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On Multiple Integral Relations Involving Generalized Mellin-Barnes Type of Contour Integral *

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Abstract

In this paper, we first evaluate two finite basic integrals involving the product of the multivariable polynomial $S_L^{h_1,...,h_r}[x_1,...,x_r]$ and two generalized Mellin Barnes type of contour integrals. Next, with the help of these basic integrals we establish two integral relations which involve general arguments and are the most general in nature. Again by suitably specializing the function f in the first integral relation, we have evaluated a multiple integral which is new and quite general in nature. Finally, we present four special cases of our main findings involving the ${}_p\overline{\psi}_q(z)$, $\overline{J}_\lambda^{\nu,\mu}(z)$ and $S_L^h[x]$, which are also believed to be new. Some known results follow as special cases of our findings.

Keywords and Phrases: $\overline{\mathrm{H}}$-*function, multivariable polynomials, generalized Wright's Hyper-geometric function, generalized Wright's Bessal function.*

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1. Introduction

A lot of research work has recently come up on the study and development of a function that is more general than the Fox's H-function, popularly known as $\overline{\text{H}}$-function. It was introduced by Inayat-Hussain [4, 5] and now stands on fairly firm footing through the research contributions of various authers [1-5, 8].

The $\overline{\text{H}}$-function is defined and represented in the following manner [2].

$$\overline{H}_{p,q}^{m,n}[z] = \overline{H}_{p,q}^{m,n}\left[z\middle|_{(b_j,\beta_j)_{1,m},(b_j,\beta_j,B_j)_{m+1,q}}^{(a_j,\alpha_j,A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}}\right] = \frac{1}{2\pi i}\int_L z^{\xi}\overline{\phi}(\xi)d\xi \ (z \neq 0) \tag{1.1}$$

where

$$\overline{\phi}(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j\xi)\prod_{j=1}^{n}\{\Gamma(1 - a_j + \alpha_j\xi)\}^{A_j}}{\prod_{j=m+1}^{q}\{\Gamma(1 - b_j + \beta_j\xi)\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j - \alpha_j\xi)} \tag{1.2}$$

It may be noted that the $\overline{\phi}(\xi)$ contains fractional powers of some of the gamma function and m, n, p, q are integers such that $1 \leq m \leq q, 1 \leq n \leq p, (\alpha_j)_{1,p}, (\beta_j)_{1,q}$ are positive real numbers and $(A_j)_{1,n}, (B_j)_{m+1,q}$ may take non-integer values, which we assume to be positive for standardization purpose. $(a_j)_{1,p}$ and $(b_j)_{1,q}$ are complex numbers.

The nature of contour L, sufficient conditions of convergence of defining integral (1) and other details about the $\overline{\text{H}}$-function can be seen in the papers [2, 3].

The behavior of the $\overline{\text{H}}$-function for small values of $|z|$ follows easily from a result given by Rathie [8]:

$\overline{H}_{p,q}^{m,n}[z] = o\left(|z|^{\alpha}\right)$; where

$$\alpha = \underbrace{\min}_{1\leq j\leq m} \text{Re}\left(\frac{b_j}{\beta_j}\right), \ |z| \to 0 \tag{1.3}$$

The following series representation for the $\overline{\text{H}}$-function given by Saxena et al. [9] will be required later on:

$$\overline{H}^{m,n}_{p,q}\left[z\Big|^{(a_j,\alpha_j,A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}}_{(b_j,\beta_j)_{1,m},(b_j,\beta_j,B_j)_{m+1,q}}\right]=\sum_{t=0}^{\infty}\sum_{h=1}^{m}\overline{f}(\zeta)z^{\zeta} \tag{1.4}$$

where

$$\overline{f}(\zeta)=\frac{\prod\limits_{\substack{j=1\\j\neq h}}^{m}\Gamma(b_j-\beta_j\zeta)\prod\limits_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j\zeta)\}^{A_j}}{\prod\limits_{j=m+1}^{q}\{\Gamma(1-b_j+\beta_j\zeta)\}^{B_j}\prod\limits_{j=n+1}^{p}\Gamma(a_j-\alpha_j\zeta)}\frac{(-1)^t}{t!\beta_h}, \tag{1.5}$$

$$\zeta=\frac{b_h+t}{\beta_h} \tag{1.6}$$

$$\mu_1=\sum_{j=1}^{m}|B_j|+\sum_{j=m+1}^{q}|b_jB_j|-\sum_{j=1}^{n}|a_jA_j|-\sum_{j=n+1}^{p}|A_j|>0 \tag{1.7}$$

$$0<|z|<\infty.$$

The following functions which follow as special cases of the $\overline{\mathrm{H}}$-function will be required in the sequel [3]:

