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Radii of Starlikeness, Parabolic Starlikeness and Strong Starlikeness for Janowski Starlike Functions with Complex Parameters*

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Abstract

For complex constants A and B with $|B| \leq 1$, $A \neq B$, let $S^*[A, B]$ be the class consisting of normalized analytic functions f satisfying $\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$. The radius of starlikeness, the radius of strong-starlikeness, and the radius of parabolic-starlikeness are obtained for functions in $S^*[A, B]$. Consequences of these results are also discussed.

Keywords and Phrases: *Starlike functions, Janowski starlike functions, strongly starlike functions, parabolic starlike functions, subordination, radius problem.*

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1. Introduction and Motivation

Let $\mathcal{A}$ denote the class of all analytic functions f defined on the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$, $f'(0) = 1$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. For f and g in $\mathcal{A}$, the function f is subordinate to g, written as $f(z) \prec g(z)$, if there is an analytic function w satisfying $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in \Delta$. In the event that g is univalent on Δ, then f subordinates g is equivalent to $f(\Delta) \subset g(\Delta)$ and $f(0) = g(0)$.

Let ϕ be an analytic function with positive real part on Δ, $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $S^*(\phi)$ denote the class of functions f in $\mathcal{S}$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \phi(z).$$

The class $S^*(\phi)$ was introduced by Ma and Minda [8]. The class $S^*[\beta]$ consisting of starlike functions of order β, $0 \leq \beta < 1$ and the class $S^*[\alpha, \beta]$ of Janowski starlike functions are special cases of $S^*(\phi)$, where $\phi(z) := (1+(1-2\beta)z)/(1-z)$ and $\phi(z) := (1 + \alpha z)/(1 + \beta z)$ $(-1 \leq \beta < \alpha \leq 1)$ respectively. For $0 < \alpha \leq 1$, $S^*((\frac{1+z}{1-z})^\alpha)$ is the class of strongly starlike functions f satisfying $|\arg(zf'(z)/f(z))| < \alpha\pi/2$.

Every convex function f in Δ maps the circle $|z| = r < 1$ onto a convex arc. However, it need not map every circular arc about a center in Δ onto a convex arc. This motivated the investigation of uniformly convex functions. A function $f \in \mathcal{S}$ is uniformly convex [6] if f maps every circular arc γ contained in Δ with center $\zeta \in \Delta$, onto a convex arc. Denote the class of all uniformly convex functions by UCV. Ma and Minda [7] and Ronning [19], independently showed that a function f is uniformly convex if and only if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \Delta).$$

Thus, a function $f \in UCV$ if the quantity $1 + (zf''(z)/f'(z))$ lies in the parabolic region $\Omega = \{u + iv : v^2 < 2u - 1\}$.

A corresponding class S_p consisting of parabolic starlike functions f, where $f(z) = zg'(z)$ for g in UCV, was introduced in [19]. Clearly a function f is in

S_p if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \Delta).$$

A survey of these functions may be found in [18], while radius problems associated with the classes UCV and S_p are found in [4, 12, 14, 16, 20, 21]. For further properties of uniformly convex functions, see [1, 2, 3, 6, 9, 11, 13, 15, 17, 18, 22, 25].

The radius of a property $\mathcal{P}$ in a set of functions $\mathcal{M}$, denoted by $R_{\mathcal{P}}(\mathcal{M})$, is the largest number R such that every function in the set $\mathcal{M}$ has the property $\mathcal{P}$ in each disk $\Delta_r = \{z \in \Delta : |z| < r\}$ for every $r < R$. For example, the radius of convexity in the class $\mathcal{S}$ is $2 - \sqrt{3}$.

Let the class $S^*[A, B]$ be defined by

$$S^*[A, B] := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad A, B \in \mathbb{C}, A \neq B, |B| \leq 1, z \in \Delta \right\}.$$

For $A = 1 - 2\beta$, $\beta > 1$ and $B = -1$, denote the class $S^*[1 - 2\beta, -1]$ by $\mathbb{M}(\beta)$. Equivalently, $\mathbb{M}(\beta)$ can be expressed in the form

$$\mathbb{M}(\beta) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \beta, \quad z \in \Delta \right\}.$$

