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Quasi-Mean Value Theorems for Symmetrically Differentiable Functions*

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Abstract

In this paper, we give a survey of results related the various quasi-mean value theorems for symmetrically differentiable functions and present some new results. The symmetric derivative of a real function is discussed and it's elementary properties are pointed out. Some results leading to the quasi-Lagrange mean value theorem for the symmetrically differentiable functions are presented along with some generalizations. We also present several results concerning the quasi-Flett mean value theorem for the symmetrically differentiable functions. A new result that eliminate the boundary condition in the quasi-Flett mean theorem is also included. The quasi-Flett mean value theorem of Cauchy like is surveyed along with some related results. A new result that eliminate the boundary condition is presented related to the quasi-Flett mean value theorem of Cauchy like for the symmetrically differentiable functions. Further, by identifying several other new auxiliary functions, we present corresponding new quasi-mean value theorems which are variant of quasi-Lagrange mean value theorem, quasi-Flett mean value theorem, and quasi-Flett mean value theorem of Cauchy like for the symmetrically differentiable functions.

Keywords and Phrases: *Auxiliary function, Darboux property, Flett's mean value theorem, Lagrange mean value theorem, Lagrange mean value theorem of Cauchy like, Quasi-Flett mean value theorem, Quasi-Flett mean value theorem of Cauchy like, Quasi-Lagrange mean value theorem, Symmetric derivative.*

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1. Symmetric Differentiation of Real Functions

The derivative of a real function f, that is a function from the real line $\mathbb{R}$ into itself, at a point x is given by the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists. *The limit exists will mean it exists and is finite.*

Over the years, mathematicians have proposed many generalizations of derivatives by altering the right side of the definition of derivative. If we replace the difference quotient $\frac{f(x+h)-f(x)}{h}$ by a central difference quotient $\frac{f(x+h)-f(x-h)}{2h}$, then we obtain a generalized derivative. This generalized derivative is called the symmetric derivative of the function f.

Definition 1. A real function f on an interval $[a, b]$ is said to be symmetrically differentiable at a point x in $]a, b[$ if the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists. The limit exists will mean it exists and is finite.

We shall denote this limit as $f^s(x)$. If a function is *symmetrically differentiable* at every point of an interval, then we say it is symmetrically differentiable on that interval. One can easily show that if a function is differentiable, then it is also symmetrically differentiable. However, the converse is not true since $f(x) = |x|$ is symmetrically differentiable at 0 but it is not differentiable at 0.

We know from our knowledge of ordinary differentiation that every differentiable function is continuous. Now we wonder if the same is true for symmetrically differentiable functions. The function $f(x) = \frac{1}{x^2}$ is symmetrically differentiable at zero but it is not defined at zero. This illustrates that, contrary to the notion of ordinary differentiability, a discontinuous function can have a symmetric derivative.

A continuous symmetrically differentiable function satisfies the sum, difference, product and quotient rules of ordinary derivatives.

Note that by generalizing the definition of differentiability to symmetric differentiability we lost some aesthetic properties like continuity and smoothness of curves. If a function has a finite jump discontinuity at a point, then it still may be symmetrically differentiable.

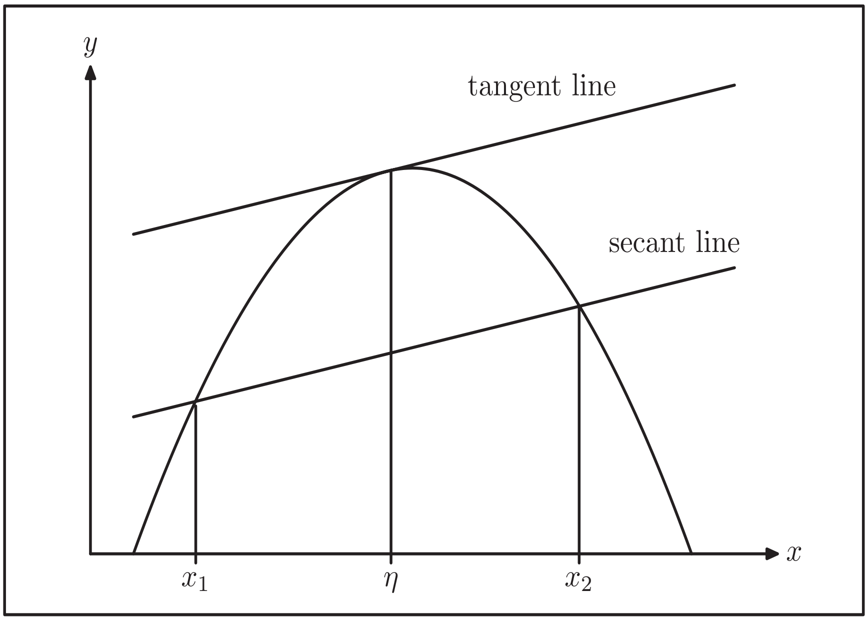

Figure 1: Geometrical Interpretation of the Mean Value Theorem.

2. A Quasi-Lagrange Mean Value Theorem

Lagrange's mean value theorem is a very important result in analysis. It originated from Rolle's theorem, which was proved by the French mathematician Michel Rolle (1652-1719) for polynomials in 1691. Joseph Lagrange (1736-1813) presented his mean value theorem in his book *Theorie des functions analytiques* in 1797. It received further recognition when Augustin Louis Cauchy (1789-1857) proved his mean value theorem in his book *Equationnes differentielles ordinaires*. Most of the results in Cauchy's book were established using the mean value theorem. Since the discovery of Lagrange's mean value theorem, many papers have appeared directly or indirectly dealing with Lagrange's theorem.

Lagrange's mean value theorem states that if f is continuous on $[x_1, x_2]$ and is differentiable on $]x_1, x_2[$, then there exists a mean value $\eta \in]x_1, x_2[$ such that

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(\eta). \tag{1}$$

The mean value theorem has the following geometric interpretation. The tangent line to the graph of the function f at the point $(\eta, f(\eta))$ is parallel to the secant line joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. This is illustrated in Figure 1.

