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Generalized Order Statistics from Kumaraswamy Distribution and its Characterization*

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Abstract

We give explicit expressions and some recurrence relations for single and product moments of generalized order statistics from Kumaraswamy distribution. The results include as particular cases the above relations for moments of reversed order statistics and records. Further, using a recurrence relation for single moments we obtain characterization of Kumaraswamy distribution.

Keywords and Phrases: *Generalized order statistics, Records, single moments, Product moments, Recurrence relations, Kumaraswamy distribution and characterization.*

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1. Introduction

A random variable X is said to have Kumaraswamy distribution (Kumaraswamy (1980)) if its pdf is given by

$$f(x) = \alpha\,\beta\,x^{\alpha-1}[1-x^{\alpha}]^{\beta-1}, \quad 0 \le x \le 1, \ \alpha,\ \beta > 0 \tag{1.1}$$

and the corresponding df is

$$\overline{F}(x) = [1-x^{\alpha}]^{\beta}. \tag{1.2}$$

Therefore, in view of (1.1) and (1.2), we have

$$\overline{F}(x) = \frac{[x^{-(\alpha-1)} - x]}{\alpha\,\beta} f(x), \tag{1.3}$$

The Kumaraswamy distribution was originally conceived to model hydrological phenomena and has been used for this and also for other purposes. See, for example, Sundar and Subbiah (1989), Fletcher and Ponnambalam (1996), Seifi et al. (2000), Ganji et al. (2006), Sanchez et al. (2007) and Courard-Hauri (2007).

In probability theory Kumaraswamy's double bounded distribution is as versatile as the beta distribution, but much simpler to use especially in simulation studies as it has a simple closed form for both the pdf and cdf.

The concept of generalized order statistics (gos) was introduced by Kamps (1995). A variety of order models of random variables is contained in this concept.

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (iid) random variables (rv) with distribution function (df) $F(x)$ and probability density function (pdf) $f(x)$. Assuming that $k > 0$, $n \in N$, $m \in \Re$ and $\gamma_r = k + (n-r)(m+1) > 0$. If the random variables $X(r,n,m,k)$, $r = 1,2,\ldots,n$, possess a joint pdf of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i} f(x_i)\right)(1-F(x_n))^{k-1} f(x_n), \tag{1.4}$$

on the cone $F^{-1}(0) < x_1 \le \ldots \le x_n < F^{-1}(1)$,

then they are called generalized order statistics of a sample from a distribution with df $F(x)$. Note that in the case $m = 0$, $k = 1$, this model reduces to the joint pdf of the ordinary order statistics, and when $m = -1$ we get the joint pdf of the $k-$th upper record values.On using (1.4), the pdf of the $r-$th gos is given by

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}(\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \tag{1.5}$$

and the joint pdf of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}(\bar{F}(x))^m f(x) g_m^{r-1}(F(x))$$

$$\times[h_m(F(y)) - h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s - 1} f(y), \quad x < y, \tag{1.6}$$

where

$$\bar{F}(x) = 1 - F(x), \quad C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad r = 1, 2, \ldots, n,$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \ne -1 \\ -\ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0), \quad x \in [0,1).$$

Many authors utilized the gos in their work, such as Kamps and Gather (1997), Keseling (1999), Cramer and Kamps (2000), Ahsanullah (2000), Pawlas and Szynal (2001), Ahmed and Fawzy (2003), Ahmed (2007), Khan, et al. (2007), Khan et al. (2010) have established recurrence relations for moments of generalized order statistics from Erlang-Truncated exponential distribution. Kamps (1998) investigated the importance of recurrence relations of order statistics in characterization. Recurrence relations for moments of $k-$records were investigated, among others, by Grudzien and Szynal (1997), Pawlas and Szynal (1998, 1999).

In the present study, we have established explicit expressions and some recurrence relations for single and product moments of generalized order statistics from the Kumaraswamy distribution. Further its various deductions and particular cases are discussed and a characterization of Kumaraswamy distribution has been obtained on using a recurrence relation for single moments.