(I)

$$_p\overline{\Psi}_q\left[\begin{array}{l}(a_j,\alpha_j;A_j)_{1,p}\\(b_j,\beta_j;B_j)_{1,q}\end{array};z\right]=\overline{H}^{1,p}_{p,q+1}\left[-z\Big|\begin{array}{c}(1-a_j,\alpha_j;A_j)_{1,p}\\(0,1),(1-b_j,\beta_j;B_j)_{1,q}\end{array}\right] \tag{1.8}$$

(II)

$$\overline{J}^{\nu,\mu}_{\lambda}(z)=\overline{H}^{1,0}_{0,2}\left[-z\Big|\begin{array}{c}-\\(0,1),(-\lambda,\nu;\mu)\end{array}\right] \tag{1.9}$$

A general class of multivariable polynomials is defined by Srivastava and Garg [11]:

$$S^{h_1,\ldots,h_r}_{L}[x_1,\ldots x_r]$$

$$=\sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\leq L}(-L)_{h_1k_1+\ldots+h_rk_r}A(L;k_1,\ldots,k_r)\frac{x^{k_1}}{k_1!}\cdots\frac{x^{k_r}}{k_r!} \tag{1.10}$$

where $h_1, ..., h_r$ arbitrary positive integers and the coefficients $A(L; k_1, ..., k_r)$, $(L; h_i \in N; i = 1, ..., r)$ are arbitrary constant, real or complex.

Evidently by the case $r = 1$ of polynomials (1.10) would correspond to the polynomials given by Srivastava [12]

$$S_L^h[x] = \sum_{k=0}^{[L/h]} \frac{(-L)_{hk}}{k!} A(L; k) x^k \qquad (L \in N = \{0, 1, 2, ...\}) \tag{1.11}$$

where h is an arbitrary positive integer and the coefficient $A_{L,k}(L, k \geq 0)$ are arbitrary constant, real or complex.

In our investigation, we also require the following result:

From Mac-Robert [6], we have

$$\int_0^{\frac{\pi}{2}} \cos(2u\theta)(\sin\theta)^\nu d\theta = \frac{\sqrt{\pi}\,\Gamma\left(u - \frac{\nu}{2}\right)\Gamma\left(\frac{1}{2} + \frac{\nu}{2}\right)}{2\Gamma\left(-\frac{\nu}{2}\right)\Gamma\left(1 + u + \frac{\nu}{2}\right)} \tag{1.12}$$

where u is an integer and $\mathrm{Re}(\nu + 1) > 0$.

2. Basic Integrals

First Integral

$$\int_0^{\frac{\pi}{2}} \cos(2u\theta)(\sin\theta)^\nu \overline{H}_{p_1,q_1}^{m_1,n_1}\left[a(\sin\theta)^\sigma \Big|_{(b_j,\beta_j)_{1,m_1},(b_j,\beta_j,B_j)_{m_1+1,q_1}}^{(a_j,\alpha_j,A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}}\right] \times$$

$$\overline{H}_{p_2,q_2}^{m_2,n_2}\left[bz^\rho(\sin\theta)^\omega \Big|_{(d_j,\delta_j)_{1,m_2},(d_j,\delta_j,D_j)_{m_2+1,q_2}}^{(c_j,\gamma_j,C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}}\right] d\theta$$

$$= \frac{\sqrt{\pi}}{2} \sum_{t_1=0}^{\infty} \sum_{h_1=1}^{m_1} \overline{f}(\zeta) a^\zeta \overline{H}_{p_2+2,q_2+2}^{m_2+1,n_2+1}\left[bz^\rho \Big|_Q^P\right] \tag{2.1}$$

where

$$P = \left(\frac{1}{2} - \frac{\nu}{2} - \frac{\sigma\zeta}{2}, \frac{\omega}{2}; 1\right), (c_j, \gamma_j; C_j)_{1,n_2}, (c_j, \gamma_j)_{n_2+1,p_2}, \left(-\frac{\nu}{2} - \frac{\sigma\zeta}{2}, \frac{\omega}{2}\right)$$

$$Q = \left(-\frac{\nu}{2} - \frac{\sigma\zeta}{2} + u, \frac{\omega}{2}\right), (d_j, \delta_j)_{1,m_2}, (d_j, \delta_j; D_j)_{m_2+1,q_2}, \left(-\frac{\nu}{2} - \frac{\sigma\zeta}{2} - u, \frac{\omega}{2}; 1\right) \tag{2.2}$$