In this section, we establish a quasi-Lagrange mean value theorem for functions with symmetric derivatives. We will further show that every continuous function whose symmetric derivative has the Darboux property obeys the ordinary mean value theorem of Lagrange.

The ordinary mean value theorem is not true for symmetric derivatives as illustrated in the following example. The function $f(x) = |x|$ does not satisfy the ordinary mean value theorem on the interval $[-1, 2]$. The symmetric derivative of the function $f(x) = |x|$ is given by

$$f^s(x) = \begin{cases} \frac{|x|}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Note that the range of $f^s(x)$ is the set $\{0, -1, 1\}$. The slope of the secant line is $\frac{f(2)-f(-1)}{3}$ which is $\frac{1}{3}$. Since the range of $f^s(x)$ does not contain this value therefore there is no η such that

$$f^s(\eta) = \frac{1}{3}.$$

The following lemma due to Aull [1] is instrumental in proving the so called quasi-mean value theorem.

Lemma 1. *Let f be continuous on the interval $[a, b]$ and let f be symmetrically differentiable on $]a, b[$. If $f(b) > f(a)$, then there exists a point $\eta \in]a, b[$ such that*

$$f^s(\eta) \geq 0.$$

Further, if $f(b) < f(a)$, then there exists a point $\xi \in]a, b[$ such that

$$f^s(\xi) \leq 0.$$

Proof. Suppose $f(b) > f(a)$. Let k be a real number such that $f(a) < k < f(b)$. The set

$$\{x \in [a, b] \mid f(x) > k\}$$

is bounded from below by a. Since it is a subset of $\mathbb{R}$ it has a greatest lower bound, say, η. Since f is continuous and k satisfies $f(a) < k < f(b)$, therefore η is different from a and b. Let $]\eta - h,\, \eta + h[$ be an arbitrary neighborhood of η in $[a, b]$. Since η is the greatest lower bound of the set $\{x \in [a, b] \mid f(x) > k\}$, there are points in $]\eta - h,\, \eta + h[$ such that

$$f(x + h) > k$$

and

$$f(x - h) \le k.$$

Therefore

$$f^s(\eta) = \lim_{h \to 0} \frac{f(\eta + h) - f(\eta - h)}{2h} \ge 0.$$

Similarly, it can be shown that if $f(a) > f(b)$, then there exists $\xi \in]a, b[$ such that

$$f^s(\xi) \le 0.$$

The proof of this lemma is now complete.

The following theorem of Aull [1] can be considered as a version of Rolle's theorem for symmetrically differentiable functions.

Theorem 1. *Let f be continuous on $[a, b]$ and symmetrically differentiable on $]a, b[$. Suppose $f(a) = f(b) = 0$. Then there exist η and ξ in $]a, b[$ such that*

$$f^s(\eta) \ge 0$$

and

$$f^s(\xi) \le 0.$$

Proof. If $f \equiv 0$, then the theorem is obviously true. Hence, we assume that $f \not\equiv 0$. Since f is continuous and $f(a) = f(b) = 0$, there are points x_1 and x_2 such that

$$f(x_1) > 0 \qquad \text{and} \qquad f(x_2) < 0 \tag{2}$$

or

$$f(x_1) < 0 \qquad \text{and} \qquad f(x_2) > 0 \tag{3}$$

or

$$f(x_1) > 0 \qquad \text{and} \qquad f(x_2) > 0 \tag{4}$$

or

$$f(x_1) < 0 \qquad \text{and} \qquad f(x_2) < 0. \tag{5}$$

If the inequalities in (2) are true, then we apply Lemma 1 to f on the interval $[a, x_1]$ to obtain

$$f^s(\eta) \geq 0$$

for some $\eta \in]a, x_1[\subset]a, b[$. Again applying Lemma 1 to f on the interval $[a, x_2]$, we obtain

$$f^s(\xi) \leq 0$$

for some $\xi \in]a, x_2[\subset]a, b[$. The other cases can be handled in a similar manner and the proof of the theorem is now complete.

Remark 1. Notice that if we work on the function $f(x) - f(a)$ instead of $f(x)$, then the condition $f(a) = f(b) = 0$ can be relaxed to $f(a) = f(b)$.

Now we prove the quasi-mean value theorem for symmetrically differentiable functions which was originally proved by Aull [1].

Theorem 2. *Let f be continuous on $[a, b]$ and symmetrically differentiable on $]a, b[$. Then there exist η and ξ in $]a, b[$ such that*

$$f^s(\eta) \leq \frac{f(b) - f(a)}{b - a} \leq f^s(\xi).$$

Proof. Define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Then g is continuous on $[a, b]$ and symmetrically differentiable on $]a, b[$. Further, $g(a) = g(b) = 0$. Applying Theorem 1 to g on the interval $[a, b]$, we get

$$g^s(\eta) \leq 0 \qquad \text{and} \qquad g^s(\xi) \geq 0. \tag{6}$$

From (6) and the definition of g, we obtain

$$f^s(\eta) \leq \frac{f(b) - f(a)}{b - a} \leq f^s(\xi)$$

and the proof of the theorem is now complete.

Remark 2. Larson [4] have shown that in Theorem 2 continuity of f can be replaced by measurability.

From the above theorem, we see that the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$ can not be equal to the value of the symmetric derivative of every symmetrically differentiable function f at an intermediate point. Now the question arises what additional condition or conditions should be imposed on the symmetric derivative of f so that the regular mean value theorem will hold for functions that are symmetrically differentiable. We will show that if the symmetric derivative of f has the Darboux property then the regular mean value theorem will hold.

Definition 2. A real valued function f defined on the interval $[a, b]$ is said to have the *Darboux property* if whenever η and ξ are in $[a, b]$, and y is any number between $f(\eta)$ and $f(\xi)$, then there exists a number γ between η and ξ such that $y = f(\gamma)$.

