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Products of Composition Operators and Volterra-type Integral Operators from Logarithmic Bloch Spaces into Bloch-type Spaces *

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Abstract

Let φ be an analytic self-map of the unit disk $\mathbb{D}$, $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$ and $g \in H(\mathbb{D})$. The boundedness and compactness of the products of composition operators and Volterra-type integral operators from the logarithmic Bloch spaces into the Bloch-type spaces are investigated in this paper.

Keywords and Phrases: *Bloch-type space, logarithmic Bloch space, composition operator, Volterra-type integral operator.*

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1. Introduction

Let $\mathbb{D}$ denote the open unit disc of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. Every analytic self-map φ of the unit disk $\mathbb{D}$ induces through composition a linear composition operator C_φ from $H(\mathbb{D})$ to itself. It is a well-known consequence of Littlewood's subordination principle ([27]) that the formula $C_\varphi(f) = f \circ \varphi$ defines a bounded linear operator on the classical Hardy and Bergman spaces. That is, $C_\varphi : H^p \to H^p$ and $C_\varphi : A^p \to A^p$ are bounded operators. Some characterizations of the boundedness and compactness of the composition operator between various Banach spaces of analytic functions can be found in [4, 6, 8, 12, 33, 41, 52, 53, 54]. Recently, R. Yoneda in [48] gave some necessary and sufficient conditions for a composition operator C_φ to be bounded and compact on the logarithmic Bloch space defined as follows

$$\mathcal{B}_{\log} = \{f \in H(\mathbb{D}) : \|f\| = \sup_{z\in\mathbb{D}}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|f'(z)| < \infty\}.$$

The space $\mathcal{B}_{\log}$ is a Banach space under the norm $\|f\|_{\mathcal{B}_{\log}} = |f(0)| + \|f\|$. It is obvious that there are unbounded $\mathcal{B}_{\log}$ functions. For example, consider the function $f(z) = \log\log\frac{e}{1-z}$. There are also bounded function that they do not belong in $\mathcal{B}_{\log}$. In fact, the interpolating Blaschke products do not belong in $\mathcal{B}_{\log}$. S. Ye in [44] characterized the boundedness and compactness of the weighted composition operator uC_φ between the logarithmic Bloch space $\mathcal{B}_{\log}$ and the β-Bloch space $\mathcal{B}^\beta$ on the unit disk, as well as the boundedness and compactness of the weighted composition operator uC_φ between the little logarithmic Bloch space $\mathcal{B}_{\log,0}$ and the little β-Bloch space $\mathcal{B}_0^\beta$ on the unit disk. A function $f \in H(\mathbb{D})$ is said to belong to the Bloch-type space (or β-Bloch space), denoted by $\mathcal{B}^\beta = \mathcal{B}^\beta(\mathbb{D})$, if

$$B_\beta(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|f'(z)| < \infty.$$

The space $\mathcal{B}^\beta$ becomes a Banach space with the norm $\|f\|_\beta = |f(0)| + B_\beta(f)$. It is easily proved that for $0 < \alpha < 1$, $\mathcal{B}^\alpha \subsetneqq \mathcal{B}_{\log} \subsetneqq \mathcal{B}^1$. Let $\mathcal{B}_0^\beta$ denote the subspace of $\mathcal{B}^\beta$ consisting of those $f \in \mathcal{B}^\beta$ such that

$$\lim_{|z|\to 1}(1-|z|^2)^\beta|f'(z)| = 0.$$

This space is called the little Bloch-type space. For $\beta = 1$, we obtain the well-known classical Bloch space and the little Bloch space, simply denoted by $\mathcal{B}$ and $\mathcal{B}_0$. Let $\mathcal{B}_{\log,0}$ denote the subspace of $\mathcal{B}_{\log}$ consisting of those $f \in \mathcal{B}_{\log}$ such that

$$\lim_{|z|\to 1}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|f'(z)| = 0.$$

For more information about the $\mathcal{B}^\beta$ see [55]. S. Ye in [43] proved that $\mathcal{B}_{\log,0}$ is a closed subspace of $\mathcal{B}_{\log}$. P. Galanopoulos in [7] characterized the boundedness and compactness of the composition operator $C_\varphi : \mathcal{B}_{\log} \to Q^p_{\log}$ and the boundedness and compactness of the weighted composition operator $uC_\varphi : \mathcal{B}_{\log} \to \mathcal{B}_{\log}$. S. Li in [13] characterized the boundedness and compactness of the weighted composition operator uC_φ from Bergman spaces A^p_β into the logarithmic Bloch space $\mathcal{B}_{\log}$ on the unit disk. S. Ye in [45] characterized the boundedness and compactness of the weighted composition operator uC_φ from the general function space $F(p, q, s)$ into the logarithmic Bloch space $\mathcal{B}_{\log}$ on the unit disk. Some characterizations of the weighted composition operator between various Bloch-type spaces can be found in [5, 19, 28, 30, 31, 32]. S. Li and S. Stević in [20] studied the boundedness and compactness of the following two Volterra-type integral operators