Second Integral

$$\int_0^{\frac{\pi}{2}} \cos(2u\theta)(\sin\theta)^{\nu} S_L^{h_1,\ldots,h_r}\left[c_1(\sin\theta)^{\eta_1}, \ldots, c_r(\sin\theta)^{\eta_r}\right] \times$$
$$\overline{H}_{p_1,q_1}^{m_1,n_1}\left[a(\sin\theta)^{\sigma}\Big|^{(a_j,\alpha_j,A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}}_{(b_j,\beta_j)_{1,m_1},(b_j,\beta_j,B_j)_{m_1+1,q_1}}\right] \times$$
$$\overline{H}_{p_2,q_2}^{m_2,n_2}\left[bz^{\rho}(\sin\theta)^{\omega}\Big|^{(c_j,\gamma_j,C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}}_{(d_j,\delta_j)_{1,m},(d_j,\delta_j,D_j)_{m_2+1,q_2}}\right] d\theta$$
$$= \frac{\sqrt{\pi}}{2} \sum_{t_1=0}^{\infty} \sum_{h_1=1}^{m_1} \sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\leq L} (-L)_{h_1k_1+\ldots+h_rk_r} \times$$
$$A(L;k_1,\ldots,k_r) \prod_{i=1}^{r} \frac{c_i^{k_i}}{k_i!} \overline{f}(\zeta) a^{\zeta} \overline{H}_{p_2+2,q_2+2}^{m_2+1,n_2+1}\left[bz^{\rho}\Big|^{P'}_{Q'}\right] \tag{2.3}$$

$$P' = \left(\frac{1}{2} - \frac{\nu}{2} - \frac{\sigma\zeta}{2} - \sum_{i=1}^{r}\frac{\eta_i k_i}{2}, \frac{\omega}{2}; 1\right), (c_j, \gamma_j; C_j)_{1,n_2}, (c_j, \gamma_j)_{n_2+1,p_2},$$
$$\left(-\frac{\nu}{2} - \frac{\sigma\zeta}{2} - \sum_{i=1}^{r}\frac{\eta_i k_i}{2}, \frac{\omega}{2}\right)$$
$$Q' = \left(-\frac{\nu}{2} - \frac{\sigma\zeta}{2} - \sum_{i=1}^{r}\frac{\eta_i k_i}{2} + u, \frac{\omega}{2}\right), (d_j, \delta_j)_{1,m_2}, (d_j, \delta_j; D_j)_{m_2+1,q_2},$$
$$\left(-\frac{\nu}{2} - \frac{\sigma\zeta}{2} - \sum_{i=1}^{r}\frac{\eta_i k_i}{2} - u, \frac{\omega}{2}; 1\right) \tag{2.4}$$

where $\overline{f}(\zeta)$ is given by (1.5)

The above results (2.1) and (2.3) are valid for:

(I) $\text{Re}(\nu+1) > 0, u = 0, 1, 2, ...,$ and $\sigma > 0, \omega > 0, \rho > 0$ and $\eta_i > 0$,

$$|arg(a)| < \frac{1}{2}\Omega\pi, \quad |arg(b)| < \frac{1}{2}\Omega'\pi, \text{Re}(\Omega) > 0, \quad \text{Re}(\Omega') > 0. \tag{2.5}$$

$$\Omega = \sum_{j=1}^{m} \beta_j + \sum_{j=1}^{n} A_j\alpha_j - \sum_{j=m+1}^{q} B_j\beta_j - \sum_{j=n+1}^{p} \alpha_j \tag{2.6}$$

$$\Omega' = \sum_{j=1}^{m_2} \delta_j + \sum_{j=1}^{n_2} C_j\gamma_j - \sum_{j=m_2+1}^{q_2} D_j\delta_j - \sum_{j=n_2+1}^{p_2} \gamma_j > 0 \tag{2.7}$$

(II)

$$\text{R}e\left[1+\nu+\sigma\left(\frac{b_j}{\beta_j}\right)+\omega\left(\frac{d_j}{\delta_j}\right)\right] > 0 \tag{2.8}$$

and the various $\overline{\text{H}}$-function occurring in equation (2.1) and (2.3) satisfy the conditions corresponding appropriately to those given by papers [2, 3] and (1.7).