The following theorem is again due to Aull [1].

Theorem 3. *Let f be continuous on $[a, b]$ and symmetrically differentiable on $]a, b[$. If the symmetric derivative of f has the Darboux property, then there exists γ in $]a, b[$ such that*

$$f^s(\gamma) = \frac{f(b) - f(a)}{b - a}.$$

Proof. By the above theorem, we obtain η and ξ in $]a, b[$ such that

$$f^s(\eta) \leq \frac{f(b) - f(a)}{b - a} \leq f^s(\xi).$$

Since $f^s(x)$ has the Darboux property, there exists a γ in $]a, b[$ such that

$$f^s(\gamma) = \frac{f(b) - f(a)}{b - a}.$$

This completes the proof.

Since symmetrically differentiable functions are not necessarily differentiable, the question arises what additional conditions should be imposed on the function to make it differentiable. It is known that continuity of the function along with symmetric differentiability does not imply differentiability.

Next, we show by means of the quasi-mean value theorem that if $f(x)$ and $f^s(x)$ are both continuous, then f is differentiable. The following result is due to Aull [1].

Theorem 4. *Let $f(x)$ be continuous and symmetrically differentiable on $]a,b[$. If the symmetric derivative of f is continuous on $]a,b[$, then $f'(x)$ exists and*

$$f'(x) = f^s(x).$$

Proof. Let $x \in]a,b[$ and choose h to be sufficiently small so that $a < x+h < b$. Since f^s is continuous, it has the Darboux property. Applying Theorem 3 (that is, the mean value theorem) to f on $[x, x+h]$, we have

$$f^s(\eta) = \frac{f(x+h) - f(x)}{h}$$

for some $\eta \in]x, x+h[$. Taking limit of both sides as $h \to 0$ and knowing that the limit of the left side exists, one obtains

$$f^s(x) = f'(x).$$

This completes the proof.

This above theorem can be made even stronger. We leave the proof of the following theorem to the reader.

Theorem 5. *Let $f^s(x)$ be continuous at a point $x = a$ and let $f(x)$ be continuous in a neighborhood of a. Then $f'(a)$ exists and*

$$f'(a) = f^s(a).$$

This shows that the continuity of $f^s(x)$ at a point a and continuity of $f(x)$ in a neighborhood of a suffice for the existence of $f'(a)$.

3. Quasi-Flett Mean Value Theorem and Generalizations

In 1958, T.M. Flett proved that if $f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ and satisfies $f'(a) = f'(b)$, then there exists η in the open interval $]a,b[$ such that $f(\eta) - f(a) = (\eta - a)f'(\eta)$ [3]. Flett's conclusion implies that the tangent at

$(\eta, f(\eta))$ passes through the point $(a, f(a))$. This is illustrated in Figure 2. A result that does not depend on the hypothesis $f'(a) = f'(b)$ was proved in [2]. If $f : [a, b] \to \mathbb{R}$ is a differentiable function, then there exists a point $\eta \in]a, b[$ such that

$$f(\eta) - f(a) = (\eta - a)\, f'(\eta) - \frac{1}{2}\,\frac{f'(b) - f'(a)}{b - a}\,(\eta - a)^2.$$

In this section, we present some results that can be considered as a Flett mean value theorem and its generalizations for symmetrically differentiable functions.

From here after, for f differentiable on $[a, b]$, we adopt the convention $f'(a) = f^s(a)$ and $f'(b) = f^s(b)$.

The following result was proved by Reich [5]. This results generalizes the quasi-mean value theorem for functions with symmetric derivatives.

Theorem 6. *Let f be continuous on the interval $[a, b]$ and symmetrically differentiable on the open interval $]a, b[$. Suppose f is differentiable at the right end point b of the interval $[a, b]$ and $[f(b) - f(a)]\, f'(b) \le 0$. Then there exist points η, ξ in $]a, b]$ such that*

$$f^s(\eta) \ge 0 \qquad \text{and} \qquad f^s(\xi) \le 0.$$

Proof. If $f'(b) = 0$ then letting $\eta = b$ and $\xi = b$, we have $f^s(\eta) = 0$ and $f^s(\xi) = 0$ using the convention $f^s(b) = f'(b)$.

If $f(b) = f(a)$, then applying Theorem 2 to f on $[a, b]$, we obtain

$$f^s(\xi) \le 0 \le f^s(\eta)$$

for some η and ξ in $]a, b[$.

Suppose $[f(b) - f(a)]\, f'(b) < 0$. This implies that either $f'(b) < 0$ and $f(b) > f(a)$ or $f'(b) > 0$ and $f(b) < f(a)$. In the first case, since f is continuous on $[a, b]$ and $f(b) > f(a)$ with f decreasing at b, there exists a point y in $]a, b[$ such that

$$f(y) > f(b) > f(a).$$

Hence applying Lemma 1 to f on the interval $[y, b]$, we obtain $f^s(\xi) \le 0$ for some $\xi \in]y, b[$. Again applying Lemma 1 to f on the interval $[a, b]$, we get