$$J_g f(z) = \int_0^z f(\xi)g'(\xi)\,\mathrm{d}\xi$$

and

$$I_g f(z) = \int_0^z f'(\xi)g(\xi)\,\mathrm{d}\xi$$

on the Zygmund space, for any $g \in H(\mathbb{D})$. Y. Yu and Y. Liu in [50] characterized the boundedness and compactness of operators I_g and J_g from the logarithmic Bloch spaces into the Bergman-type spaces. S. Ye and J. Gao in [46] characterized the boundedness and compactness of operators J_g between the logarithmic Bloch spaces and the Bloch-type spaces. Boundedness and compactness of the operators J_g and I_g, some one-dimensional, as well as their n-dimensional extensions, acting on various function spaces were investigated in [1, 2, 3, 9, 10, 11, 14, 16, 17, 18, 20, 21, 24, 34, 35, 36, 40, 42, 47, 49, 50, 51]. S. Stević in [37] introduce the following integral-type operator on the space $H(\mathbb{B})$ of all holomorphic functions on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$

$$P^g_\varphi(f)(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{\mathrm{d}t}{t}, z \in \mathbb{B},$$

where $g \in H(\mathbb{B})$, $g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$, and investigated the boundedness and compactness of the operator P_φ^g from the weighted Bergman space $A_\alpha^p(\mathbb{B})$ to the Bloch-type spaces. S. Stević in [38] continued to investigate the boundedness and compactness of the operator P_φ^g from the logarithmic Bloch space $\mathcal{B}_{\log}(\mathbb{B})$ and the little logarithmic Bloch space $\mathcal{B}_{\log,0}(\mathbb{B})$ to the Bloch-type space $\mathcal{B}_\mu(\mathbb{B})$ or the little Bloch-type space $\mathcal{B}_{\mu,0}(\mathbb{B})$. S. Stević in [39] continued to investigate operator P_φ^g from the Bloch space $\mathcal{B}(\mathbb{B})$ and the little Bloch space $\mathcal{B}_0(\mathbb{B})$ to the Bloch-type space $\mathcal{B}_\mu(\mathbb{B})$ or the little Bloch-type space $\mathcal{B}_{\mu,0}(\mathbb{B})$ on the unit ball and calculated the essential norm of the operators $P_\varphi^g : \mathcal{B}(\mathbb{B})(\text{or } \mathcal{B}_0(\mathbb{B})) \to \mathcal{B}_\mu(\mathbb{B})(\text{or } \mathcal{B}_{\mu,0}(\mathbb{B}))$ in an elegant way. Y. Liu and Y. Yu in [26] investigated integral-type operator C_φ^g from Bloch-type spaces into logarithmic Bloch spaces, where

$$C_\varphi^g f(z) = \int_0^z f'(\varphi(w))g(w)\,\mathrm{d}w, \text{ for } f, g \in H(\mathbb{D}).$$

Products of composition operators and integral-type operators have been recently introduced by S. Li and S. Stević in [15, 22, 23, 25], where they characterized the boundedness and compactness of these operators between various spaces.

Here, we shall be interested in characterizing the products of composition operators and Volterra-type integral operators, which are defined by

$$(C_\varphi J_g f)(z) = \int_0^{\varphi(z)} f(w)g'(w)\,\mathrm{d}w, \quad (C_\varphi I_g f)(z) = \int_0^{\varphi(z)} f'(w)g(w)\,\mathrm{d}w$$

on $H(\mathbb{D})$. More precisely, the boundedness and compactness of the operators $C_\varphi J_g$ from the logarithmic Bloch spaces into the Bloch-type spaces are studied in this paper.

In this paper, positive constants are denoted by C. They may differ from one occurrence to the next.

2. The Boundedness of $C_\varphi J_g : \mathcal{B}_{\log}(\text{or } \mathcal{B}_{\log,0}) \to \mathcal{B}^\beta(\text{or } \mathcal{B}_0^\beta)$

In this section, we study the boundedness of $C_\varphi J_g : \mathcal{B}_{\log}(\text{or } \mathcal{B}_{\log,0}) \to \mathcal{B}^\beta(\text{or } \mathcal{B}_0^\beta)$. For this purpose, we start this section by stating some lemmas which are used in the proofs of main results of this paper. They are incorporated in the lemmas which follow.

Lemma 1. ([44, 46]) *Let* $f \in \mathcal{B}_{\log}$, *then*

$$|f(z)| \leq C\left(2 + \log\log\frac{2}{1-|z|}\right)\|f\|_{\mathcal{B}_{\log}}.$$

Lemma 2. ([44, 46]) *Let* $f(z) = \frac{(1-|z|)\log\frac{2}{1-|z|}}{|1-z|\log\frac{4}{|1-z|}}$, $z \in \mathbb{D}$, *then* $|f(z)| < 2$.

Lemma 3. ([44, 46]) *Let* $g_t(z) = \frac{(1-|z|)\log\frac{2}{1-|z|}}{(1-|tz|)\log\frac{2}{1-|tz|}}$, $t \in [0,1]$, $z \in \mathbb{D}$, *then* $|g_t(z)| < 2$.