Proof of the Basic Integrals: To prove the basic integral, first we express $\overline{\text{H}}^{m_1,n_1}_{p_1,q_1}$ occurring on the L.H.S. of (2.1) in the series form given by (1.4), change the order of summation and integration which is permissible under the condition stated, put the value of $\overline{\text{H}}^{m_2,n_2}_{p_2,q_2}$ in terms of Mellin-Barness contour integral by the appliction of (1.1) change the order of integration of θ and ξ integrals, we get:

$$\sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1} \overline{f}(\zeta)a^{\zeta}\frac{1}{2\pi i}\int_L b^{\xi}z^{\rho\xi}\overline{\phi}(\xi)\left\{\int_0^{\frac{\pi}{2}} \cos(2u\theta)(\sin\theta)^{\nu+\sigma k+\omega\xi}d\theta\right\}d\xi \tag{2.9}$$

Further using the result (1.12) the above integral becomes

$$\sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1} \overline{f}(\zeta)a^{\zeta}\frac{1}{2\pi i}\int_L b^{\xi}z^{\rho\xi}\overline{\phi}(\xi)\frac{\sqrt{\pi}\Gamma\left(\frac{\nu+\sigma\zeta+\omega\xi+1}{2}\right)\Gamma\left(u-\frac{\nu+\sigma\zeta+\omega\xi}{2}\right)}{2\Gamma\left(-\frac{\nu+\sigma\zeta+\omega\xi}{2}\right)\Gamma\left(1+u+\frac{\nu+\sigma\zeta+\omega\xi}{2}\right)}d\xi \tag{2.10}$$

Then interpreting with the help of equation (1.1), (2.10) provides equation (2.1). Proceeding on parallel lines (2.3) can be obtained by using relation (1.10).

3. Double Integral Relations

First Integral Relation

$$\int_0^\infty \int_0^\infty \cos\left(2u\tan^{-1}\left(\frac{y}{x}\right)\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right)^\nu \times$$
$$\overline{H}^{m_1,n_1}_{p_1,q_1}\left[a\frac{y^\sigma}{(x^2+y^2)^{\sigma/2}}\middle|^{(a_j,\alpha_j,A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}}_{(b_j,\beta_j)_{1,m_1},(b_j,\beta_j,B_j)_{m_1+1,q_1}}\right]\times$$
$$\overline{H}^{m_2,n_2}_{p_2,q_2}\left[b(x^2+y^2)^{\rho-\omega/2}y^\omega\middle|^{(c_j,\gamma_j,C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}}_{(d_j,\delta_j)_{1,m_2},(d_j,\delta_j,D_j)_{m_2+1,q_2}}\right]f(x^2+y^2)dxdy$$
$$=\frac{\sqrt{\pi}}{4}\sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1}\overline{f}(\zeta)a^\zeta\int_0^\infty \overline{H}^{m_2+1,n_2+1}_{p_2+2,q_2+2}\left[bz^\rho\middle|^{P}_{Q}\right]f(z)dz \tag{3.1}$$

Where $\overline{f}(\zeta)$ and P, Q are given by (1.5) and (2.2) respectively.

Second Integral Relation

$$\int_0^\infty \int_0^\infty \cos\left(2u\tan^{-1}\left(\frac{y}{x}\right)\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right)^\nu \times$$
$$S_L^{h_1,\ldots,h_r}\left[c_1\frac{(y)^{\eta_1}}{(x^2+y^2)^{\eta_1/2}},\ldots,c_r\frac{(y)^{\eta_r}}{(x^2+y^2)^{\eta_r/2}}\right]\times$$
$$\overline{H}^{m_1,n_1}_{p_1,q_1}\left[a\frac{y^\sigma}{(x^2+y^2)^{\sigma/2}}\middle|^{(a_j,\alpha_j,A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}}_{(b_j,\beta_j)_{1,m_1},(b_j,\beta_j,B_j)_{m_1+1,q_1}}\right]\times$$
$$\overline{H}^{m_2,n_2}_{p_2,q_2}\left[b(x^2+y^2)^{\rho-\omega/2}y^\omega\middle|^{(c_j,\gamma_j,C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}}_{(d_j,\delta_j)_{1,m},(d_j,\delta_j,D_j)_{m_2+1,q_2}}\right]f(x^2+y^2)dxdy$$
$$=\frac{\sqrt{\pi}}{4}\sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1}\sum_{k_1,\ldots,k_r=0}^{h_1k_1+\ldots+h_rk_r\le L}(-L)_{h_1k_1+\ldots+h_rk_r}A(L;k_1,\ldots,k_r)\times$$
$$\prod_{i=1}^{r}\frac{c_i^{k_i}}{k_i!}\overline{f}(\zeta)a^\zeta\int_0^\infty \overline{H}^{m_2+1,n_2+1}_{p_2+2,q_2+2}\left[bz^\rho\middle|^{P'}_{Q'}\right]f(z)dz \tag{3.2}$$

Where $\overline{f}(\zeta)$ and P', Q' are given by (1.5) and (2.4) respectively.