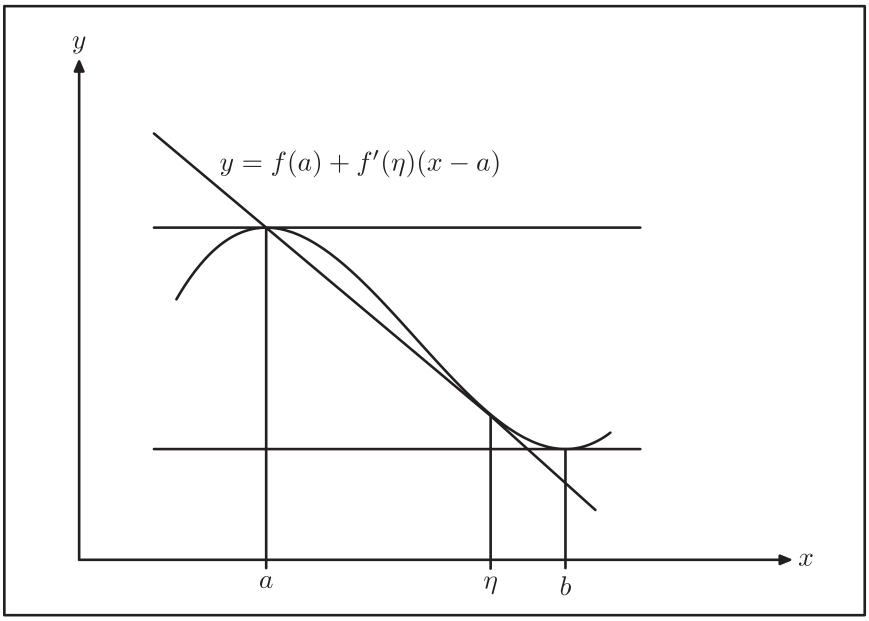

Figure 2: Geometric interpretation of Flett's theorem.

$f^s(\eta) \geq 0$ for some $\eta \in]a,b[$. The case when $f'(b) > 0$ and $f(b) < f(a)$ can be handled in a similar manner. This completes the proof of the theorem.

In the next theorem, we present a generalized version of Flett's mean value theorem for symmetrically differentiable functions due to Reich [5].

Theorem 7. *Let f be continuous on the interval $[a,b]$ and symmetrically differentiable on the open interval $]a,b[$. Suppose f is differentiable at the end points a and b of the interval $[a,b]$ and*

$$\left[f'(b) - \frac{f(b)-f(a)}{b-a}\right]\left[f'(a) - \frac{f(b)-f(a)}{b-a}\right] \geq 0.$$

Then there are points η, ξ in $]a,b]$ such that

$$(\eta - a)\, f^s(\eta) \geq f(\eta) - f(a)$$

and

$$(\xi - a)\, f^s(\xi) \leq f(\xi) - f(a).$$

Proof. Define $h : [a, b] \to \mathbb{R}$ by

$$h(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} & \text{if} \quad x \in \,]a, b] \\ f'(a) & \text{if} \quad x = a. \end{cases} \tag{7}$$

Then evidently, h is continuous on $[a, b]$ and symmetrically differentiable on $]a, b]$. Further, we have

$$h^s(x) = -\frac{h(x)}{x-a} + \frac{f^s(x)}{x-a} \tag{8}$$

for all $x \in \,]a, b]$. In view of

$$\left[f'(b) - \frac{f(b)-f(a)}{b-a}\right] \left[f'(a) - \frac{f(b)-f(a)}{b-a}\right] \geq 0,$$

we see that $[h(b) - h(a)]\, h'(b) \leq 0$. Thus by Theorem 6, we obtain

$$h^s(\xi) \leq 0 \leq h^s(\eta)$$

for some ξ, $\eta \in \,]a, b]$. By (7) and (8), the last inequalities yield

$$(\eta - a)\, f^s(\eta) \geq f(\eta) - f(a)$$

and

$$(\xi - a)\, f^s(\xi) \leq f(\xi) - f(a).$$

This completes the proof of the theorem.

The Theorem 7 is a generalization of quasi-Flett mean value theorem since the latter can be deduced from the former.

Corollary 1. *Let f be continuous on the interval $[a, b]$ and symmetrically differentiable on the open interval $]a, b[$. Further, let f be differentiable at the end points a and b of the interval $[a, b]$. Suppose $f'(a) = f'(b)$. Then there exist $\eta, \xi \in]a, b[$ such that*

$$(\eta - a)\, f^s(\eta) \geq f(\eta) - f(a)$$

and

$$(\xi - a)\, f^s(\xi) \leq f(\xi) - f(a).$$

Proof. Define $h : [a, b] \to \mathbb{R}$ as in the Theorem 7. That is

$$h(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} & \text{if } x \in \,]a, b] \\ f'(a) & \text{if } x = a. \end{cases} \tag{9}$$

Then h is continuous on the interval $[a, b]$ and symmetrically differentiable on $]a, b]$. Differentiating h, we get

$$h^s(x) = -\frac{f(x) - f(a)}{(x-a)^2} + \frac{f^s(x)}{x - a} \tag{10}$$

for all x in $a < x \leq b$.

First, consider the case when

$$f(b) - f(a) = (b - a)\, f'(b). \tag{11}$$

Using (9) and (11), we have

$$\begin{aligned} h(b) - h(a) &= \frac{f(b) - f(a)}{b - a} - f'(a) && \text{by (9)} \\ &= f'(b) - f'(a) && \text{by (11)} \\ &= 0 && \text{by hypothesis} \end{aligned}$$

Hence, we have $h(b) = h(a)$. Applying Theorem 2 to the symmetrically differentiable function to h, we get $h^s(\eta) \geq 0$ and $h^s(\xi) \leq 0$ for some $\eta, \xi \in \,]a, b]$. These two inequalities, by (9) and (10), yield

$$(\eta - a)\, f^s(\eta) \geq f(\eta) - f(a)$$

and

$$(\xi - a)\, f^s(\xi) \leq f(\xi) - f(a).$$

Next, we consider the case

$$f(b) - f(a) \neq (b - a)\, f'(b). \tag{12}$$

Hence, we have $\left[f'(b) - \frac{f(b)-f(a)}{b-a}\right] > 0$ or $\left[f'(b) - \frac{f(b)-f(a)}{b-a}\right] < 0$. Therefore, in either case using the fact that $f'(b) = f'(a)$, we get

$$\left[f'(b) - \frac{f(b) - f(a)}{b - a}\right] \left[f'(a) - \frac{f(b) - f(a)}{b - a}\right] > 0,$$

and by Theorem 7 we have the asserted results. Further, it can be shown that $\eta \neq b$ and $\xi \neq b$.

In the following theorem we eliminate the end points restriction from the quasi-Flett mean value theorem using the auxiliary function used in [2].