2. Relations for Single Moments

We shall first establish the explicit expression for $E[X^j(r,n,m,k)]$. Using (1.5), we have when $m \neq -1$

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}\int_0^1 x^j [\overline{F}(x)]^{\gamma_r - 1} f(x)\, g_m^{r-1}(F(x))\, dx \tag{2.1}$$

Further, on using the binomial expansion, we can rewrite (2.1) as

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u}$$

$$\times \int_0^1 x^j [\overline{F}(x)]^{\gamma_r+(m+1)u-1} f(x)\, dx \tag{2.2}$$

on using the (1.1) and (1.2) we get

$$E[X^j(r,n,m,k)] = \frac{\alpha\beta C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{u=0}^{r-1}\sum_{v=0}^{\beta(\gamma_r+(m+1)u)-1}(-1)^{u+v}\binom{r-1}{u}$$

$$\times \binom{\beta(\gamma_r+(m+1)u)-1}{v}\frac{1}{[j+\alpha(v+1)]} \tag{2.3}$$

and when $m = -1$ that

$$E[X^j(r,n,-1,k)] = \frac{\alpha\beta^r k^r}{(r-1)!}\sum_{u=0}^{\beta k-1}(-1)^u\binom{\beta k-1}{u}\int_0^1 x^{j+\alpha(u+1)-1}[-\ln(1-x^\alpha)]^{r-1} dx \tag{2.4}$$

Using the logarithmic expansion

$$[-\ln(1-t)]^j = \left(\sum_{p=1}^{\infty}\frac{t^p}{p}\right)^j = \sum_{p=0}^{\infty} a_p(j)\,t^{j+p}, \quad |t|<1,$$

where $a_p(j)$ is the coefficient of t^{j+p} in the expansion of $\left(\sum_{p=1}^{\infty}\frac{t^p}{p}\right)^j$ [see Balakrishnan and Cohan (1991), Shawky and Bakoban (2008)], equation (2.4) can be expressed as

$$E[X^j(r,n,-1,k)] = \frac{\alpha\,\beta^r k^r}{(r-1)!}\sum_{u=0}^{\beta k-1}\sum_{p=0}^{\infty}(-1)^u a_p(r-1)\binom{\beta k-1}{u}\int_0^1 x^{j+\alpha(u+p+1)+r-2}dx$$

$$= \frac{\alpha\,\beta^r k^r}{(r-1)!}\sum_{u=0}^{\beta k-1}\sum_{p=0}^{\infty}(-1)^u a_p(r-1)\binom{\beta k-1}{u}$$

$$\times\frac{1}{[j+\alpha(a+p+1)+r-1]}. \tag{2.5}$$

Special Cases

i) Putting $m=0$, $k=1$ in (2.3) the explicit formula for the single moments of order statistics of the Kumaraswamy distribution can be obtained as

$$E(X_{r:n}^{j}) = \alpha\,\beta\,C_{r:n}\sum_{u=0}^{r-1}\sum_{v=0}^{\beta(n-r+u+1)-1}(-1)^{u+v}\binom{r-1}{u}\binom{\beta(n-r+u+1)-1}{v}$$

$$\times\frac{1}{[j+\alpha(v+1)]},$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}.$$

ii) Putting $k=1$ in (2.5), the explicit expression for the moments of upper record values for the Kumaraswamy distribution can be obtained as

$$E[X^j(r,n,-1,k)] = \frac{\alpha\,\beta^r}{(r-1)!}\sum_{u=0}^{\beta-1}\sum_{p=0}^{\infty}(-1)^u a_p(r-1)\binom{\beta-1}{u}$$

$$\times\frac{1}{[j+\alpha(a+p+1)+r-1]}.$$

Recurrence relations for single moments of *gos* from (1.2) can be obtained in the following theorem