Conditions of validity of (3.1) and (3.2) easily follows from those given in (2.1) and (2.3)

Derivation of the Integral Relations: To prove the result (3.1), we start with result (2.1). We first replace z by r^2, multiply its both sides by $rf(r^2)$ and then integrate the resulting equation with respect to r over the semi-infinite ray $(0, \infty)$. We thus obtain:

$$\int_0^\infty rf(r^2)\int_0^{\frac{\pi}{2}} \cos(2u\theta)(\sin\theta)^\nu \overline{H}_{p_1,q_1}^{m_1,n_1}\left[a(\sin\theta)^\sigma\Big|_{(b_j,\beta_j)_{1,m_1},(b_j,\beta_j,B_j)_{m_1+1,q_1}}^{(a_j,\alpha_j,A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}}\right]\times$$

$$\overline{H}_{p_2,q_2}^{m_2,n_2}\left[br^{2\rho}(\sin\theta)^\omega\Big|_{(d_j,\delta_j)_{1,m_2},(d_j,\delta_j,D_j)_{m_2+1,q_2}}^{(c_j,\gamma_j,C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}}\right]d\theta dr$$

$$= \int_0^\infty rf(r^2)\frac{\sqrt{\pi}}{2}\sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1}\overline{f}(\zeta)a^\zeta\overline{H}_{p_2+2,q_2+2}^{m_2+1,n_2+1}\left[br^{2\rho}\Big|_{Q}^{P}\right]dr \tag{3.3}$$

Now on changing the polar coordinates occurring on the left-hand side of (3.3) into cartesian coordinates by means of the sustitutions

$$x = r\cos\theta, y = r\sin\theta, r^2 = x^2 + y^2, \theta = \tan^{-1}\frac{y}{x}$$

and putting $r^2 = z$ in the right hand side of (3.3), we easily get the desired result (3.1) after a little simplifications.

The result (3.2) can be established on the same lines as given above.

4. Applications

By choosing suitably the function f in the integral relations obtained in the previous section, a large number of interesting double integrals can be easily evaluated. We shall, however, present here only one integral by way of illustration:

Thus if in (3.1), we set $f(z) = z^{\mu-1}\mathrm{H}_{P,Q}^{M,N}\left[wz\Big|_{(f_j,F_j)_{1,Q}}^{(e_j,E_j)_{1,P}}\right]$ and making use of the known result given by Gupta and Soni [3] we obtain:

$$\int_0^\infty \int_0^\infty \cos\left(2u\tan^{-1}\left(\frac{y}{x}\right)\right) y^\nu (x^2+y^2)^{\mu-\nu/2-1} H_{P,Q}^{M,N}\left[w(x^2+y^2)\Big|_{(f_j,F_j)_{1,Q}}^{(e_j,E_j)_{1,P}}\right] \times$$

$$\overline{H}_{p_1,q_1}^{m_1,n_1}\left[a\frac{y^\sigma}{(x^2+y^2)^{\sigma/2}}\Big|_{(b_j,\beta_j)_{1,m_1},(b_j,\beta_j,B_j)_{m_1+1,q_1}}^{(a_j,\alpha_j,A_j)_{1,n_1},(a_j,\alpha_j)_{n_1+1,p_1}}\right] \times$$

$$\overline{H}_{p_2,q_2}^{m_2,n_2}\left[b(x^2+y^2)^{\rho-\omega/2}y^\omega\Big|_{(d_j,\delta_j)_{1,m_2},(d_j,\delta_j,D_j)_{m_2+1,q_2}}^{(c_j,\gamma_j,C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2}}\right] f(x^2+y^2)dxdy$$

$$= w^{-\mu}\frac{\sqrt{\pi}}{4}\sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1}\overline{f}(\zeta)a^\zeta \overline{H}_{p_2+Q+2,q_2+P+2}^{m_2+N+1,n_2+M+1}\left[bw^{-\rho}\Big|_{Q^*}^{P^*}\right] \tag{4.1}$$