Theorem 8. *Let f be continuous on the interval $[a,b]$ and symmetrically differentiable on the open interval $]a,b[$. Suppose f is differentiable at the end points a and b of the interval $[a,b]$. Then there are points η, ξ in $]a,b[$ such that*

$$(\eta - a)\, f^s(\eta) \geq f(\eta) - f(a) + \frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(\eta - a)^2 \tag{13}$$

and

$$(\xi - a)\, f^s(\xi) \leq f(\xi) - f(a) + \frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(\xi - a)^2. \tag{14}$$

Proof. Consider the function

$$g(x) = f(x) - \frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(x-a)^2.$$

Then g is symmetrically differentiable on $]a,b[$ and

$$g^s(x) = f^s(x) - \frac{f'(b)-f'(a)}{b-a}(x-a).$$

Since f is differentiable at the end points of the interval $[a,b]$, thus g is also differentiable at the end points of the interval $[a,b]$. Hence $g'(b) = g'(a)$. Now applying the quasi-Flett mean value theorem, we have the asserted result.

Now the question arises what additional condition or conditions should be imposed on the symmetric derivative of f so that the regular Flett's mean value theorem will hold for functions that are symmetrically differentiable. We will show that if the symmetric derivative of f has the Darboux property then the regular mean value theorem will hold.

Theorem 9. *Let f be continuous on the interval $[a,b]$ and symmetrically differentiable on the open interval $]a,b[$. Suppose f is differentiable at the end points a and b of the interval $[a,b]$. If the symmetric derivative of f has the Darboux property, then there exists γ in $]a,b[$ such that*

$$f(\gamma) - f(a) = (\gamma - a) f^s(\gamma) + \frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(\gamma - a)^2.$$

Proof. Define $\psi :]a, b] \to \mathbb{R}$ as

$$\psi(x) = \frac{f^s(x)}{x-a} - \frac{f(x) - f(a)}{(x-a)^2}.$$

Clearly, $\psi(x)$ is continuous since $f^s(x)$ and $f(x)$ are continuous. Hence $\psi(x)$ has the Darboux property. By Theorem 8 we have the inequalities (13) and (14). These two inequalities can be written as

$$\psi(\xi) \leq \frac{1}{2}\frac{f'(b) - f'(a)}{b-a} \leq \psi(\eta).$$

Since $\psi(x)$ has Darboux property, there exits a $\gamma \in]a, b[$ such that

$$\psi(\gamma) = \frac{1}{2}\frac{f'(b) - f'(a)}{b-a}$$

and hence we have the asserted result.

4. Quasi-Flett Mean Value Theorem of Cauchy Like and Generalizations

Recently, Wachnicki [9] have proved the following Flett mean value theorem of Cauchy like.

Theorem 10. *Let $f, g : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Suppose $g'(x) \neq 0$ for $x \in [a, b]$. Then there exists a point $\eta \in]a, b[$ such that*

$$\frac{f(\eta) - f(a)}{g(\eta) - g(a)} = \frac{f'(\eta)}{g'(\eta)} - \left[\frac{f'(b)}{g'(b)} - \frac{f'(a)}{g'(a)}\right] \frac{g(\eta) - g(a)}{g(b) - g(a)}. \tag{15}$$

In this section, we prove a theorem similar to the theorem of Wachnicki [9] for functions with symmetric derivatives. First, we begin with two results originally due to Reich [5].

Theorem 11. *Let f and g be continuous on the interval $[a, b]$ and symmetrically differentiable on the open interval $]a, b[$. Further, let f and g be both differentiable at the end points a and b of the interval $[a, b]$ with $g'(a) \neq 0 \neq g'(b)$. Suppose $g(x) \neq g(a)$ for all $x \in]a, b]$ and*

$$\left[\frac{f'(a)}{g'(a)} - \frac{f(b) - f(a)}{g(b) - g(a)}\right] [f'(b)[g(b) - g(a)] - [f(b) - f(a)]f'(b)] \geq 0. \tag{16}$$

Then exist points η, ξ *in* $]a,b]$ *such that*

$$[g(\eta)-g(a)]\, f^s(\eta) \geq [f(\eta)-f(a)]\, g^s(\eta)$$

and

$$[g(\xi)-g(a)]\, f^s(\xi) \leq [f(\xi)-f(a)]\, g^s(\xi).$$

Proof. Define $h:[a,b]\to\mathbb{R}$ by

$$h(x)=\begin{cases}\frac{f(x)-f(a)}{g(x)-g(a)} & \text{if} \quad x\in\,]a,b]\\ \frac{f'(a)}{g'(a)} & \text{if} \quad x=a.\end{cases}$$

Then h is continuous on $[a,b]$ and symmetrically differentiable on $]a,b[$. Further h is also differentiable at the end point b. Hence

$$h^s(x)=-\frac{f(x)-f(a)}{(g(x)-g(a))^2}g^s(x)+\frac{1}{g(x)-g(a)}f^s(x).$$

In view of (16) it is easy to check that $[h(b)-h(a)]h'(b)\leq 0$. Hence by Theorem 6 we have

$$h^s(\eta)\geq 0 \qquad \text{and} \qquad h^s(\xi)\leq 0$$

These inequalities implies the asserted results.