The class $\mathbb{M}(\beta)$ was investigated by Uralegaddi *et al.* [26], while a subclass of $\mathbb{M}(\beta)$ was investigated by Owa and Srivastava [10]. For a fixed nonzero complex number a, let $S^*[A, B, a]$ denote the family of Janowski starlike functions of complex order a consisting of analytic functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{a}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1 + Az}{1 + Bz}, \quad (-1 \leq B < A \leq 1, z \in \Delta). \tag{1.1}$$

In this paper, several radius problems are investigated. Specifically, we compute the radii of starlikeness, strong starlikeness, and parabolic starlikeness for the class $S^*[A, B]$. For this purpose, we first determine the image of the disk $|z| \leq r$ under a function subordinated to a particular bilinear transformation. The $S^*[\beta]$ and $\mathbb{M}(\beta)$ radii are then computed by finding lower and upper bounds for the quantity $\operatorname{Re}(zf'(z)/f(z))$. The $S^*[A, B]$-radius of the class $S^*[C, D]$ is computed by using the corresponding superordinate function.

The radius of strong starlikeness and parabolic starlikeness cannot be computed by estimating the real part of the quantity $zf'(z)/f(z)$ as these classes are not associated with half-planes. Therefore, we determine the circular disk containing the image of $zf'(z)/f(z)$ and use it to compute these radii. Several known results relating to radii problems are shown to be simple consequences of the results obtained. Unless explicitly stated otherwise, it is assumed that the complex constants A and B satisfy $A \neq B$ and $|B| \leq 1$.

2. Radius of Starlikeness of order β

If $f \prec g$, then $|f'(0)| \leq |g'(0)|$ and $f(\Delta_r) \subset g(\Delta_r)$, where Δ_r is the disk $|z| \leq r < 1$. This is called the Lindelöf subordination principle. Let p be analytic in Δ with $p(0) = 1$. Consider

$$p(z) \prec q(z) := \frac{1 + Az}{1 + Bz}.$$

For $r < 1$, the image of the disk $|z| \leq r$ under the map q is clearly a circular disk. Solving for z in terms of q, the inequality $|z| \leq r$ becomes

$$|q(z) - 1| \leq r|A - q(z)B|.$$

This inequality may be expressed in the form

$$\left| q(z) - \frac{1 - A\overline{B}r^2}{1 - |B|^2r^2} \right| \leq \frac{|B - A|r}{1 - |B|^2r^2}.$$

By using the Lindelöf subordination principle, it then follows that

$$\left| p(z) - \frac{1 - A\overline{B}r^2}{1 - |B|^2r^2} \right| \leq \frac{|B - A|r}{1 - |B|^2r^2} \quad (|z| \leq r < 1). \tag{2.1}$$

Theorem 2.1. *Let $\beta \geq 0$, and $f \in S^*[A, B]$. Then*

(1) $f \in S^*[\beta]$ in $|z| \leq R(\beta)$ for $0 \leq \beta < 1$, and

(2) $f \in \mathbb{M}(\beta)$ in $|z| \leq R(\beta)$ for $\beta > 1$,

where

$$R(\beta) := \min\left\{\frac{2|(1-\beta)|}{|B-A|+|(2\beta-1)B-A|}, 1\right\}.$$

In particular, the radius of starlikeness of order β for functions in the class $S^*[A,B,a]$, $(-1 \le B < A \le 1)$, is

$$R_a(\beta) = \min\left\{\frac{2(1-\beta)}{|a|\,|A-B|+|aA+(2-2\beta-a)B|}, 1\right\}.$$

These results are sharp.

Proof.

Let $f \in S^*[A,B]$. With $p(z) = zf'(z)/f(z)$, the inequality (2.1) yields

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \ge \operatorname{Re}\left(\frac{1-A\overline{B}r^2}{1-|B|^2r^2}\right) - \frac{|B-A|r}{1-|B|^2r^2}.$$

The last inequality shows that

$$\operatorname{Re}\frac{zf'(z)}{f(z)} - \beta \ge \frac{1-\beta-|B-A|r-(\operatorname{Re}A\overline{B}-\beta|B|^2)r^2}{1-|B|^2r^2} \ge 0$$

provided $1-\beta-|B-A|r-(\operatorname{Re}A\overline{B}-\beta|B|^2)r^2 \ge 0$. Solving for the positive real root yields

$$r = R_\beta = \frac{2(1-\beta)}{|B-A|+|(2\beta-1)B-A|}.$$

Therefore the $S^*[\beta]$-radius for the class $S^*[A,B]$ is $R(\beta) = \min\{1, R_\beta\}$.