where

$$P^* = \left(\frac{1}{2}-\frac{\nu}{2}-\frac{\sigma\zeta}{2},\frac{\omega}{2};1\right),(c_j,\gamma_j;C_j)_{1,n_2},(1-f_j-F_j\mu,F_j\rho;1)_{1,M},$$
$$(c_j,\gamma_j)_{n_2+1,p_2},\left(-\frac{\nu}{2}-\frac{\sigma\zeta}{2},\frac{\omega}{2}\right),(1-f_j-F_j\mu,F_j\rho;1)_{M+1,Q}$$

$$Q^* = \left(-\frac{\nu}{2}-\frac{\sigma\zeta}{2}+u,\frac{\omega}{2}\right),(d_j,\delta_j)_{1,m_2},(1-e_j-E_j\mu,E_j\rho)_{1,N},$$
$$(d_j,\delta_j;D_j)_{m_2+1,q_2},\left(-\frac{\nu}{2}-\frac{\sigma\zeta}{2}-u,\frac{\omega}{2};1\right),(1-e_j-E_j\mu,E_j\rho;1)_{N+1,P} \tag{4.2}$$

provided the the following conditions are satisfied:
$\rho > 0, |arg(a)| < \frac{1}{2}\Omega\pi$, where Ω is given by (2.6) and $|arg(w)| < \frac{1}{2}\Omega''\pi$

$$\Omega'' \equiv \sum_{j=1}^{N}(E_j) - \sum_{j=N+1}^{P}(E_j) + \sum_{j=1}^{M}(F_j) - \sum_{j=M+1}^{Q}(F_j) > 0 \tag{4.3}$$

$$-\rho \underbrace{\min}_{1\le j\le m_2}\left[\mathrm{Re}\left(\frac{d_j}{\delta_j}\right)\right] - \underbrace{\min}_{1\le j\le M}\left[\mathrm{Re}\left(\frac{f_j}{F_j}\right)\right] < \mathrm{Re}(\mu)$$
$$< \rho C_j \underbrace{\min}_{1\le j\le n_2}\left[1-\mathrm{Re}\left(\frac{c_j}{\gamma_j}\right)\right] + \underbrace{\min}_{1\le j\le N}\left[1-\mathrm{Re}\left(\frac{e_j}{E_j}\right)\right] \tag{4.4}$$

5. Special Cases

On account of the most general nature of $\overline{\mathrm{H}}$-function and $S_L^{h_1,\ldots,h_r}[...]$ occurring in our main results given by (2.1), (2.3), (3.1) and (3.2) a large number of integrals and integral relations involving simpler functions of one and several variables can be easily obtained as their special cases. We however give here only four special cases by way of illustration:

(I) If we take $m_1 = 1, n_1 = p_1, q_1 = q_1 + 1$ in (3.1) then the $\overline{\mathrm{H}}^{m_1,n_1}_{p_1,q_1}$-function occurring therein breaks up into the ${}_{p_1}\overline{\Psi}_{q_1}(.)$ given by (1.8) and the integral relation (3.1) takes the following form after a little simplification which is also believed to be new:

$$\int_0^\infty \int_0^\infty \cos\left(2u\tan^{-1}\left(\frac{y}{x}\right)\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right)^\nu \times$$

$${}_{p_1}\overline{\Psi}_{q_1}\left[-a\frac{y^\sigma}{(x^2+y^2)^{\sigma/2}}\right]\overline{H}^{m_2,n_2}_{p_2,q_2}\left[b(x^2+y^2)^{\rho-\omega/2}y^\omega\right] f(x^2+y^2)dxdy$$

$$= \frac{\sqrt{\pi}}{4}\sum_{t_1=0}^{\infty}\overline{F}(t_1)a^{t_1}\frac{(-1)^{t_1}}{t_1!}\int_0^\infty \overline{H}^{m_2+1,n_2+1}_{p_2+2,q_2+2}\left[bt^\rho\Big|_{Q_1}^{P_1}\right] f(t)dt \tag{5.1}$$

where

$$\overline{F}(t_1) = \frac{\prod_{j=1}^{p}\{\Gamma(a_j+\alpha_j t_1)\}^{A_j}}{\prod_{j=1}^{q}\{\Gamma(b_j+\beta_j t_1)\}^{B_j}} \tag{5.2}$$