Lemma 4. ([44, 46]) *Let* $f \in \mathcal{B}_{\log,0}$, *then*

$$\lim_{|z|\to 1}\frac{|f(z)|}{\log\log\frac{2}{1-|z|}} = 0.$$

Theorem 1. *Suppose* $0 < \beta < \infty$, φ *is an analytic self-map of* $\mathbb{D}$ *and* $g \in H(\mathbb{D})$. *Then the following statements are equivalent:*

(a) $C_\varphi J_g : \mathcal{B}_{\log} \to \mathcal{B}^\beta$ *is bounded.*

(b) $C_\varphi J_g : \mathcal{B}_{\log,0} \to \mathcal{B}^\beta$ *is bounded.*

(c)

$$M_1 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)| < \infty. \tag{2.1}$$

and

$$M_2 = \sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)|\left(\log\log\frac{2}{1-|\varphi(z)|}\right) < \infty. \tag{2.2}$$

Proof. We first prove that $(c) \Rightarrow (a)$. Suppose that (2.1) and (2.2) hold. For $z \in \mathbb{D}$ and $f \in \mathcal{B}_{\log}$, by Lemma 1 we have

$$
\begin{aligned}
&|(1-|z|^2)^\beta (C_\varphi J_g f)'(z)| = (1-|z|^2)^\beta |f(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
&\le C\|f\|_{\mathcal{B}_{\log}} (1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \left(2+\log\log\frac{2}{1-|\varphi(z)|}\right) \\
&\le 2C\|f\|_{\mathcal{B}_{\log}} (1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \\
&+C\|f\|_{\mathcal{B}_{\log}} (1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \left(\log\log\frac{2}{1-|\varphi(z)|}\right) \\
&\le (2CM_1 + CM_2)\|f\|_{\mathcal{B}_{\log}}.
\end{aligned}
$$

On the other hand, note that the quantity $\max_{|w|\le|\varphi(0)|} |g'(w)|$ is finite since the set $|w| \le |\varphi(0)|$ is compact in view of the fact $|\varphi(0)| < 1$. By Lemma 1 we have that

$$
\begin{aligned}
&|(C_\varphi J_g f)(0)| = \left|\int_0^{\varphi(0)} f(w) g'(w)\, \mathrm{d}w\right| \\
&\le \max_{|w|\le|\varphi(0)|} |f(w)| \max_{|w|\le|\varphi(0)|} |g'(w)| \\
&\le C\left(2+\log\log\frac{2}{1-|\varphi(0)|}\right)\|f\|_{B_{\log}},
\end{aligned}
$$

thus

$$
\begin{aligned}
&\|C_\varphi J_g f\|_\beta = |(C_\varphi J_g f)(0)| + B(C_\varphi J_g f) \\
&\le C\|f\|_{\mathcal{B}_{\log}},
\end{aligned}
$$

so that $C_\varphi J_g : \mathcal{B}_{\log} \to \mathcal{B}^\beta$ is bounded.

$(a) \Rightarrow (b)$. This implication is clear.

$(b) \Rightarrow (c)$. Assume that $C_\varphi J_g : \mathcal{B}_{\log,0} \to \mathcal{B}^\beta$ is bounded. By taking the function given by $f(z) = 1$ we obtain (2.1). For $w \in \mathbb{D}$, set

$$
f_w(z) = 2 + \log\log\frac{4}{1-\overline{\varphi(w)}z}.
$$

Since

$$
f_w'(z) = \frac{\overline{\varphi(w)}}{(1-\overline{\varphi(w)}z)\log\frac{4}{1-\overline{\varphi(w)}z}},
$$

we obtain that for each $w \in \mathbb{D}$,

$$
\begin{aligned}
&(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|f_w'(z)| \\
&= (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)\left|\frac{\overline{\varphi(w)}}{(1-\overline{\varphi(w)}z)\log\frac{4}{1-\overline{\varphi(w)}z}}\right| \\
&\le \left|\frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)}{(1-|\varphi(w)|)\log 2}\right| \\
&\to 0 \text{ (as } |z|\to 1).
\end{aligned}
$$

By Lemmas 2 and 3 we have

$$
\begin{aligned}
&\sup_{z\in\mathbb{D}}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|f_w'(z)| \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)\left|\frac{\overline{\varphi(w)}}{(1-\overline{\varphi(w)}z)\log\frac{4}{1-\overline{\varphi(w)}z}}\right| \\
&\le \sup_{z\in\mathbb{D}}\left|\frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)}{(1-\overline{\varphi(w)}z)\log\frac{4}{1-\overline{\varphi(w)}z}}\right| \\
&\le 2\sup_{z\in\mathbb{D}}\frac{(1-|z|)\left(\log\frac{2}{1-|z|}\right)}{(1-|\varphi(w)z|)\log\frac{2}{(1-|\varphi(w)z|)}}\frac{(1-|\varphi(w)z|)\log\frac{2}{(1-|\varphi(w)z|)}}{|1-\overline{\varphi(w)}z|\log\frac{4}{|1-\overline{\varphi(w)}z|}} \\
&\le 8,
\end{aligned}
$$

it follows that $\sup_{w\in\mathbb{D}}\|f_w\| \le 8$, and $f_w \in \mathcal{B}_{\log,0}$ for each fixed $w \in \mathbb{D}$. From this and the boundedness of $C_\varphi J_g : \mathcal{B}_{\log,0} \to \mathcal{B}^\beta$, we have that the following inequality holds

$$
\begin{aligned}
&(1-|w|^2)^\beta|g'(\varphi(w))||\varphi'(w)|\left(\log\log\frac{2}{1-|\varphi(w)|}\right) \\
&\le (1-|w|^2)^\beta|g'(\varphi(w))||f_w(\varphi(w))||\varphi'(w)| \\
&= (1-|w|^2)^\beta|(C_\varphi J_g f_w)'(w)| \\
&\le \|C_\varphi J_g f_w\|_\beta \le \|C_\varphi J_g\|\|f_w\|_{\mathcal{B}_{\log}} \le C < \infty,
\end{aligned}
$$

we obtain (2.2), finishing the proof of the implication.