It is easily seen computationally that the result is sharp for the function $f \in \mathcal{A}$ given by

$$f(z) = \begin{cases} z(1+Bz)^{(A-B)/B}, & B \ne 0 \\ ze^{Az}, & B = 0. \end{cases} \tag{2.2}$$

For the function $f \in S^*[A,B]$, the inequality (2.1) again gives

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \frac{1-A\overline{B}r^2}{1-|B|^2r^2}\right) \le \frac{|B-A|r}{1-|B|^2r^2} \quad (|z| \le r < 1)$$

and therefore

$$\begin{aligned}\operatorname{Re}\frac{zf'(z)}{f(z)}-\beta &\leq \operatorname{Re}\left(\frac{1-A\overline{B}r^2}{1-|B|^2r^2}\right)+\frac{|B-A|r}{1-|B|^2r^2}-\beta\\ &=\frac{1-\beta+|B-A|r-(\operatorname{Re}A\overline{B}-\beta|B|^2)r^2}{1-|B|^2r^2}\leq 0\end{aligned}$$

provided $1-\beta+|B-A|r-(\operatorname{Re}A\overline{B}-\beta|B|^2)r^2\leq 0$. This inequality is satisfied for $0\leq r\leq R$ where

$$r=R_{\mathbb{M}}=\frac{2(\beta-1)}{|B-A|+|(2\beta-1)B-A|}.$$

Therefore the $\mathbb{M}(\beta)$-radius for $f\in S^*[A,B]$ is $R_{\mathbb{M}}(\beta)=\min\{R_{\mathbb{M}},1\}$. The result is sharp for the function f given in (2.2).

Next let $f\in S^*[A,B,a]$. Then

$$1+\frac{1}{a}\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+Az}{1+Bz},$$

and hence

$$\frac{zf'(z)}{f(z)}\prec\frac{1+[B+a(A-B)]z}{1+Bz}.$$

It follows that

$$S^*[A,B,a]=S^*[B+a(A-B),B].$$

Thus, the radius of starlikeness of order β for functions in the class $S^*[A,B,a]$ follows from the radius of starlikeness of order β for the class $S^*[A,B]$. This radius is sharp for the extremal function

$$f(z)=\begin{cases} z(1+Bz)^{a(A-B)/B} & B\neq 0,\\ ze^{aAz} & B=0.\end{cases}$$

The case when A and B are real numbers yields the following corollary:

Corollary 2.1. [16] *Let $A,B\in\mathbb{R}$, $A<B$ and $0\leq\beta<1$. If $f\in S^*[A,B]$, then the function f belongs to $S^*[\beta]$ in $|z|\leq R(\beta)$ where*

$$R(\beta)=\begin{cases} 1 & \left(\frac{1+A}{1+B}\geq\beta\right)\\ \frac{1-\beta}{\beta B-A} & \left(\frac{1+A}{1+B}\leq\beta\right).\end{cases}$$

Corollary 2.2. [16] *Let* $A, B \in \mathbb{R}$, $A < B < 1$ *and* $\beta > 1$. *If* $f \in S^*[A, B]$, *then the function* f *is in* $\mathbb{M}(\beta)$ *for* $|z| \leq R_{\mathbb{M}}(\beta)$ *where*

$$R_{\mathbb{M}}(\beta) = \begin{cases} 1 & (\frac{1-A}{1-B} \leq \beta) \\ \frac{\beta-1}{\beta B - A} & (\frac{1-A}{1-B} \geq \beta). \end{cases}$$

The idea of estimating the real part of $zf'(z)/f(z)$ cannot be used to find $S^*[A, B]$-radius of the class $S^*[C, D]$. We can use the disk containing the image of $zf'(z)/f(z)$ to find the radius. However, we shall obtain the radius by making use of the superordinate function.

Theorem 2.2. *Let* $A, B, C, D \in \mathbb{C}$, $A \neq B$, $|B| \leq 1$, $D \neq C$, *and* $|D| \leq 1$. *If* $f \in S^*[C, D]$, *then the* $S^*[A, B]$*-radius,* $R_{[A,B]}$ *of* f, *is given by*

$$R_{[A,B]} = \min\left\{\frac{|A - B|}{|C - D| + |AD - BC|}, 1\right\}.$$

Proof.