$$P_1 = \left(\frac{1}{2}-\frac{\nu}{2}-\frac{\sigma t_1}{2},\frac{\omega}{2};1\right),(c_j,\gamma_j;C_j)_{1,n_2},(c_j,\gamma_j)_{n_2+1,p_2},\left(-\frac{\nu}{2}-\frac{\sigma t_1}{2},\frac{\omega}{2}\right)$$

$$Q_1 = \left(-\frac{\nu}{2}-\frac{\sigma t_1}{2}+u,\frac{\omega}{2}\right),(d_j,\delta_j)_{1,m_2},(d_j,\delta_j;D_j)_{m_2+1,q_2},$$
$$\left(-\frac{\nu}{2}-\frac{\sigma t_1}{2}-u,\frac{\omega}{2};1\right) \tag{5.3}$$

The conditions of validity of (5.1) easily follow from those given in (3.1).

(II) If we take $m_1 = 1, n_1 = p_1 = 0, q_1 = 2$ in (3.1) take the $\overline{\mathrm{H}}^{m_1,n_1}_{p_1,q_1}$-function occurring therein break up into the $\overline{J}^{\nu,\mu}_{\lambda}(z)$ given by (1.9) and the integral relation (3.1) takes the following form after a little simplification which is also believed to be new:

$$\int_0^\infty \int_0^\infty \cos\left(2u\tan^{-1}\left(\frac{y}{x}\right)\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right)^\nu \times$$
$$\overline{J}^{\nu,\mu}_{\lambda}\left[-a\frac{y^\sigma}{(x^2+y^2)^{\sigma/2}}\right]\overline{H}^{m_2,n_2}_{p_2,q_2}\left[b(x^2+y^2)^{\rho-\omega/2}y^\omega\right]f(x^2+y^2)dxdy$$
$$=\frac{\sqrt{\pi}}{4}\sum_{t_1=0}^{\infty}\overline{F}(t_1)a^{t_1}\frac{(-1)^{t_1}}{t_1!}\int_0^\infty \overline{H}^{m_2+1,n_2+1}_{p_2+2,q_2+2}\left[bt^\rho\Big|^{P_1}_{Q_1}\right]f(t)dt \tag{5.4}$$

where P_1, Q_1 are given by (5.3) and

$$\overline{F}(t_1)=\frac{1}{\{\Gamma(1+\lambda+\nu t_1)\}^\mu} \tag{5.5}$$

The conditions of validity of (5.4) easily follow from those of given in (3.1).

(III) If we take $m_1 = 1, n_1 = p_1, q_1 = q_1 + 1$ and $r = 1$ in (3.2) take the $\overline{\mathrm{H}}^{m_1,n_1}_{p_1,q_1}$-function occurring therein breaks up into the ${}_{p_1}\overline{\Psi}_{q_1}(.)$ given by (1.8) and the polynomials $S^{h_1,\ldots,h_r}_{L}[.]$ reduces to $S^h_L[.]$ given by (1.11) and the integral relation (3.2) takes the following form after a little simplification which is also believed to be new:

$$\int_0^\infty \int_0^\infty \cos\left(2u\tan^{-1}\left(\frac{y}{x}\right)\right)\left(\frac{y}{\sqrt{x^2+y^2}}\right)^\nu S^h_L\left[c\frac{(y)^\eta}{(x^2+y^2)^{\eta/2}}\right]\times$$
$${}_{p_1}\overline{\Psi}_{q_1}\left[-a\frac{y^\sigma}{(x^2+y^2)^{\sigma/2}}\right]\overline{H}^{m_2,n_2}_{p_2,q_2}\left[b(x^2+y^2)^{\rho-\omega/2}y^\omega\right]f(x^2+y^2)dxdy$$
$$=\frac{\sqrt{\pi}}{4}\sum_{t_1=0}^{\infty}\sum_{k=0}^{h}(-L)_{hk}c^kA(L;k)\overline{F}(t_1)a^{t_1}\int_0^\infty \overline{H}^{m_2+1,n_2+1}_{p_2+2,q_2+2}\left[bt^\rho\Big|^{P_1}_{Q_1}\right]f(t)dt \tag{5.6}$$

where

$$\overline{F}(t_1) = \frac{\prod_{j=1}^{p}\{(a_j)_{t_1}\}^{A_j}}{\prod_{j=1}^{q}\{(b_j)_{t_1}\}^{B_j}} \tag{5.7}$$

The conditions of validity of (5.6) easily follow from those given in (3.2).

