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Harmonic Univalent Functions Starlike or Convex of Complex Order*

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Abstract

We consider a generalized class $\overline{SH}(m,b)$ which is constructed by modified Salagean operator. The object of the present paper is to study some relations between classes of harmonic univalent functions which are starlike or convex of complex order and other classes of harmonic univalent functions and to give an answer to a conjecture due to Owa [9] in harmonic case.

Keywords and Phrases: *Harmonic, Starlike, Convex, Modified salagean operator.*

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense- preserving in D is that $|h'(z)| > |g'(z)|$ in D. See [4].

Denote by SH the class of functions $f = h+\overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) =$

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$f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in SH$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad |b_1| < 1. \tag{1}$$

Note that SH reduces to S, the class of normalized analytic univalent functions, if the co-analytic part of $f = h + \overline{g}$ is identically zero.

In 1984 Clunie and Sheil-Small [4] investigated the class SH as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on SH and its subclasses such as Avcı and Zlotkiewicz [1], Silverman [13], Silverman and Silvia [14], Jahangiri [5] studied the harmonic univalent functions. More recently, Ahuja et al. [2] investigated the convolutions of special harmonic univalent functions, Jahangiri et al. [7] make use of the Alexander integral transforms of certain analytic functions (which are starlike or convex of positive order) with a view to investigating the construction of sense-preserving, univalent, and close to convex harmonic functions, Yalcin [19] investigated the properties of a generalized class of harmonic univalent functions by using modified Salagean operator and, Ahuja [3] studied on connections between harmonic mappings and hypergeometric functions.

The differential operator D^m $(m \in \mathbb{N}_0)$ was introduced by Salagean [11]. For $f = h + \overline{g}$ given by (1), Jahangiri et al. [6] defined the modified Salagean operator of f as

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)},$$

where

$$D^m h(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k \quad \text{and} \quad D^m g(z) = \sum_{k=1}^{\infty} k^m b_k z^k.$$

We let the subclasses $\overline{SH}$ consisting of harmonic functions $f_m = h + \overline{g}_m$ in SH so that h and g_m are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \; g_m(z) = (-1)^m \sum_{k=1}^{\infty} b_k z^k, \quad a_k, \; b_k \geq 0. \tag{2}$$

We let $\overline{SH}(m, b)$ denote the subclass of $\overline{SH}$ consisting of functions $f_m = h + \overline{g}_m \in \overline{SH}$ that satisfy the condition

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{D^{m+1} f(z)}{D^m f(z)} - 1\right)\right\} > 0, \; b \in \mathbb{C}/\{0\}. \tag{3}$$

We further let $\overline{SH_1}(m,b)$ denote the subclass of $\overline{SH}$ consisting of functions $f_m = h + \overline{g}_m \in \overline{SH}$ that satisfy the condition

$$\sum_{k=1}^{\infty} k^m \left([2(k-1+|b|)]\, a_k + [k+1+|k+1-2b|]\, b_k \right) \leq 4\,|b|\,. \tag{4}$$

Denote by $\overline{SH_2}(m,b)$ the subclass of $\overline{SH}$ consisting of functions $f_m = h+\overline{g}_m \in \overline{SH}$ that satisfy the condition

$$\sum_{k=1}^{\infty} k^m \left(\left[(k-1)\frac{\operatorname{Re} b}{|b|} + |b| \right] a_k + \left[(k+1)\frac{\operatorname{Re} b}{|b|} - |b| \right] b_k \right) \leq 2\,|b|\,. \tag{5}$$

The classes $\overline{SH}(0,1-\alpha)$ and $\overline{SH}(1,1-\alpha)$, $(0 \leq \alpha < 1)$ (α is real) are the classes of harmonic univalent functions starlike and convex of order α with negative coefficients introduced and studied by Silverman [13].

The classes of analytic functions(normalized) starlike and convex of order α $(0 \leq \alpha < 1)$ in U were introduced by Robertson [10] (see also Srivastava and Owa [15]) and the classes of analytic functions starlike or convex of complex order were studied by Nasr and Aouf [8] and Wiatrowski [18], respectively.

If we take the co-analytic part of $f_m = h+\overline{g}_m$ of the form (2) is identically zero and specialize the parametres m and b, we obtain the following subclasses studied by various authors:

(i) $\overline{SH}(0,1-\alpha) = T^*(\alpha)$ and $\overline{SH}(1,1-\alpha) = C(\alpha)$, $\alpha \in [0,1)$ (Silverman [12]);

(ii) $\overline{SH}(0,b) = T_n^*(b)$, $\overline{SH_1}(0,b) = O_n^*(b)$, $\overline{SH_2}(0,b) = P_n^*(b)$ (Owa and Salagean [16]);

(iii) $\overline{SH}(m,b) = T_{n,m}(b)$, $\overline{SH_1}(m,b) = O_{n,m}(b)$, $\overline{SH_2}(m,b) = P_{n,m}(b)$ (Owa and Salagean [17]).