Theorem 2.1. *For Kumaraswamy distribution given in (1.1) and* $n \in N$, $m \in \Re$, $2 \le r \le n$

$$E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)]$$

$$= \frac{j}{\alpha\,\beta\gamma_r}\{E[X^{j-\alpha}(r,n,m,k)] - E[X^j(r,n,m,k)]\}. \tag{2.6}$$

Proof. From (1.5), we have

$$E[X^j(r,n,m,k)] = \frac{C_{r-1}}{(r-1)!}\int_0^1 x^j[\overline{F}(x)]^{\gamma_r-1} f(x)\, g_m^{r-1}(F(x))\,dx. \tag{2.7}$$

Integrating by parts taking $[\overline{F}(x)]^{\gamma_r-1} f(x)$ as the part to be integrated, we get

$$E[X^j(r,n,m,k)] = E[X^j(r-1,n,m,k)] + \frac{j\,C_{r-1}}{\gamma_r(r-1)!}\int_0^1 x^{j-1}[\overline{F}(x)]^{\gamma_r} g_m^{r-1}(F(x))\,dx$$

the constant of integration vanishes since the integral considered in (2.7) is a definite integral. On using (1.3), we obtain

$$E[X^j(r,n,m,k)] - E[X^j(r-1,n,m,k)]$$

$$= \frac{j\,C_{r-1}}{\alpha\beta\,\gamma_r(r-1)!}\left\{\int_0^1 x^{j-\alpha}[\overline{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x))\,dx\right.$$

$$\left. -\int_0^1 x^j[\overline{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x))\,dx\right\}$$

and hence the result.

Remark 2.1. Setting $m = 0$, $k = 1$ in Theorem 2.1, we obtain recurrence relations for single moments of order statistics of the Kumaraswamy distribution in the form

$$E(X_{r:n}^j) - E(X_{r-1:n}^j) = \frac{j}{\alpha\,\beta(n-r+1)}\{E(X_{r:n}^{j-\alpha}) - E(X_{r:n}^j)\}.$$

Remark 2.2. Putting $m = -1$, $k \ge 1$ in (2.6) , we get the recurrence relations for single moments of upper k – records of the Kumaraswamy distribution in the form

$$E(X^{j}_{U(n)})^{k} - E(X^{j}_{U(n-1)})^{k} = \frac{j}{\alpha\beta k}\{E(X^{j-\alpha}_{U(n)})^{k} - E(X^{j}_{U(n)})^{k}\}.$$

3. Relations for Product Moments

On using (1.6) and binomial expansion, the explicit expression for the product moments of *gos* $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s < n$, can be obtained when $m \ne -1$ as

$$E[X^{i}(r,n,m,k)\ X^{j}(s,n,m,k)]$$

$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1} (-1)^{u+v} \binom{r-1}{u}\binom{s-r-1}{v}$$

$$\times \int_{0}^{1} x^{i} [\overline{F}(x)]^{(s-r+u-v)(m+1)-1} f(x) I(x)\, dx, \qquad x > y, \tag{3.1}$$

where

$$I(x) = \int_{x}^{1} y^{j} [\overline{F}(y)]^{\gamma_s+(m+1)v-1} f(y)\, dy. \tag{3.2}$$

$$= \alpha\lambda \sum_{c=0}^{\beta(\gamma_r+(m+1)b)-1} (-1)^{c} \binom{\beta(\gamma_r+(m+1)v)-1}{c} \frac{[1-x^{j+\alpha(c+1)}]}{[j+\alpha(c+1)]}.$$