Corollary 2. *Let f and g be continuous on the interval $[a,b]$ and symmetrically differentiable on the open interval $]a,b[$. Further, let f and g be both differentiable at the end points a and b of the interval $[a,b]$ with $g'(a)\neq 0\neq g'(b)$ with $g'(b)>0$. Suppose $g(x)\neq g(a)$ for all $x\in\,]a,b]$ and*

$$\frac{f'(a)}{g'(a)}=\frac{f'(b)}{g'(b)}. \tag{17}$$

Then exist points η, ξ *in* $]a,b[$ *such that*

$$[g(\eta)-g(a)]\, f^s(\eta) \geq [f(\eta)-f(a)]\, g^s(\eta)$$

and

$$[g(\xi)-g(a)]\, f^s(\xi) \leq [f(\xi)-f(a)]\, g^s(\xi).$$

Proof. Define $h : [a,b] \to \mathbb{R}$ by

$$h(x) = \begin{cases} \frac{f(x)-f(a)}{g(x)-g(a)} & \text{if} \quad x \in \,]a,b] \\ \frac{f'(a)}{g'(a)} & \text{if} \quad x = a. \end{cases} \tag{18}$$

Then h is symmetrically differentiable on $]a,b[$. Further h is also differentiable at the end point b. Hence

$$h^s(x) = -\frac{f(x)-f(a)}{(g(x)-g(a))^2}\, g^s(x) + \frac{1}{g(x)-g(a)}\, f^s(x). \tag{19}$$

Now consider two cases. Suppose

$$[g(b)-g(a)]f'(b) = [f(b)-f(a)]g'(b). \tag{20}$$

Then

$$\begin{aligned} h(b)-h(a) &= \frac{f(b)-f(a)}{g(b)-g(a)} - \frac{f'(a)}{g'(a)} && \text{by (18)} \\ &= \frac{f'(b)}{g'(b)} - \frac{f'(a)}{g'(a)} && \text{by (20)} \\ &= 0 && \text{by hypothesis (17).} \end{aligned}$$

Applying Theorem 2 for the symmetrically differentiable function to h, we have $h^s(\eta) \geq 0$ and $h^s(\xi) \leq 0$ for some η and ξ in $]a,b]$. These inequalities yield the asserted results.

Next we consider the case

$$[g(b)-g(a)]f'(b) \neq [f(b)-f(a)]g'(b). \tag{21}$$

Hence, we have

$$[g(b)-g(a)]f'(b) - [f(b)-f(a)]g'(b) > 0 \tag{22}$$

or

$$[g(b)-g(a)]f'(b) - [f(b)-f(a)]g'(b) < 0. \tag{23}$$

The inequality (22) can be rewritten as

$$f'(b) - \frac{[f(b)-f(a)]}{g(b)-g(a)} g'(b) > 0. \tag{24}$$

In view of (20), the inequality (24) yields

$$\left[\frac{f'(a)}{g'(a)}g'(b) - \frac{[f(b)-f(a)]}{g(b)-g(a)}g'(b)\right][[g(b)-g(a)]f'(b) - [f(b)-f(a)]g'(b)] > 0.$$

Since $g'(b) > 0$, we have

$$\left[\frac{f'(a)}{g'(a)} - \frac{[f(b)-f(a)]}{g(b)-g(a)}\right][[g(b)-g(a)]f'(b) - [f(b)-f(a)]g'(b)] > 0. \qquad (25)$$

Similarly, using (23) and (20), we again arrive at (25). Hence by Theorem 11, we have the asserted result. It can be shown that $\eta \neq b$ and $\xi \neq b$ and the proof of the corollary is complete.

In the following theorem, we eliminate the end point condition used in Theorem 11 using an idea from the recent paper [9].

Theorem 12. *Let f and g be continuous on the interval $[a,b]$ and symmetrically differentiable on the open interval $]a,b[$. Further, let f and g be both differentiable at the end points a and b of the interval $[a,b]$ with $g'(a) \neq 0 \neq g'(b)$. Suppose $g(x) \neq g(a)$ for all $x \in]a,b]$. Then exist points η, ξ in $]a,b[$ such that*

$$[g(\eta)-g(a)]\, f^s(\eta) \geq \left[f(\eta) - f(a) + \frac{1}{2}\left[\frac{f'(b)}{g'(b)} - \frac{f'(a)}{g'(a)}\right]\left(\frac{(g(\eta)-g(a))^2}{g(b)-g(a)}\right)\right] g^s(\eta)$$

and

$$[g(\xi)-g(a)]\, f^s(\xi) \leq \left[f(\xi) - f(a) + \frac{1}{2}\left[\frac{f'(b)}{g'(b)} - \frac{f'(a)}{g'(a)}\right]\left(\frac{(g(\xi)-g(a))^2}{g(b)-g(a)}\right)\right] g^s(\xi).$$

Proof. Define a function $\psi : [a,b] \to \mathbb{R}$ by

$$\psi(x) = f(x) - \frac{1}{2}\left[\frac{f'(b)}{g'(b)} - \frac{f'(a)}{g'(a)}\right]\frac{(g(x)-g(a))^2}{g(b)-g(a)}. \qquad (26)$$

Then clearly ψ is symmetrically differentiable on $]a,b[$ and differentiable at the end points of $[a,b]$. Moreover ψ is continuous on $[a,b]$. It is easy to check that

$$\psi'(b) = \frac{f'(a)}{g'(a)}\, g'(b)$$

and

$$\psi'(a) = f'(a).$$

Hence

$$\frac{\psi'(b)}{g'(b)} = \frac{\psi'(a)}{g'(a)}.$$

Now applying the Corollary 2 to functions ψ and g on the interval $[a, b]$, we have

$$[g(\eta) - g(a)]\, \psi^s(\eta) \geq [\psi(\eta) - \psi(a)]\, g^s(\eta)$$

and

$$[g(\xi) - g(a)]\, \psi^s(\xi) \leq [\psi(\xi) - \psi(a)]\, g^s(\xi)$$

for some $\eta, \xi \in]a, b]$. Together with the definition of ψ in (26) and from these two inequalities we have the asserted result.

5. Some Additional Auxiliary Functions and Related Quasi-Mean Value Theorems

The mean value theorem was first discovered by Joseph Louis Lagrange (1736-1813) but the idea of applying the Rolle's theorem to a suitably contrived auxiliary function was given by Ossian Bonnet (1819-1892). The auxiliary function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \tag{27}$$

plays the central role in deriving the Lagrange Mean value theorem from the celebrated Rolle's theorem. This auxiliary function is also used in deriving the quasi-Lagrange mean value theorem for the symmetrically differentiable function. There are other auxiliary functions one can use to get variants of the quasi-Lagrange mean theorem for the symmetrically differentiable functions.