Let P and Q be functions defined by

$$P(z) = \frac{1 + Az}{1 + Bz} \quad \text{and} \quad Q(z) = \frac{1 + Cz}{1 + Dz}.$$

Since

$$\frac{zf'(z)}{f(z)} \prec Q(z),$$

the $S^*[A, B]$-radius is the number R $(0 < R \leq 1)$ such that $Q(Rz) \prec P(z)$ for z in Δ. Define the function H by

$$H(z) = P^{-1}(Q(z)).$$

Since

$$P^{-1}(w) = \frac{w - 1}{A - Bw},$$

a computation yields

$$H(z) = \frac{(C - D)z}{(A - B) + (AD - BC)z}.$$

Therefore

$$|H(Rz)| \leq \frac{|C-D|R}{|A-B| - |AD-BC|R} \leq 1$$

for

$$|z| = R \leq \frac{|A-B|}{|C-D| + |AD-BC|}.$$

If $-1 \leq B < A \leq 1$ and $-1 \leq D < C \leq 1$, the $S^*[C,D,b]$-radius of the functions in the class $S^*[A,B,a]$ is

$$R_{[C,D,b]} = \min\left\{ \frac{|b|\,(C-D)}{|a|\,(A-B) + |BD(a-b) + bBC - aAD|}, 1 \right\}.$$

This radius reduces to a result in [4, Theorem 2.3, p. 306] in the special case when $a = b = 1$ and A, B, C, D are real numbers satisfying $-1 \leq B < A \leq 1$ and $-1 \leq D < C \leq 1$.

Our method works even in a more general setting. For a univalent function ϕ with $\phi(0) = 1$ and a nonzero complex number a, let $S^*_a(\phi)$ be the class of functions $f \in \mathcal{A}$ satisfying the subordination

$$1 + \frac{1}{a}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z).$$

A function in the class $S^*_a(\phi)$ is called a starlike function of complex order a with respect to ϕ. Note that the radius of starlikeness of complex order b with respect to ψ for the class $S^*_a(\phi)$ is computed by finding the largest radius $r < 1$ satisfying

$$\left|\psi^{-1}\left(\frac{a[\phi(rz) - 1] + b}{b}\right)\right| \leq 1.$$

Corollary 2.3. *Let $A, B, C, D \in \mathbb{C}$, $A \neq B$, $|B| \leq 1$, $D \neq C$, and $|D| \leq 1$. Then the class $S^*[C,D]$ is a subclass of $S^*[A,B]$ if and only if*

$$|AD - BC| \leq |A-B| - |C-D|.$$

In the special case where the parameters A, B, C, D are real, the class $S^*[A,B,a]$ is a subclass of $S^*[C,D,b]$ if and only if

$$|BD(a-b) + bBC - aAD| \leq |b|\,(C-D) - |a|\,(A-B).$$

The above corollary is an extension of the fact that $S^*[\alpha] \subset S^*[\beta]$ if and only if $\alpha \geq \beta$.

3. Radius of Parabolic Starlikeness

In [17] the class S_p of parabolic starlike functions was generalized by introducing a parameter β, $-1 \leq \beta < 1$. The class $S_p(\beta)$ is a subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$ satisfying

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) - \beta \quad (z \in \Delta).$$

Observe that the values of the functional $zf'(z)/f(z)$ lie in the parabolic region

$$\Omega := \left\{u + iv : v^2 < 2(1-\beta)\left(u - \frac{\beta+1}{2}\right)\right\}. \tag{3.1}$$

In this section, the $S_p(\beta)$-radius of $S^*[A,B]$ for $A, B \in \mathbb{R}$, $A < B$ and $|B| \leq 1$ is determined.

Theorem 3.1. *Let $\beta < 1$, $A < B$, and $|B| \leq 1$. Let R_1 be given by*

$$R_1 := \min\left\{\frac{2(1-\beta)}{B - A + \sqrt{(B-A)^2 + 4B^2(1-\beta)^2}}, 1\right\},$$

R_2 be the largest number in $(0,1]$ such that $1 \geq (B(1+\beta) - 2A)r + \beta$ for all $r \in [0, R_2]$, and R_3 be the largest number in $(0,1]$ such that $A + B(1-2\beta) \geq 2B^3(1-\beta)r^2$ for all $r \in [0, R_3]$. If $f \in S^*[A,B]$, then f satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right| + \beta \quad (|z| < R),$$

where

$$R = \begin{cases} R_2 & \text{if } R_2 \leq R_1 \\ R_3 & \text{if } R_2 > R_1. \end{cases}$$

Proof.