Owa [9] gave a necessary and sufficient coefficient condition of analytic functions starlike of complex order b $(b \in \mathbb{C}/\{0\})$, and then, Owa and Salagean [16] gave an answer to this conjecture , and they derived that the classes satisfying necessary and sufficient coefficient conditions are not coincided in all $b \in \mathbb{C}/\{0\}$. The main purpose of this paper is to give an answer to this conjecture due to Owa in harmonic case. Here, we define three subclasses and we investigate the relations between these classes.

2. Main Results

Theorem 1. $\overline{SH_1}(m,b) \subseteq \overline{SH}(m,b)$.

Proof.

Let $f \in \overline{SH_1}(m,b)$. According to the condition (2) we only need to show that if (4) holds then

$$\operatorname{Re}\left\{\frac{(b-1)D^m f(z) + D^{m+1} f(z)}{bD^m f(z)}\right\} > 0,$$

where $b \in \mathbb{C}/\{0\}$. Using the fact that $\operatorname{Re} w > 0$ if and only if $|1+w| > |1-w|$, it suffices to show that

$$\left|(2b-1)D^m f(z) + D^{m+1} f(z)\right| - \left|D^m f(z) - D^{m+1} f(z)\right| > 0. \tag{6}$$

Substituting for $D^m f(z)$ and $D^{m+1} f(z)$ in (6) yields

$$\begin{aligned}
&\left|2bz - \sum_{k=2}^{\infty} k^m (k-1+2b) a_k z^k + (-1)^m \sum_{k=1}^{\infty} k^m (2b-1-k) b_k \overline{z^k}\right| \\
&- \left|\sum_{k=2}^{\infty} k^m (k-1) a_k z^k + (-1)^{2m} \sum_{k=1}^{\infty} k^m (k+1) b_k \overline{z^k}\right| \\
\geq\ & 2|b||z| - \sum_{k=2}^{\infty} k^m (k-1+2|b|) a_k |z|^k - \sum_{k=1}^{\infty} k^m |k+1-2b| b_k |z|^k \\
&- \sum_{k=2}^{\infty} k^m (k-1) a_k |z|^k - \sum_{k=1}^{\infty} k^m (k+1) b_k |z|^k \\
\geq\ & 2|b| - 2\sum_{k=2}^{\infty} k^m (k-1+|b|) a_k - \sum_{k=1}^{\infty} k^m (k+1+|k+1-2b|) b_k \geq 0.
\end{aligned}$$

Theorem 2. $\overline{SH}(m,b) \subseteq \overline{SH_2}(m,b)$.

Proof.

Let $f \in \overline{SH}(m,b)$. From (3) we have

$$\operatorname{Re}\left\{\frac{1}{b}\left(\frac{-\sum_{k=2}^{\infty}k^m(k-1)a_kz^k-(-1)^{2m}\sum_{k=1}^{\infty}k^m(k+1)b_k\overline{z^k}}{z-\sum_{k=2}^{\infty}k^m a_kz^k+(-1)^{2m}\sum_{k=1}^{\infty}k^m b_k\overline{z^k}}\right)\right\}>-1.$$

If we choose z on the real axis and $z \to 1^-$ we get

$$\frac{\sum_{k=2}^{\infty}k^m(k-1)a_k+\sum_{k=1}^{\infty}k^m(k+1)b_k}{1-\sum_{k=2}^{\infty}k^m a_k+\sum_{k=1}^{\infty}k^m b_k}\,\frac{\operatorname{Re} b}{|b|^2}\le 1,$$

and so,

$$\sum_{k=2}^{\infty}k^m(k-1)a_k+\sum_{k=1}^{\infty}k^m(k+1)b_k\le\frac{|b|^2}{\operatorname{Re} b}\left(1-\sum_{k=2}^{\infty}k^m a_k+\sum_{k=1}^{\infty}k^m b_k\right),$$

which is equivalent to (5). Thus, $f \in \overline{SH_2}(m,b)$.

Theorem 3. *If $b \in (0,1]$ then $\overline{SH_1}(m,b) = \overline{SH}(m,b) = \overline{SH_2}(m,b)$.*

Proof.

If $b \in (0,1]$ then the inequalities (4) and (5) are equivalent; hence $\overline{SH_1}(m,b) = \overline{SH_2}(m,b)$. By using Theorem 1 and Theorem 2, from this assertion we obtain the conclusion of the present theorem.

Theorem 4. *If $b \in (-\infty,0)$ or $\operatorname{Re} b \in (-1/2,0)$, then $\overline{SH_2}(m,b) \not\subseteq \overline{SH}(m,b)$.*

Proof.

Case I: $b \in [-1,0)$.
Let

$$f_\alpha(z) = z - \alpha\frac{z^2}{2^m} \tag{7}$$

and let $\alpha > 0$. We have

$$\sum_{k=1}^{\infty} k^m \left[(k-1)\frac{\operatorname{Re} b}{|b|} + |b|\right] a_k = -b + (-(b+1)\alpha) < 2\,|b|\,, \tag{8}$$

and then $f_\alpha(z) \in \overline{SH_2}(m,b)$.