Remark 1. In [25], for $\alpha \in (0,1)$ and $0 < \beta < \infty$, S. Li and S. Stević proved that the operator $C_\varphi J_g : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \to \mathcal{B}^\beta$ is bounded if and only if the operator $C_\varphi J_g : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \to \mathcal{B}^\beta$ is compact if and only if (2.1) holds.

Theorem 2. *Suppose* $0 < \beta < \infty$, φ *is an analytic self-map of* $\mathbb{D}$ *and* $g \in H(\mathbb{D})$. *Then* $C_\varphi J_g : \mathcal{B}_{\log,0} \to \mathcal{B}_0^\beta$ *is bounded if and only if* $C_\varphi J_g : \mathcal{B}_{\log,0} \to \mathcal{B}^\beta$ *is bounded and*

$$\lim_{|z|\to 1}(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| = 0. \tag{2.3}$$

Proof. Assume that $C_\varphi J_g : \mathcal{B}_{\log,0} \to \mathcal{B}_0^\beta$ is bounded, Then clearly $C_\varphi J_g : \mathcal{B}_{\log,0} \to \mathcal{B}^\beta$ is bounded. Taking the test function $f(z) = 1 \in \mathcal{B}_{\log,0}$, we obtain that (2.3).

Conversely, assume (2.3) holds and $C_\varphi J_g : \mathcal{B}_{\log,0} \to \mathcal{B}^\beta$ is bounded. From this it follows that for any $\epsilon > 0$, there exists a $\delta \in (0,1)$, such that $\delta < |z| < 1$ implies

$$C(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| < \frac{\epsilon}{2}, \tag{2.4}$$

and

$$\frac{|f(z)|}{\log\log\frac{2}{1-|z|}} < \frac{\epsilon}{2M_2}, \tag{2.5}$$

for each function $f \in \mathcal{B}_{\log,0}$ by Lemma 4. Hence writing $\mathbb{D}_1 = \{z \in \mathbb{D} : \delta < |z| < 1\}$, using (2.4), (2.5) and Theorem 1, we deduce that

$$\begin{aligned}
&\sup_{\{z\in\mathbb{D}_1:|\varphi(z)|\le\delta\}}(1-|z|^2)^\beta|(C_\varphi J_g)'(z)| \\
&= \sup_{\{z\in\mathbb{D}_1:|\varphi(z)|\le\delta\}}(1-|z|^2)^\beta|f(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
&\le \sup_{|u|\le\delta}|f(u)| \sup_{\{z\in\mathbb{D}_1:|\varphi(z)|\le\delta\}}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)| \\
&\le C\sup_{z\in\mathbb{D}_1}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)| \\
&< \frac{\epsilon}{2},
\end{aligned} \tag{2.6}$$

and

$$
\begin{aligned}
&\sup_{\{z\in\mathbb{D}_1:\delta<|\varphi(z)|<1\}}(1-|z|^2)^\beta|(C_\varphi J_g)'(z)|\\
&=\sup_{\{z\in\mathbb{D}_1:\delta<|\varphi(z)|<1\}}(1-|z|^2)^\beta|f(\varphi(z))||g'(\varphi(z))||\varphi'(z)|\\
&\le\frac{\epsilon}{2M_2}\sup_{\{z\in\mathbb{D}_1:\delta<|\varphi(z)|<1\}}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)|\left(\log\log\frac{2}{1-|\varphi(z)|}\right)\\
&\le\frac{\epsilon}{2}. \qquad (2.7)
\end{aligned}
$$

From (2.6) and (2.7), we get that $C_\varphi J_g f\in\mathcal{B}_{\log,0}$, that is $C_\varphi J_g:\mathcal{B}_0^\beta\to\mathcal{B}_{\log,0}$ is bounded which finishes the proof.

Remark 2. In [25], for $\alpha\in(0,1)$ and $0<\beta<\infty$, S. Li and S. Stević proved that the operator $C_\varphi J_g:\mathcal{B}^\alpha(\mathcal{B}_0^\alpha)\to\mathcal{B}_0^\beta$ is bounded if and only if (2.3) holds.

Theorem 3. *Suppose* $0<\beta<\infty$, φ *is an analytic self-map of* $\mathbb{D}$ *and* $g\in H(\mathbb{D})$. *If*

$$\lim_{|z|\to1}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)|\left(2+\log\log\frac{2}{1-|\varphi(z)|}\right)=0, \qquad (2.8)$$

then $C_\varphi J_g:\mathcal{B}_{\log}\to\mathcal{B}_0^\beta$ is bounded.