Each of these four following functions

$$g_1(x) = [f(x) - f(a)]\,(x - b) - [f(x) - f(b)]\,(x - a), \tag{28}$$

$$g_2(x) = [f(x) - f(a)]\,(x - b) + [f(x) - f(b)]\,(x - a), \tag{29}$$

$$g_3(x) = [f(x) - f(a)]\,[f(x) - f(b)] - (x - a)\,(x - b), \tag{30}$$

$$g_3(x) = [f(x) - f(a)]\,[f(x) - f(b)] + (x - a)\,(x - b). \tag{31}$$

satisfies the boundary condition $g_i(a) = g_i(b)$ for $(i = 1, 2, 3, 4)$ whenever $f(a) = f(b)$ and thus they can be used as auxiliary functions. The above four auxiliary functions was due to Tong (see [7] and [8]). Based on these auxiliary functions now we present the corresponding quasi-Lagrange mean value theorems. Since the proof of these theorems are straightforward we omit their proofs.

The auxiliary function $g_1(x)$ yields the Theorem 2. However, the auxiliary function $g_2(x)$ yields the following theorem.

Theorem 13. *Let f be continuous on $[a,b]$ and symmetrically differentiable on $]a,b[$. Then there exist η and ξ in $]a,b[$ such that*

$$\left(\eta - \frac{a+b}{2}\right) f^s(\eta) \leq \left[\frac{f(a)+f(b)}{2} - f(\eta)\right] \tag{32}$$

$$\left(\xi - \frac{a+b}{2}\right) f^s(\xi) \geq \left[\frac{f(a)+f(b)}{2} - f(\xi)\right]. \tag{33}$$

The following two theorems are obtained by taking the auxiliary functions $g_3(x)$ and $g_4(x)$, respectively.

Theorem 14. *Let f be continuous on $[a,b]$ and symmetrically differentiable on $]a,b[$. Then there exist η and ξ in $]a,b[$ such that*

$$f^s(\eta)\left(f(\eta) - \frac{f(a)+f(b)}{2}\right) \leq -\left(\eta - \frac{a+b}{2}\right) \tag{34}$$

$$f^s(\xi)\left(f(\xi) - \frac{f(a)+f(b)}{2}\right) \geq -\left(\xi - \frac{a+b}{2}\right). \tag{35}$$

Theorem 15. *Let f be continuous on $[a,b]$ and symmetrically differentiable on $]a,b[$. Then there exist η and ξ in $]a,b[$ such that*

$$f^s(\eta)\left(f(\eta) - \frac{f(a)+f(b)}{2}\right) \leq \left(\eta - \frac{a+b}{2}\right) \tag{36}$$

$$f^s(\xi)\left(f(\xi) - \frac{f(a)+f(b)}{2}\right) \geq \left(\xi - \frac{a+b}{2}\right). \tag{37}$$

Notice that if $\psi(x) = \left(x - \frac{a+b}{2}\right) f^s(x) + f(x)$, then the inequalities (32) and (34) in Theorem 13 can be written as

$$\psi(\eta) \leq \frac{f(a)+f(b)}{2} \leq \psi(\xi).$$

If $\psi(x)$ is assumed to have the Darboux property, then there exists a γ between η and ξ such that

$$\psi(\gamma) = \frac{f(a) + f(b)}{2}$$

and this in turn gives

$$\left(\gamma - \frac{a+b}{2}\right) f^s(\gamma) = \left[\frac{f(a) + f(b)}{2} - f(\gamma)\right]. \tag{38}$$

Thus we have the following theorem.

Theorem 16. *Let f be continuous on $[a, b]$ and symmetrically differentiable on $]a, b[$. If the symmetric derivative of f has the Darboux property, then there exists γ in $]a, b[$ such that*

$$\left(\gamma - \frac{a+b}{2}\right) f^s(\gamma) = \left[\frac{f(a) + f(b)}{2} - f(\gamma)\right]. \tag{39}$$

This theorem has the following geometrical interpretation due to Tong [7]. Let us call the point $\left(\frac{a+b}{2}, \frac{f(a)+f(b)}{2}\right)$ the mid-center of $f(x)$ on the interval $[a, b]$ and denote it by M. If $f(x)$ is continuous on $[a, b]$, symmetrically differentiable on $]a, b[$, and $f^s(x)$ has the Darboux property, then there exists at least one point $G = (\gamma, f(\gamma))$ such that the line $x = \gamma$ bisects the angle between the tangent line of the curve $y = f(x)$ at point G, and the line passing through G and the mid-center of $f(x)$ on $[a, b]$. This is illustrated in Figure 3.

The following function (see [2] and also [6])

$$h(x) = f(x) - \frac{1}{2}\frac{f'(b) - f'(a)}{b - a}(x - a)^2 \tag{40}$$

was used as an auxiliary function in eliminating the boundary condition $f'(a) = f'(b)$ in the Flett mean value theorem.