Since

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},$$

it follows from the inequality (2.1) that

$$\left|\frac{zf'(z)}{f(z)} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(B-A)r}{1-B^2r^2}, \quad (|z| \leq r < 1). \tag{3.2}$$

By letting $w(z) = \frac{zf'(z)}{f(z)}$, the points on the boundary of the disk in (3.2) are given by

$$\operatorname{Re} w(z) = \frac{(1 - ABr^2) + (B - A)r\cos\theta}{1 - B^2r^2}, \quad \operatorname{Im} w(z) = \frac{(B - A)r\sin\theta}{1 - B^2r^2}. \tag{3.3}$$

Since

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \left|\frac{zf'(z)}{f(z)} - 1\right| + \beta$$

is equivalent to

$$(\operatorname{Im} w(z))^2 < 2(1 - \beta)\left(\operatorname{Re} w(z) - \frac{1 + \beta}{2}\right), \tag{3.4}$$

substituting (3.3) into (3.4) and simplifying leads to

$$\begin{aligned} T(x) \;:=\; & (B - A)^2r^2x^2 + 2(1 - \beta)(B - A)(1 - B^2r^2)rx \\ & + 2(1 - \beta)(1 - B^2r^2)(1 - ABr^2) \\ & - (1 - \beta^2)(1 - B^2r^2)^2 - (B - A)^2r^2 \geq 0, \end{aligned}$$

where $x = \cos\theta$. We need to find $r = R$ such that $T(x) \geq 0$ for all $x \in [-1, 1]$. Since

$$T'(x) = 2(B - A)^2r^2x + 2(1 - \beta)(B - A)(1 - B^2r^2)r,$$

we see that $T'(x) = 0$ for

$$x = x_0 = -\frac{(1 - \beta)(1 - B^2r^2)}{(B - A)r}.$$

Since $\beta < 1$, $A < B$ and $|B| \leq 1$, then $x_0 < 0$. If $x_0 \leq -1$, it is required that $T(-1) \geq 0$ and if $-1 < x_0 < 0$, then $T(x_0) \geq 0$. Note that $x_0 \leq -1$ is equivalent to

$$r \leq \frac{2(1 - \beta)}{B - A + \sqrt{(B - A)^2 + 4B^2(1 - \beta)^2}}.$$

The condition $T(-1) \geq 0$ is equivalent to

$$1 \geq (B(1 + \beta) - 2A)r + \beta,$$

while $T(x_0) \geq 0$ yields

$$A + B(1 - 2\beta) \geq 2B^3(1 - \beta)r^2.$$

For $A, B \in \mathbb{R}$, $A < B$ and $|B| \leq 1$, let R_1, R_2 and R_3 be as in the hypothesis. If $R_2 \leq R_1$, then the disk (3.2) will be inside the parabolic region (3.1) if and only if $r \leq R_2$. If $R_2 > R_1$, then the disk (3.2) will be inside the parabolic region (3.1) if and only if $r \leq R_3$. This completes the proof. In the special case $\beta = 0$, the following result is obtained:

Corollary 3.1. [16] *For $A, B \in \mathbb{R}$, $A < B$ and $|B| \leq 1$, let R_1 be given by*

$$R_1 := \min\left\{1, \frac{2}{B - A + \sqrt{(B-A)^2 + 4B^2}}\right\}.$$

and let R_2 be the largest number in $(0,1]$ such that $1 \geq (B-2A)r$ for all $r \in [0, R_2]$ and R_3 be the largest number in $(0,1]$ such that $A + B \geq 2B^2r^2$ for all $r \in [0, R_3]$. If $f \in S^*[A, B]$, then the S_p-radius is given by

$$R = \begin{cases} R_2 & \text{if } R_2 \leq R_1 \\ R_3 & \text{if } R_2 > R_1. \end{cases}$$

4. Radius of Strong Starlikeness

Recall that a function $f \in \mathcal{A}$ is strongly starlike of order γ, $0 < \gamma \leq 1$, if f satisfies the subordination