Proof. For any $f\in\mathcal{B}_{\log}$, we have

$$
\begin{aligned}
&(1-|z|^2)^\beta|f(\varphi(z))||g'(\varphi(z))||\varphi'(z)|\\
&\le C\|f\|_{\mathcal{B}_{\log}}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)|\left(2+\log\log\frac{2}{1-|\varphi(z)|}\right)\\
&\to0\ \ (\text{as}\ \ |z|\to1),
\end{aligned}
$$

thus, $C_\varphi J_g f\in\mathcal{B}_0^\beta$. Since (2.8) implies (2,1) and (2.2), by Theorem 1, $C_\varphi J_g:\mathcal{B}_{\log}\to\mathcal{B}^\beta$ is bounded, we obtain that the operator $C_\varphi J_g:\mathcal{B}_{\log}\to\mathcal{B}_0^\beta$ is bounded. The proof is completed.

Theorem 4. *Suppose* $0<\beta<\infty$, φ *is an analytic self-map of* $\mathbb{D}$ *and* $g\in H(\mathbb{D})$. *If* $C_\varphi J_g:\mathcal{B}_{\log}\to\mathcal{B}_0^\beta$ *is bounded, then*

$$\lim_{|z|\to1}(1-|z|^2)^\beta|g'(\varphi(z))||\varphi'(z)|\left(2+\log\log\frac{2}{1-\varphi(z)}\right)=0. \qquad (2.9)$$

Proof. If $C_\varphi J_g : \mathcal{B}_{\log} \to \mathcal{B}_0^\beta$ is bounded, we use the fact that for each function $f \in \mathcal{B}_{\log}$, the analytic function $C_\varphi J_g f \in \mathcal{B}_0^\beta$. Then taking the test function $f(z) = 1 \in \mathcal{B}_{\log}$ and $f(z) = \log\log \frac{2}{1-z} \in \mathcal{B}_{\log}$ we obtain that

$$\lim_{|z|\to 1} (1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| = 0, \tag{2.10}$$

and

$$\lim_{|z|\to 1} (1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \left(\log\log \frac{2}{1-\varphi(z)}\right) = 0. \tag{2.11}$$

(2.10) and (2.11) imply (2.9) holds.

Theorem 5. *Suppose* $0 < \beta < \infty$, φ *is an analytic self-map of* $\mathbb{D}$, $g \in H(\mathbb{D})$ *and* $\mathcal{U}_{\log} = \{f \in \mathcal{B}_{\log} : f'$ *is uniformly continuous on* $\mathbb{D}\}$. *Then* $C_\varphi J_g : \mathcal{U}_{\log} \to \mathcal{B}_0^\beta$ *is bounded if and only if* $C_\varphi J_g : \mathcal{U}_{\log} \to \mathcal{B}^\beta$ *is bounded and* (2.3) *holds.*

Proof. Necessity. If $C_\varphi J_g : \mathcal{U}_{\log} \to \mathcal{B}_0^\beta$ is bounded, then $C_\varphi J_g : \mathcal{U}_{\log} \to \mathcal{B}^\beta$ is bounded. Taking the function $f(z) = 1 \in \mathcal{U}_{\log}$ we get (2.3).

Sufficiency. Suppose that $C_\varphi J_g : \mathcal{U}_{\log} \to \mathcal{B}^\beta$ is bounded and (2.3) holds. For each polynomial $p(z)$ the following inequality holds

$$\begin{aligned}&(1-|z|^2)^\beta |(C_\varphi J_g p)'(z)| = (1-|z|^2)^\beta |p(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\ &\le \|p\|_\infty (1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)|.\end{aligned}$$

(2.3) imply that $C_\varphi J_g p \in \mathcal{B}_0^\beta$. For any $f \in \mathcal{U}_{\log}$, let $f_t(z) = f(tz)$ $(0 < t < 1)$. From Lemma 3 we have

$$\begin{aligned}&(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|f_t'(z)| = (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|tf'(tz)| \\ &\le t\|f\|_{\mathcal{B}_{\log}} \frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)}{(1-|tz|^2)\left(\log\frac{2}{1-|tz|}\right)} \\ &\le C\|f\|_{\mathcal{B}_{\log}} \frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)}{(1-|t|^2)\left(\log\frac{2}{1-|t|}\right)} \\ &\to 0 \quad (\text{as} \ \ |z| \to 1),\end{aligned}$$

and for every $\epsilon > 0$, there is a $\delta > 0$ such that when $1 - \delta < t < 1$,

$$\begin{aligned}
&(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|(f_t-f)'(z)| = (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|tf'(tz)-f'(z)| \\
&\le (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)(|tf'(tz)-f'(tz)|+|f'(tz)-f'(z)|) \\
&\le (1-t)\|f\|_{\mathcal{B}_{\log}}\frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)}{(1-|tz|^2)\left(\log\frac{2}{1-|tz|}\right)} + (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|f'(tz)-f'(z)| \\
&\le C(1-t)\|f\|_{\mathcal{B}_{\log}} + C|f'(tz)-f'(z)| \\
&< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence $f_t \in \mathcal{B}_{\log,0}$ and