Each of these four functions

$$h_1(x) = 2\, f(x) - \frac{f'(b) - f'(a)}{b - a}(x - a)\,(x - b), \tag{41}$$

$$h_2(x) = 2\, f(x) - \frac{f'(b) + f'(a)}{b - a}(x - a)\,(x - b), \tag{42}$$

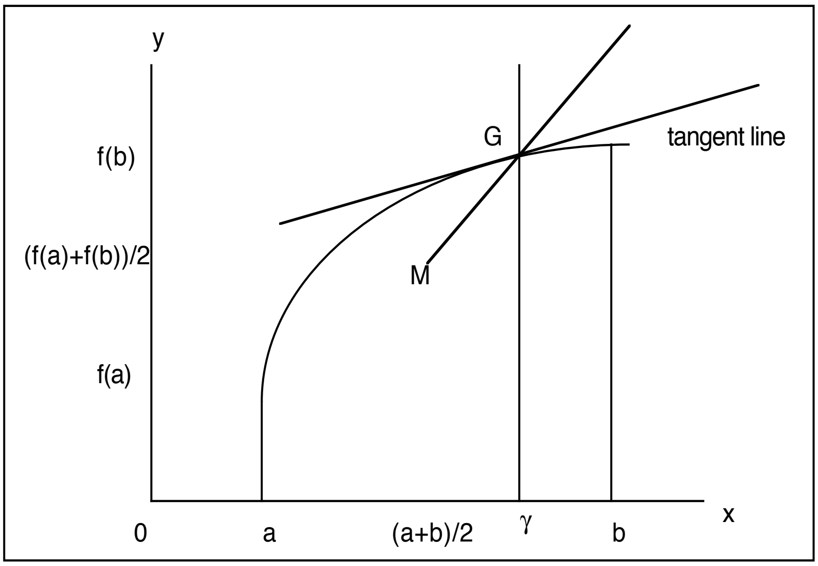

Figure 3: Geometrical Interpretation of Theorem 16.

$$h_3(x) = 2\,f(x) - \frac{f'(b)}{b-a}(x-a)^2 + \frac{f'(a)}{b-a}(x-b)^2, \tag{43}$$

$$h_4(x) = 2\,f(x) + \frac{f'(b)}{b-a}(x-a)^2 - \frac{f'(a)}{b-a}(x-b)^2 \tag{44}$$

satisfies $h_i'(a) = h_i'(b)$ for $(i = 1, 2, 3, 4)$ whenever $f'(a) = f'(b)$. Thus each of these functions can be used for finding a variant of the quasi-Flett mean value theorem for the symmetrically differentiable functions.

The auxiliary functions $h_1(x)$ and $h_3(x)$ yield the Theorem 8. However, the auxiliary functions $h_2(x)$ and $h_4(x)$ yield the following theorem.

Theorem 17. *Let f be continuous on the interval $[a, b]$ and symmetrically differentiable on the open interval $]a, b[$. Suppose f is differentiable at the end points a and b of the interval $[a, b]$. Then there are points η, ξ in $]a, b[$ such that*

$$(\eta - a)\, f^s(\eta) \geq f(\eta) - f(a) - \frac{1}{2}\frac{f'(b) - f'(a)}{b-a}(\eta - a)^2 \tag{45}$$

and

$$(\xi - a)\, f^s(\xi) \leq f(\xi) - f(a) - \frac{1}{2}\frac{f'(b) - f'(a)}{b-a}(\xi - a)^2. \tag{46}$$

Similar results can be obtained by taking the auxiliary function $(b-a)\,h_i(x)$ for $(i=1,2,3,4)$ instead of $h_i(x)$.

The following function (see [9])

$$k(x) = f(x) - \frac{1}{2}\left[\frac{f'(b)}{g'(b)} - \frac{f'(a)}{g'(a)}\right]\frac{(g(x)-g(a))^2}{g(b)-g(a)} \tag{47}$$

was used in [9] as an auxiliary function in eliminating the condition $\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}$ in the Flett mean value theorem of Cauchy like.

Each of these four functions

$$k_1(x) = 2\,f(x) - \left[\frac{f'(b)}{g'(b)} - \frac{f'(a)}{g'(a)}\right]\frac{(g(x)-g(a))(g(x)-g(b))}{g(b)-g(a)}, \tag{48}$$

$$k_3(x) = 2\,f(x) + \left[\frac{f'(b)}{g'(b)} - \frac{f'(a)}{g'(a)}\right]\frac{(g(x)-g(a))(g(x)-g(b))}{g(b)-g(a)}, \tag{49}$$

$$k_3(x) = 2\,f(x) - \frac{f'(b)}{g'(b)}\frac{(g(x)-g(a))^2}{g(b)-g(a)} + \frac{f'(a)}{g'(a)}\frac{(g(x)-g(b))^2}{g(b)-g(a)}, \tag{50}$$

$$k_3(x) = 2\,f(x) + \frac{f'(b)}{g'(b)}\frac{(g(x)-g(a))^2}{g(b)-g(a)} - \frac{f'(a)}{g'(a)}\frac{(g(x)-g(b))^2}{g(b)-g(a)}, \tag{51}$$

satisfies $\frac{k_i'(a)}{g'(a)} = \frac{k_i'(b)}{g'(b)}$ for $(i=1,2,3,4)$ whenever $\frac{f'(a)}{g'(a)} = \frac{f'(b)}{g'(b)}$. Thus each of these functions can be used for finding a variant of the quasi-Flett mean value theorem of Cauchy like for the symmetrically differentiable functions.

The auxiliary function $k_1(x)$ and $k_3(x)$ yield the Theorem 12. However, the auxiliary function $k_2(x)$ and $k_4(x)$ yield the following theorem.

Theorem 18. *Let f and g be continuous on the interval $[a,b]$ and symmetrically differentiable on the open interval $]a,b[$. Further, let f and g be both differentiable at the end points a and b of the interval $[a,b]$ with $g'(a) \neq 0 \neq g'(b)$. Suppose $g(x) \neq g(a)$ for all $x \in]a,b]$. Then exist points η, ξ in $]a,b[$ such that*

$$[g(\eta)-g(a)]\,f^s(\eta) \geq \left[f(\eta) - f(a) - \frac{1}{2}\left[\frac{f'(b)}{g'(b)} - \frac{f'(a)}{g'(a)}\right]\left(\frac{(g(\eta)-g(a))^2}{g(b)-g(a)}\right)\right] g^s(\eta)$$

and

$$[g(\xi)-g(a)]\,f^s(\xi) \leq \left[f(\xi) - f(a) - \frac{1}{2}\left[\frac{f'(b)}{g'(b)} - \frac{f'(a)}{g'(a)}\right]\left(\frac{(g(\xi)-g(a))^2}{g(b)-g(a)}\right)\right] g^s(\xi).$$

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