Let now

$$F(z) = 1 + \frac{1}{b}\left(\frac{D^{m+1} f_\alpha(z)}{D^m f_\alpha(z)} - 1\right),\ z \in U.$$

Then, by a simple computation and by using the fact that

$$D^m f_\alpha(z) = z - 2^m \alpha 2^{-m} z^2 = z - \alpha z^2$$

we obtain

$$F(z) = 1 + \frac{\alpha z}{b(\alpha z - 1)} = 1 + \varphi(z),$$

where

$$\varphi(z) = \frac{\alpha z}{b(\alpha z - 1)}. \tag{9}$$

For $\alpha > 1$ we have $\varphi(U) = \mathbb{C}_\infty - U(c,\rho)$, where U is the disk with the center

$$c = \frac{\alpha^2}{b(\alpha^2 - 1)} \tag{10}$$

and the radius

$$\rho = \frac{\alpha}{b(1-\alpha^2)}. \tag{11}$$

We have $F(U) = \mathbb{C}_\infty - U(c+1,\rho)$ and we deduce that $\operatorname{Re} F(z) > 0$ for all $z \in U$ does not hold.

We have obtained that for $\alpha > 1$, $f_\alpha(z) \notin \overline{SH}(m,b)$ and in this case $\overline{SH_2}(m,b) \nsubseteq \overline{SH}(m,b)$.

Case II: $b \in (-\infty,-1)$.

We consider the function $f_\alpha(z)$ defined by (7) for $\alpha \in \left(1, \frac{b}{1+b}\right)$. In this case the inequality (8) holds too and this implies that $f_\alpha(z) \in \overline{SH_2}(m,b)$.

We also obtain that $f_\alpha(z) \notin \overline{SH}(m,b)$ like in Case I.

Case III: $\operatorname{Re} b \in (-1/2, 0)$.

Let now $f_1 = z - 2^{-m} z^2$. Then $f_1 \in \overline{SH_2}(m,b)$ because the inequality

$$\sum_{k=1}^{\infty} k^m \left[(k-1)\frac{\operatorname{Re} b}{|b|} + |b|\right] a_k = 2\,|b| + \operatorname{Re} b/\,|b| \leq 2\,|b|$$

holds for all b when $\operatorname{Re} b < 0$.

Now let $r = \operatorname{Re} b < 0$ and let s be a negative real number such that

$$1 + 2r(1 - s) > 0.$$

If we choose z_0 one of the rooth of the equation

$$z = \frac{b(1-s)}{1 + b(1-s)},$$

then $z_0 \in U$ and for $f_1$we have

$$1 + \frac{1}{b}\left(\frac{D^{m+1} f_1(z_0)}{D^m f_1(z_0)} - 1\right) = s < 0$$

hence $f_1 \notin \overline{SH}(m, b)$.

Theorem 5. *If $b \in (-\infty, 0)$ or $b \in (1, \infty)$ then $\overline{SH}(m, b) \nsubseteq \overline{SH_1}(m, b)$.*

Proof.

Case I: $b \in (-\infty, 0)$.

Let f_α be given by (7), where $\alpha > |b| / (1 + |b|)$. Then

$$\sum_{k=1}^{\infty} k^m \left[2(k - 1 + |b|)\right] a_k = 2(|b| + (1 + |b|)\alpha) > 4\,|b| \tag{12}$$

and this implies $f_\alpha \notin \overline{SH_1}(m, b)$ for $m \in \mathbb{N}_0$ and $b \in (-\infty, 0)$.

We have

$$F(z) = 1 + \frac{1}{b}\left(\frac{D^{m+1} f_\alpha(z)}{D^m f_\alpha(z)} - 1\right) = 1 + \varphi(z),$$

where φ is given by (9).

From $\varphi(U) = U(c, \rho)$ where c and ρ given by (10) and (11), we obtain

$$\operatorname{Re} F(z) \geq \frac{(1 + b)\alpha + b}{b(\alpha + 1)} \tag{13}$$

If $b \in (-\infty, -1)$ and $\alpha \in \left(\frac{|b|}{1+|b|}, 1\right)$, then

$$\frac{(1+b)\alpha + b}{b(\alpha+1)} > 0 \tag{14}$$

and if $b \in (-1, 0)$ and $\alpha \in (\frac{|b|}{1+|b|}, \frac{|b|}{|1-|b||}) \cap (0, 1)$, then (14) also holds. By combining (14) with (13) and the definition of $\overline{SH}(m, b)$, we obtain that

$$f_\alpha \in \overline{SH}(m, b) \text{ for } \alpha \in \left(\frac{|b|}{1+|b|}, \frac{|b|}{|1-|b||}\right) \cap (0, 1), \text{ and } b \in (-\infty, 0).$$

Case II: $b \in (1, \infty)$.

We consider the function f_α defined by (7) for $\alpha \in \left(\frac{b}{1+b}, 1\right)$. In this case the inequality (12) holds too and this implies that $f_\alpha \notin \overline{SH_1}(m, b)$.

We also obtain that $f_\alpha \in \overline{SH}(m, b)$ for $\alpha \in \left(\frac{b}{1+b}, 1\right)$.

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