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\gamma},$$

or equivalently

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| \leq \frac{\pi}{2}\gamma.$$

In other words, the values of $zf'(z)/f(z)$ are in the sector $|y| \leq \tan(\gamma\pi/2)x$, $x \geq 0$. In this section we compute the radius of strong starlikeness for the class $S^*[A, B]$. The following lemma will be required:

Lemma 4.1. *[4] If $R_a \leq (\operatorname{Re} a)\sin(\pi\gamma/2) - (\operatorname{Im} a)\cos(\pi\gamma/2)$, $\operatorname{Im} a \geq 0$ for $a \in \mathbb{C}$, then the disk $|w - a| \leq R_a$ is contained in the sector $|\arg w| \leq \pi\gamma/2$, $0 < \gamma \leq 1$.*

Theorem 4.1. *Let $0 < \gamma \leq 1$ and $\operatorname{Im}(A\overline{B}) \leq 0$. If $f \in S^*[A,B]$, then the function f is strongly starlike of order γ in $|z| < R(\gamma)$ where $R(\gamma) = \min\{1, R_\gamma\}$, and*

$$R_\gamma = \frac{2\sin(\pi\gamma/2)}{|B-A| + [|B-A|^2 + 4\sin^2(\pi\gamma/2)\operatorname{Re}(A\overline{B}) - 4\cos(\pi\gamma/2)\sin(\pi\gamma/2) Im\ (A\overline{B})]^{\frac{1}{2}}}\ .$$

Proof.

The inequality (2.1) yields

$$\left|\frac{zf'(z)}{f(z)} - a\right| \leq R_a,$$

where

$$a = \frac{1 - A\overline{B}r^2}{1-|B|^2r^2} \quad \text{and} \quad R_a = \frac{|B-A|r}{1-|B|^2r^2}.$$

The condition in Lemma 4.1 is satisfied if

$$\left[\operatorname{Im}(A\overline{B})\cos(\frac{\pi\gamma}{2}) - \operatorname{Re}(A\overline{B})\sin(\frac{\pi\gamma}{2})\right] r^2 - |B-A|r + \sin(\frac{\pi\gamma}{2}) \geq 0.$$

Since $\sin(\frac{\pi\gamma}{2}) \geq 0$, the above quadratic inequality yields
$r = R_\gamma = \frac{2\sin(\pi\gamma/2)}{|B-A| + [|B-A|^2 + 4\sin^2(\pi\gamma/2)\operatorname{Re}(A\overline{B}) - 4\cos(\pi\gamma/2)\sin(\pi\gamma/2)\operatorname{Im}(A\overline{B})]^{\frac{1}{2}}}.$
Thus f is strongly starlike of order γ in $|z| < R(\gamma)$ where $R(\gamma) = \min\{1, R_\gamma\}$.

Silvia [24] defined the class $SP(\alpha, A, B)$ consisting of functions f in $\mathcal{A}$ satisfying

$$e^{i\alpha}\frac{zf'(z)}{f(z)} \prec \cos\alpha \frac{1+Az}{1+Bz} + i\sin\alpha, \quad z \in \triangle,$$

with $0 \leq \alpha < 1, -1 \leq B < A \leq 1$.

Corollary 4.1. *Let $f \in SP(\alpha, A, B)$, $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$, $0 < \rho \leq 1$, and $B\sin 2\alpha \leq 0$. Then the function f is strongly starlike of order ρ in $|z| < R(\rho)$ where $R(\rho) = \min\{1, R_\rho\}$, and*

$$R_\rho = \frac{2\sin\delta}{(A-B)\cos\alpha + \sqrt{(A-B)^2\cos^2\alpha + 4B^2\sin^2\delta + 4B(A-B)\sin\delta\cos\alpha\sin(\alpha+\delta)}}$$

where $\delta = \pi\rho/2$.

Proof.

The function f is in $SP(\alpha, A, B)$ if

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(A\cos\alpha+iB\sin\alpha)e^{-i\alpha}z}{1+Bz}.$$

Replacing A with $(A\cos\alpha+Bi\sin\alpha)e^{-i\alpha}$ in Theorem 4.1 leads to the desired result.

Remark 4.1. Corollary 4.1 was also obtained by Gangadharan et al. [4]. However, there was a slight mistake in their result and a complete proof is given above. See also [23].

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