\}$ and $U = \{y \in X : d_L(P_{yA}, u_0) < t\}$ are, respectively, Q-open and P-closed.

(i) Let $z \in V$. Since $d_L(P_{yA}, u_0) = a < t$, we have $t - a > 0$. By the uniform continuity of $*$, there is $t_1 > 0$ such that

$$d_L(G * P_{zA}, P_{zA}) < t - a$$

[3] We note that sup-continuity means upper semicontinuity with respect to the partial order in $(\Delta^+, \leq)$. If T is a continuous t_I-norm, then the t_{Δ^+}-norm $*_T$ is sup-continuous and, on the other hand, a t_{Δ^+}-norm which is a concolution (cf., Schweizer and Sklar [17], pp. 319) fails to be sup-continuous.

whenever $d_L(G, u_0) < t_1$. Let $r \in N_z^Q(t)$. This means that

$$d_L(P_{zr}, u_0) = d_L(P_{rz}, u_0) < t_1$$

and

$$d_L(P_{rA}, u_0) \leq d_L(P_{rz} * P_{zA}) + d_L(P_{zA}, u_0) < t - a + a = t$$

and so $N_z^Q(t_1) \subset V$. Thus V is Q-open.

(ii) Now, we will show that U is P-closed. Let $h > 0$. By Lemma 5.6, we have

$$\begin{aligned} P_{zr} * P_{rA}(t+h) + h \geq & \text{Max } (P_{zr}(h) + P_{rA}(t) - 1, 0) + h \\ & \geq 1 - h + P_{rA}(t) - 1 + h = P_{rA}(t) \end{aligned}$$

whenever $r \in N_z^P(h)$. On the other hand, we have

$$\begin{aligned} P_{rA}(t+h) + h \geq & \text{Min } (P_{zr}(h), P_{rA}(t)) \\ & \geq P_{zr} * P_{rA}(t+h) \\ & \geq P_{zr} * P_{rA}(t) \end{aligned}$$

and hence, for all $h > 0$ and $r \in N_z^P(h)$, we obtain

$$d_L(P_{zr} * P_{rA}, P_{rA}) < h$$

by Theorem 2.4.

Next, let $x \in \overline{U}^P$ and $z \notin U$. Thus we have $d_L(P_{zA}, u_0) = b < t$, $b - t > 0$ and, by Theorem 2.4, for all $h < b - t$,

$$\begin{aligned} b = d_L(P_{zA}, u_0) &\leq d_L(P_{zr} * P_{rA}, u_0) \\ &\leq d_L(P_{zr} * P_{rA}) + d_L(P_{rA}, u_0) \\ &< b - t + d_L(P_{rA}, u_0). \end{aligned}$$

Hence $d_L(P_{rA}, u_0) > t$ if $r \in N_z^P(h)$ and $U \cap N_z^P(h) = \emptyset$. Thus $U = \overline{U}^P$. This completes the proof.

Theorem 5.8. *Let $(X, P, *)$ be a PqpM-space with the supcontinuous t_{Δ^+}-norm $*$ such that $* \geq *_W$. Then the bitopological space (X, T_P, T_Q) generated by the probabilistic quasi-pseudo-metric P is pairwise regular and pairwise normal.*

Proof. By Lemma 5.7, for each $x \in X$ and $t > 0$, the set $\{y \in X : d_L(P_{xy}, u_0) < t\}$ is Q-closed and hence each $x \in X$ has a P-base consisting of Q-closed sets. Thus T_P is regular with respect to T_Q. Similarly, T_Q is regular with respect to T_P.

Now, let A and B be disjoint subsets of X such that A is P-closed and B is Q-closed. By Lemma 5.7, we have

$$A = \{x \in X : d_L(P_{xA}, u_0) = 0\}$$

and

$$B = \{x \in X : d_L(Q_{xB}, u_0) = 0\}.$$

Then we define the sets U and V as follows:

$$U = \{x \in X : d_L(P_{xA}, u_0) < d_L(Q_{xB}, u_0)\},$$
$$V = \{x \in X : d_L(Q_{xB}, u_0) < d_L(P_{xA}, u_0)\}.$$

We observe that $A \subset U$ and $B \subset V$ and $U \cap V = \emptyset$. To complete the proof, we must show that U is Q-open and V is P-open.

First, we shall show that V is P-open. Assume that $x_0 \in V$. Then we have

$$d_L(P_{x_0A}, u_0) - d_L(Q_{x_0B}, u_0) = k > 0.$$

By Lemma 5.7, the function $d_L(Q_{xB}, u_0)$ is upper semicontinuous. Therefore, there is $t_1 > 0$ such that, if $z \in N^P_{x_0}(t_1)$, then

$$d_L(P_{xA}, u_0) > d_L(P_{x_0A}, u_0) + \frac{k}{4}$$

and

$$d_L(Q_{x_0B}, u_0) > d_L(Q_{xB}, u_0) + \frac{k}{4}$$

. Thus we have

$$d_L(P_{xA}, u_0) - d_L(Q_{xB}, u_0) - (d_L(P_{x_0A}, u_0) - d_L(Q_{x_0B}, u_0)) + \frac{k}{4} + \frac{k}{4} > 0,$$

which means that $x \in V$ and so $N^P_{x_0}(t_1) \subset V$. This implies that V is P-open. This completes the proof.

Corollary 5.9. *Let (X, p) be a quasi-pseudo-metric space. Then the bitopological space (X, T_p, T_q) generated by the quasi-pseudo-metric P is pairwise regular and pairwise normal.*

Proof. The for follows immediately from Theorem 5.8 and Corollary 4.4.

Definition 5.10. ([4]) A bitopological space (X, T_1, T_2) is said to be *pairwise perfectly normal* if it is pairwise normal, each T_1-closed set is $T_2 - G_\delta$ and each T_2-closed set is $T_1 - G_\delta$.

Theorem 5.11. *Let $(X, P, *)$ be a PqpM-space such that the t_{Δ^+}-norm $*$ is sup-continuous and $* \geq *_W$. Then the bitopological space (X, T_P, T_Q) generated by the probabilistic quasi-pseudo-metric P is pairwise perfectly normal.*

Proof. By Theorem 5.8, (X, T_P, T_Q) is pairwise normal. Let A be a P-closed set. For each $n \in \mathbb{N}$, we define

$$U_n = \{y \in X : d_L(P_{yA}, u_0) < \frac{1}{n}\}.$$

Observe that, by Lemma 5.7, for each $n \in \mathbb{N}$, the set U_n is Q-open. Since A is P-closed, we get

$$\begin{aligned} A &= \overline{A}^P \\ &= \{y \in X : d_L(P_{yA}, u_0) = 0\} \\ &= \{y \in X : d_L(Q_{Ay}, u_0) = 0\} \\ &= \bigcap_{n=1}^{\infty} U_n. \end{aligned}$$

Thus A is $Q - G_\delta$. Similarly, we show that every Q-closed set is $P - G_\delta$. This completes the proof.

Corollary 5.12. *Let (X, p) be a quasi-pseudo-metric space. Then the bitopological space (X, T_P, T_Q) generated by p is pairwise perfectly normal.*

Proof. The proof follows immediately from Theorem 5.11, Corollary 5.9 and Corollary 4.4.

Remark 5.13. The concept of a bitopological space was introduced by Kelly in [7] and separation axioms in those spaces were studied by Kelly [7], Fletcher [2], Patty [9] and Reilly [10].

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