(IV) If we reduce the $\overline{\mathrm{H}}$-function to the familiar Fox's H-function in (2.1), we arrive at the following result after a little simplification which is also believed to be new:

$$\int_0^{\frac{\pi}{2}} \cos(2u\theta)(\sin\theta)^{\nu}\mathrm{H}_{p_1,q_1}^{m_1,n_1}\left[a(\sin\theta)^{\sigma}\Big|_{(b_j,\beta_j)_{1,m_1}}^{(a_j,\alpha_j)_{1,p_1}}\right]\times$$
$$\mathrm{H}_{p_2,q_2}^{m_2,n_2}\left[bz^{\rho}(\sin\theta)^{\omega}\Big|_{(d_j,\delta_j)_{1,m_2}}^{(c_j,\gamma_j)_{1,p_2}}\right]d\theta$$
$$= \frac{\sqrt{\pi}}{2}\sum_{t_1=0}^{\infty}\sum_{h_1=1}^{m_1}\overline{f}(\zeta')a^{\zeta'}\mathrm{H}_{p_2+2,q_2+2}^{m_2+1,n_2+1}\left[bz^{\rho}\Big|_{Q'}^{P'}\right] \tag{5.8}$$

where

$$P' = \left(\frac{1}{2}-\frac{\nu}{2}-\frac{\sigma\zeta'}{2},\frac{\omega}{2}\right),(c_j,\gamma_j)_{1,p_2},\left(-\frac{\nu}{2}-\frac{\sigma\zeta'}{2},\frac{\omega}{2}\right),$$

$$Q' = \left(-\frac{\nu}{2}-\frac{\sigma\zeta'}{2}+u,\frac{\omega}{2}\right),(d_j,\delta_j)_{1,q_2},\left(-\frac{\nu}{2}-\frac{\sigma\zeta'}{2}-u,\frac{\omega}{2};1\right) \tag{5.9}$$

If we take $H_{p_1,q_1}^{m_1,n_1}[.] = 1$ and $b = \rho = 1$ in the equation (5.8),we obtain the known result given by Shrivastava [10].

If we reduce $\overline{\mathrm{H}}$-function to the familiar Fox's H-function in (3.1), we arrive after a little simplification at the known result given by Prasad and Ram [7].

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References

[1] R. G. Buschman and H. M. Srivastava, The $\overline{\mathrm{H}}$-function associated with a certain class of Feynman integrals, *J. Phys. A. Math. Gen.*, **23**(1990), 4707-4710.

[2] K. C. Gupta, R. Jain, and R. Agarwal, On existence conditions for a generalized Mellin-Barnes type integral, *Natl Acad Sci Lett.*, **30** no.5-6 (2007), 169-172.

[3] K. C. Gupta and R. C. Soni, On a basic integral formula involving the product of the $\overline{\mathrm{H}}$-function and Foxs H-function, *J. Raj. Acad. Phy. Sci.*, **4** no.3 (2006), 157-164.

[4] A. A Inayat-Hussain, New properties of hypergeometric series derivable from Feynman integrals: I. Transformation and reeducation formulae, *J. Phys. A. Math. Gen.*, **20**(1987), 4109-4117.

[5] A. A. Inayat-Hussain, New properties of hypergeometric series derivable from Feynman integrals: II.A generalization of the H-function, *J. Phys. A. Math. Gen.*, **20**(1987), 4119-4128.

[6] T. M. Mac Robert, Infinite series for E-functions, *Math Z*, **71**(1959), 143-145.

[7] Y. N. Prasad and S. D. Ram, On some double integrals involving Fox H-function, *Progr. Math.(Allahabad)*, **7**(1973), 13-20.

[8] A. K. Rathie, A new generalization of generalized hypergeometric functions, *Le Mathematic he Fasc.II*, **52**(1997), 297-310.

[9] R. K. Saxena, C. Ram, and S. L. Kalla, Applications of generalized H-function in bivariate distributions, *Rew. Acad. Canar. Cienc.*, **XI V** no.1-2 (2002), 111-120.

[10] S. Shrivastava, Certain integrals and expansion formulae involving Horn function of two variables and Foxs H-function, *Proc. Nat. Acad. Sci. India Sect. A*, **79** no.1 (2009).

[11] H. M. Srivastava and M. Garg, Some integrals involving a general class of polynomials and multivariable H-functios, *Rev Roumaine Phys*, **32**(1987), 685-692.

[12] H. M. Srivastava, A contour integral involving Foxs H-function, *Indian J. Math.*, **14**(1972), 1-6.