$$\|f_t - f\|_{\mathcal{B}_{\log}} \to 0 \text{ as } t \to 1.$$

Since the set of all polynomials is dense in $\mathcal{B}_{\log,0}$([43]), the set of all polynomials is dense in $\mathcal{U}_{\log}$. Thus there is a sequence of polynomials $\{p_n\}$ such that

$$\|p_n - f\|_{\mathcal{B}_{\log}} \to 0 \text{ as } n \to \infty.$$

Since

$$\|J_gC_\varphi p_n - J_gC_\varphi f\|_\beta \le \|J_gC_\varphi\|\|p_n - f\|_{\mathcal{B}_{\log}},$$

and $\mathcal{B}_{\log,0}$ is the closed subset of $\mathcal{B}_{\log}$, we see that $J_gC_\varphi f \in \mathcal{B}_0^\beta$, thus the operator $J_gC_\varphi : \mathcal{U}_{\log} \to \mathcal{B}_0^\beta$ is bounded. The proof is completed.

3. The Compactness of $C_\varphi J_g : \mathcal{B}_{\log}(\text{or } \mathcal{B}_{\log,0}) \to \mathcal{B}^\beta(\text{or } \mathcal{B}_0^\beta)$

Now we turn to study the compactness of $C_\varphi J_g : \mathcal{B}_{\log}(\text{or } \mathcal{B}_{\log,0}) \to \mathcal{B}^\beta(\text{or } \mathcal{B}_0^\beta)$. Recall that an operator is said to be compact provided it takes bounded sets to sets with compact closure. For this purpose, we start this section by stating some useful lemmas. By standard arguments(see, for example, [6]) the following lemmas follow.

Lemma 5. *Suppose* $0 < \beta < \infty$, φ *is an analytic self-map of* $\mathbb{D}$ *and* $g \in H(\mathbb{D})$. *Let* $X = \mathcal{B}_{\log}$ *or* $\mathcal{B}_{\log,0}$, $Y = \mathcal{B}^\beta$ *or* $\mathcal{B}_0^\beta$. *Then* $C_\varphi J_g : X \to Y$ *is compact if and only if* $C_\varphi J_g : X \to Y$ *is bounded and for any bounded sequence* $\{f_n\}$ *in* X *which converges to zero uniformly on compact subsets of* $\mathbb{D}$ *as* $n \to \infty$, *we have* $\|C_\varphi J_g f_n\|_Y \to 0$ *as* $n \to \infty$.

Lemma 6. *Let* $0 < \beta < \infty$. *A closed set* K *in* $\mathcal{B}_0^\beta$ *is compact if and only if* K *is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} (1-|z|^2)^\beta |f'(z)| = 0.$$

The Lemma 6 was proved in [32]. For the case $\beta = 1$, the lemma was proved in [29].

We begin with the following necessary and sufficient condition for the compactness of $C_\varphi J_g : \mathcal{B}_{\log,0} \to \mathcal{B}_0^\beta$.

Theorem 6. *Suppose* $0 < \beta < \infty$, φ *is an analytic self-map of* $\mathbb{D}$ *and* $g \in H(\mathbb{D})$. *Then the following statements are equivalent.*

(1) $C_\varphi J_g : \mathcal{B}_{\log} \to \mathcal{B}_0^\beta$ *is compact*;

(2) $C_\varphi J_g : \mathcal{B}_{\log,0} \to \mathcal{B}_0^\beta$ *is compact*;

(3)

$$\lim_{|z|\to 1} (1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \left(2 + \log\log \frac{2}{1-|\varphi(z)|}\right) = 0. \tag{3.1}$$

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Since $C_\varphi J_g : \mathcal{B}_{\log,0} \to \mathcal{B}_0^\beta$ is compact, we obtain by Lemma 6

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathcal{B}_{\log}}\le 1} (1-|z|^2)^\beta |(C_\varphi J_g f)'(z)| = 0.$$

Thus, for any $\epsilon > 0$, there exists a $\delta \in (0,1)$, such that when $\delta < |z| < 1$,

$$\sup_{\|f\|_{\mathcal{B}_{\log}}\le 1} (1-|z|^2)^\beta |(C_\varphi J_g f)'(z)| < \frac{\epsilon}{C}.$$

Let f_z be defined in Theorem 1. It is easy to see that

$$\frac{1}{C} \leq \frac{1}{\|f_z\|_{\mathcal{B}_{\log}}}.$$

Set $h_z = f_z/\|f_z\|$, then for $\delta < |z| < 1$,

$$\begin{aligned}
&(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \left(2 + \log\log\frac{2}{1-|\varphi(z)|}\right)\\
&\leq C(1-|z|^2)^\beta |h_z(\varphi(z))||g'(\varphi(z))||\varphi'(z)|\\
&= C(1-|z|^2)^\beta |(C_\varphi J_g h_z)'(z)|\\
&\leq C \sup_{\|f\|_{\mathcal{B}_{\log}}\leq 1} |(1-|z|^2)^\beta |(C_\varphi J_g f)'(z)| < \epsilon,
\end{aligned}$$

which gives that

$$\lim_{|z|\to 1}(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \left(2 + \log\log\frac{2}{1-|\varphi(z)|}\right) = 0.$$