On substituting the above expression of $I(x)$ in (3.1), and simplifying we get

$$E[X^{i}(r,n,m,k)\ X^{j}(s,n,m,k)]$$

$$= \frac{\alpha^{2}\beta^{2} C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{c=0}^{\beta(\gamma_s+(m+1)v)-1}\sum_{d=0}^{\beta(m+1)(s-r+u-v)-1}$$

$$\times (-1)^{u+v+c+d} \binom{r-1}{u}\binom{s-r-1}{v}\binom{\beta(\gamma_s+(m+1)v)-1}{c}$$

$$\times \binom{\beta(m+1)(s-r+u-v)-1}{d} \frac{[j+\alpha(c+1)]}{[j+\alpha(c+1)][i+\alpha(d+1)][i+j+\alpha(c+d+1)]}. \tag{3.3}$$

and when $m = -1$ that

$$E[X^i(r,n,-1,k)\ X^j(s,n,-1,k)]$$

$$= \frac{k^s}{(r-1)!(s-r-1)!}\int_0^1 x^i[-\ln \overline{F}(x)]^{r-1}\frac{f(x)}{\overline{F}(x)} I(x)\,dx\,,\ x > y\,, \qquad (3.4)$$

where

$$I(x) = \int_x^1 y^j[\ln \overline{F}(x) - \ln \overline{F}(y)]^{s-r-1}[\overline{F}(y)]^{k-1} f(y)\,dy$$

$$= \alpha\beta^{u+1}\sum_{u=0}^{s-r-1}\sum_{v=0}^{\beta k-1}\sum_{p=0}^{\infty}(-1)^{2u+v}\binom{s-r-1}{u}\binom{\beta k-1}{v}[\ln \overline{F}(x)]^{s-r-1-u}$$

$$\times\frac{\left(1-x^{j+\alpha(v+p+1)+u}\right)}{[j+\alpha(v+p+1)+u]}$$

On substituting the above expression of $I(x)$ in (3.4) and simplifying the resulting equation, we obtain

$$E[X^i(r,n,-1,k)\ X^j(s,n,-1,k)]$$

$$= \frac{\alpha^2\beta^s}{(r-1)!(s-r-1)!}\sum_{u=0}^{s-r-1}\sum_{v=0}^{\beta k-1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{w=0}^{\infty}(-1)^{u+v+s-r-1}a_p(u)\,a_q(s-u-2)$$

$$\times\binom{s-r-1}{u}\binom{\beta k-1}{v}\frac{[j+\alpha(v+p+1)]}{[j+\alpha(v+p+1)+u][i+\alpha(w+q+1)+(s-u-1)]}$$

$$\times\frac{1}{[i+j+\alpha(w+q+v+p+2)+(s-u-2)]}. \qquad (3.5)$$

Special Cases

i) Putting $m=0$, $k=1$ in (3.3) the explicit formula for the product moments of order statistics of the Kumaraswamy distribution can be obtained as

$$E[X_{r:n}^i\ X_{s:n}^j] = \alpha^2\beta^2 C_{r,s:n}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}\sum_{c=0}^{\beta(n-r+v+1)-1}\sum_{d=0}^{\beta(s-r+u-v)-1}(-1)^{u+v+c+d}$$

$$\times\binom{r-1}{u}\binom{s-r-1}{v}\binom{\beta(n-r+v+1)-1}{c}\binom{\beta(s-r+u-v)-1}{d}$$

$$\times \frac{[j+\alpha(c+1)]}{[j+\alpha(c+1)][i+\alpha(d+1)][i+j+\alpha(c+d+1)]},$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

iii) Putting $k=1$ in (3.5), the explicit expression for the product moments of upper record values for the Kumaraswamy distribution can be obtained as.

$$E[X^i(r,n,-1,k)\ X^j(s,n,-1,k)]$$

$$= \frac{\alpha^2\beta^s}{(r-1)!(s-r-1)!} \sum_{u=0}^{s-r-1} \sum_{v=0}^{\beta-1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{w=0}^{\infty} (-1)^{v+2(s-r)-1} a_p(u) a_q(s-u-2)$$

$$\times \binom{s-r-1}{u}\binom{\beta-1}{v} \frac{[j+\alpha(v+p+1)]}{[j+\alpha(v+p+1)+u][i+\alpha(w+q+1)+(s-u-1)]}$$

$$\times \frac{1}{[i+j+\alpha(w+q+v+p+2)+(s-u-2)]}. \qquad .$$

Making use of (1.6), we can drive recurrence relations for product moments of *gos* from (1.1).