(3) ⇒ (1). For any bounded sequence $\{f_n\}$ in $\mathcal{B}_{\log}$ with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, we must prove that by Lemma 5

$$\|C_\varphi J_g f_n\|_\beta \to 0 \text{ as } n \to \infty.$$

We assume that $\|f_n\|_{\mathcal{B}_{\log}} \leq 1$. From (3.1), given $\epsilon > 0$, there exists a $\delta \in (0,1)$, when $\delta < |z| < 1$,

$$C(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \left(2 + \log\log\frac{2}{1-|\varphi(z)|}\right) < \frac{\epsilon}{2}, \tag{3.2}$$

then using (3.2), we get for $\delta < |z| < 1$, $n \in \mathbb{N}$

$$\begin{aligned}
&|(1-|z|^2)^\beta (C_\varphi J_g f_n)'(z)| = (1-|z|^2)^\beta |f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)|\\
&\leq C(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \left(2 + \log\log\frac{2}{1-|\varphi(z)|}\right) < \frac{\epsilon}{2}.
\end{aligned} \tag{3.3}$$

Since $\{f_n\}$ converges uniformly to 0 on a compact subset $\{\varphi(z) : |z| \leq \delta\}$ of $\mathbb{D}$ and $\sup_{|z|\leq\delta}(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \leq M_1$, we see that there exists an $N > 0$, such that for all $n \geq N$

$$\sup_{|z|\leq\delta} |f_n(\varphi(z))| < \frac{\epsilon}{2M_1}.$$

Therefore, for all $n \geq N$, $|z| \leq \delta$,

$$\begin{aligned} |(1-|z|^2)^\beta (C_\varphi J_g f_n)'(z)| &= (1-|z|^2)^\beta |f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\ &< \frac{\epsilon(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)|}{2M_1} \leq \frac{\epsilon}{2}. \end{aligned} \tag{3.4}$$

On the other hand, note that the quantity $\max_{|w|\leq|\varphi(0)|} |g'(w)|$ is finite since the set $|w| \leq |\varphi(0)|$ is compact in view of the fact $|\varphi(0)| < 1$. We have that

$$\begin{aligned} |(C_\varphi J_g f_n)(0)| &= \left| \int_0^{\varphi(0)} f_n(w) g'(w)\, \mathrm{d}w \right| \\ &\leq \max_{|w|\leq|\varphi(0)|} |f_n(w)| \max_{|w|\leq|\varphi(0)|} |g'(w)| \\ &\leq C \max_{|w|\leq|\varphi(0)|} |f_n(w)| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.5}$$

Combining (3.3), (3.4) and (3.5), we obtain

$$\|C_\varphi J_g f_n\|_\beta \to 0 \text{ as } n \to \infty.$$

The proof is complete.

Theorem 7. *Suppose* $0 < \beta < \infty$, φ *is an analytic self-map of* $\mathbb{D}$ *and* $g \in H(\mathbb{D})$. *Then* $C_\varphi J_g : \mathcal{B}_{\log} \to \mathcal{B}^\beta$ *is compact if and only if* $C_\varphi J_g : \mathcal{B}_{\log} \to \mathcal{B}^\beta$ *is bounded and*

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \left(2 + \log\log \frac{2}{1-|\varphi(z)|}\right) = 0. \tag{3.6}$$

Proof. Suppose that $C_\varphi J_g : \mathcal{B}_{\log} \to \mathcal{B}^\beta$ is bounded and (3.6) is true. For any sequence $\{f_n\}$ in $\mathcal{B}_{\log}$ such that$\|f_n\|_{\mathcal{B}_{\log}} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, it is required to show that by Lemma 5

$$\|C_\varphi J_g f_n\|_\beta \to 0 \text{ as } n \to \infty.$$

From (3.6), we have that for every $\epsilon > 0$, there exists a $\delta \in (0,1)$, such that $\delta < |\varphi(z)| < 1$ implies

$$C(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \left(2 + \log\log \frac{2}{1-|\varphi(z)|}\right) < \frac{\epsilon}{2}, \tag{3.7}$$

then using (3.7), we get for $\delta < |\varphi(z)| < 1$,

$$\begin{aligned}
&|(1-|z|^2)^\beta (C_\varphi J_g f_n)'(z)| = (1-|z|^2)^\beta |f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
&\le C(1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \left(2+\log\log\frac{2}{1-|\varphi(z)|}\right) \\
&< \frac{\epsilon}{2}. \qquad (3.8)
\end{aligned}$$

Since $C_\varphi J_g : \mathcal{B}_{\log} \to \mathcal{B}^\beta$ is bounded, using Theorem 1, we see that

$$M_1 = \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| < \infty.$$

Let $U = \{w \in \mathbb{D} : |w| \le \delta\}$, since $\{f_n\}$ converges uniformly to 0 on a compact subset U of $\mathbb{D}$, then there exists an $N > 0$, such that for all $n \ge N$

$$\sup_{w\in U} |f_n(w)| < \frac{\epsilon}{2M_1}.$$

Therefore, for all $n \ge N$

$$\begin{aligned}
&\sup_{\{|\varphi(z)|\le\delta\}} |(1-|z|^2)^\beta (C_\varphi J_g f_n)'(z)| \\
&= \sup_{\{|\varphi(z)|\le\delta\}} (1-|z|^2)^\beta |f_n(\varphi(z))||g'(\varphi(z))||\varphi'(z)| \\
&< \frac{\epsilon}{2M_1} \sup_{\{|\varphi(z)|\le\delta\}} (1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \\
&< \frac{\epsilon}{2M_1} \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |g'(\varphi(z))||\varphi'(z)| \\
&\le \frac{\epsilon}{2}. \qquad (3.9)
\end{aligned}$$