Theorem 3.1. *For the given Kumaraswamy distribution and* $n \in N$, $m \in \Re$, $1 < r \le s < n-1$

$$E[X^i(r,n,m,k)\,X^j(s,n,m,k)] - E[X^i(r,n,m,k)\,X^j(s-1,n,m,k)]$$

$$= \frac{j}{\alpha\beta\,\gamma_s}\{E[X^i(r,n,m,k)X^{j-\alpha}(s,n,m,k)]$$

$$-E[X^i(r,n,m,k)\,X^j(s,n,m,k)]\}. \qquad (3.6)$$

Proof. From (1.6), we have

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)]$$

$$= \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_0^1 x^i[\overline{F}(x)]^m f(x)\, g_m^{r-1}(F(x))\, I(x)\,dx, \qquad (3.7)$$

where

$$I(x) = \int_x^1 y^j [\overline{F}(y)]^{\gamma_s - 1} [h_m(F(y)) - h_m(F(x))]^{s-r-1} f(y)\, dy\,.$$

Solving the integral in $I(x)$ by parts and substituting the resulting expression in (3.7), we get

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)] - E[X^i(r,n,m,k)\ X^j(s-1,n,m,k)]$$

$$= \frac{j\, C_{s-1}}{\gamma_s (r-1)!(s-r-1)!} \int_0^1 \int_x^1 x^i y^{j-1} [\overline{F}(x)]^m\ f(x)\, g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\overline{F}(y)]^{\gamma_s}\, dy\, dx$$

the constant of integration vanishes since the integral in $I(x)$ is a definite integral. On using the relation (1.3), we obtain

$$E[X^i(r,n,m,k)\ X^j(s,n,m,k)] - E[X^i(r,n,m,k)\ X^j(s-1,n,m,k)]$$

$$= \frac{j\, C_{s-1}}{\alpha\, \beta\ \gamma_s (r-1)!(s-r-1)!} \left\{ \int_0^1 \int_x^1 x^i y^{j-\alpha} [\overline{F}(x)]^m\ f(x) g_m^{r-1}(F(x)) \right.$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [\overline{F}(y)]^{\gamma_s - 1}\ f(y)\, dy\, dx$$

$$- \int_0^1 \int_x^1 x^i y^j [\overline{F}(x)]^m\ f(x) g_m^{r-1}(F(x)) [h_m(F(y)) - h_m(F(x))]^{s-r-1}$$

$$\left. \times [\overline{F}(y)]^{\gamma_s - 1}\ f(y)\, dy\, dx \right\}$$

and hence the result.

Remark 3.1. Setting $m = 0$, $k = 1$ in (3.5), we obtain recurrence relations for product moments of order statistics of the Kumaraswamy distribution in the form

$$E(X_{r,s:n}^{(i,j)}) - E(X_{r,s-1:n}^{(i,j)}) = \frac{j}{\alpha\, \beta\ (n-s+1)} \{ E(X_{r,s:n}^{i,j-\alpha}) - E(X_{r,s:n}^{(i,j)}) \}\,.$$