Combining (3.5), (3.8) and (3.9), we obtain

$$\|C_\varphi J_g f_n\|_\beta \to 0 \text{ as } n \to \infty.$$

Conversely, suppose that $C_\varphi J_g : \mathcal{B}_{\log} \to \mathcal{B}^\beta$ is compact, then $C_\varphi J_g : \mathcal{B}_{\log} \to \mathcal{B}^\beta$ is bounded. Hence we only need to prove that (3.6) holds. Assume that $\{z_n\}$ is a sequence in $\mathbb{D}$ such that $\lim_{n\to\infty} |\varphi(z_n)| = 1$ (if such a sequence does not exist then (3.6) is vacuously satisfied). For each n, we choose the test functions f_n defined by

$$f_n(z) = \frac{1}{a_n}\left(\log\log\frac{4}{1-\overline{\varphi(z_n)}z}\right)^2,$$

where $a_n = \log\log\frac{4}{1-|\varphi(z_n)|^2}$. We see that f_n converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. Using Lemmas 2 and 3, we have $\|f_n\|_{\mathcal{B}_{\log}}\le C$ for all n. In view of Lemma 5 it follows that

$$\|C_\varphi J_g f_n\|_\beta \to 0 \ \text{ as } \ n\to\infty.$$

Note that

$$\begin{aligned}
&\|C_\varphi J_g f_n\|_\beta \\
&\ge (1-|z_n|^2)^\beta |f_n(\varphi(z_n))||g'(\varphi(z_n))||\varphi'(z_n)| \\
&= (1-|z_n|^2)^\beta |g'(\varphi(z_n))||\varphi'(z_n)| \left(\log\log\frac{4}{1-|\varphi(z_n)|^2}\right) \\
&\ge (1-|z_n|^2)^\beta |g'(\varphi(z_n))||\varphi'(z_n)| \left(\log\log\frac{2}{1-|\varphi(z_n)|}\right).
\end{aligned} \tag{3.10}$$

Letting $n\to\infty$ in (3.10), we obtain that

$$\lim_{n\to\infty}(1-|z_n|^2)^\beta |g'(\varphi(z_n))||\varphi'(z_n)| \left(\log\log\frac{2}{1-|\varphi(z_n)|}\right) = 0. \tag{3.11}$$

On the other hand, for each n, we choose the test functions g_n defined by

$$g_n(z) = \frac{1}{a_n}\left(\log\log\frac{4}{1-\overline{\varphi(z_n)}z}\right) - \frac{1}{a_n}\left(\log\log\frac{4}{1-\overline{\varphi(z_n)}z}\right)^2,$$

where $a_n = \log\log\frac{4}{1-|\varphi(z_n)|^2}$. We see that g_n converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. Using Lemmas 2 and 3, we have $\|g_n\|_{\mathcal{B}_{\log}}\le C$ for all n. In view of Lemma 5 it follows that

$$\|C_\varphi J_g g_n\|_\beta \to 0 \ \text{ as } \ n\to\infty.$$

Note that

$$\begin{aligned}
&\|C_\varphi J_g g_n\|_\beta \\
&\ge (1-|z_n|^2)^\beta |g_n(\varphi(z_n))||g'(\varphi(z_n))||\varphi'(z_n)| \\
&\ge (1-|z_n|^2)^\beta |g'(\varphi(z_n))||\varphi'(z_n)| \\
&-(1-|z_n|^2)^\beta |g'(\varphi(z_n))||\varphi'(z_n)| \left(\log\log\frac{4}{1-|\varphi(z_n)|^2}\right).
\end{aligned} \tag{3.12}$$

Using (3.10) and letting $n \to \infty$ in (3.12), we obtain that

$$\lim_{n\to\infty}(1-|z_n|^2)^\beta |g'(\varphi(z_n))||\varphi'(z_n)| = 0. \tag{3.13}$$

Hence we are done.

Similarly, we can obtain the following result. The proof of the following theorem will be omitted.

Theorem 8. *Suppose* $0 < \beta < \infty$, φ *is an analytic self-map of* $\mathbb{D}$, $g \in H(\mathbb{D})$ *and* $C_\varphi J_g : \mathcal{B}_{\log} \to \mathcal{B}^\beta$ *is bounded. Then* $C_\varphi J_g : \mathcal{B}_{\log,0} \to \mathcal{B}^\beta$ *is compact if and only if* (3.6) *holds.*

Remark 3. By using the same methods as in the proofs of Theorems 1-8, one can obtain the characterizations of the boundedness and compactness of the operators $C_\varphi I_g$ from logarithmic Bloch spaces into Bloch-type spaces. Let's leave such topics to the interested readers.

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