Remark 3.2. Putting $m = -1$, $k \geq 1$ in (3.5), we get the recurrence relations for product moments of upper k – records of the Kumaraswamy distribution in the form

$$E[X^i(r,n,-1,k) X^j(s,n,-1,k)] - E[X^i(r,n,-1,k) X^j(s-1,n,-1,k)]$$

$$= \frac{j}{\alpha\, \beta\ k} \{ E[X^i(r,n,-1,k) X^{j-\alpha}(s,n,-1,k))]$$

$$-E[X^i(r,n,-1,k)X^j(s,n,-1,k)]\}.$$

4. Characterization

Theorem 4.1. *Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>0$, then*

$$E[X^j(r,n,m,k)] = E[X^j(r-1,n,m,k)] + \frac{j}{\alpha\beta\gamma_r}E[X^{j-\alpha}(r,n,m,k)]$$

$$-\frac{j}{\alpha\beta\gamma_r}E[X^j(r,n,m,k)]. \qquad (4.1)$$

if and only if

$$\bar{F}(x)=[1-x^\alpha]^\beta .$$

Proof. The necessary part follows immediately from equation (2.5). On the other hand if the recurrence relation in equation (4.1) is satisfied, then on using equation (1.5), we have

$$\frac{C_{r-1}}{(r-1)!}\int_0^1 x^j[\bar{F}(x)]^{\gamma_r-1} f(x)\, g_m^{r-1}(F(x))\,dx$$

$$= \frac{(r-1)C_{r-1}}{\gamma_r(r-1)!}\int_0^1 x^j[\bar{F}(x)]^{\gamma_r+m} f(x)\, g_m^{r-2}(F(x))\,dx$$

$$+\frac{jC_{r-1}}{\alpha\beta\gamma_r(r-1)!}\int_0^1 x^{j-\alpha}[\bar{F}(x)]^{\gamma_r-1} g_m^{r-1}(F(x))f(x)dx$$

$$-\frac{jC_{r-1}}{\alpha\beta\gamma_r(r-1)!}\int_0^1 x^{j}[\bar{F}(x)]^{\gamma_r-1} g_m^{r-1}(F(x))f(x)dx \qquad (4.2)$$

Integrating the first integral on the right hand side of equation (4.2), by parts, we get

$$\frac{C_{r-1}}{(r-1)!}\int_0^1 x^j[\bar{F}(x)]^{\gamma_r-1} f(x)\, g_m^{r-1}(F(x))\,dx$$

$$=-\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^1 x^{j-1}[\bar{F}(x)]^{\gamma_r} g_m^{r-1}(F(x))\,dx$$

$$+\frac{C_{r-1}}{(r-1)!}\int_0^1 x^j[\bar{F}(x)]^{\gamma_r-1} f(x)\, g_m^{r-1}(F(x))\,dx$$

$$+\frac{j\,C_{r-1}}{\alpha\beta\,\gamma_r\,(r-1)!}\int_0^1 x^{j-\alpha}\,[\overline{F}\,(x)]^{\gamma_r-1}\,g_m^{r-1}\,(F\,(x))\,f\,(x)dx$$

$$-\frac{j\,C_{r-1}}{\alpha\beta\,\gamma_r\,(r-1)!}\int_0^1 x^{j}\,[\overline{F}\,(x)]^{\gamma_r-1}\,g_m^{r-1}\,(F\,(x))\,f\,(x)dx$$

which reduces to

$$\frac{j\,C_{r-1}}{\gamma_r\,(r-1)!}\int_0^1 x^{j-1}[\overline{F}(x)]^{\gamma_r-1}\,g_m^{r-1}(F\,(x))$$

$$\left[-[\overline{F}(x)]+\frac{(x^{-\alpha+1}-x)}{\alpha\beta}\,f\,(x)\right]dx=0 \tag{4.3}$$

Now applying a generalization of the Müntz-Szász Theorem (Hwang and Lin, 1984) to equation (4.3), we get

$$\frac{f\,(x)}{\overline{F}(x)}=\frac{\alpha\beta}{(x^{-\alpha+1}-x)}$$

which proves that

$$\overline{F}(x)=[1-x^{\alpha}\,]^{\beta},\ 0\le x\le 1\,.$$

5. Conclusion

This paper deals with the generalized order statistics from the Kumaraswamy distribution. Recurrence relations between the single and product moments are derived. Characterization of the Kumaraswamy distribution based on a recurrence relation for single moments is discussed. Special cases are also deduced.

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