Tamsui Oxford Journal of Information and Mathematical Sciences **27(4)** (2011) 429-448
Aletheia University

Fixed Point Theorems for Hybrid Maps in Symmetric Spaces*

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Received January 19, 2010, Accepted May 6, 2011.

Abstract

The purpose of this paper is to obtain coincidence and fixed point theorems for hybrid contractions consisting of single-valued and multi-valued mappings in symmetric spaces. Some special cases and applications are discussed.

Keywords and Phrases: *Fixed point, Symmetric space, Semi-metric, IT-commuting maps, Compatible maps, Weakly compatible maps, PPM-space, Symmetric PPM-space.*

*2010 *Mathematics Subject Classification.* Primary 47H10, 54H25.

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1. Introduction

Recently fixed point theorems for multivalued maps in symmetric spaces have been obtained by Aamri et al. [1], Aamri and Moutawakil [2], Moutawakil [25] and Singh and Prasad [41]. The purpose of this paper is to obtain coincidence and fixed point theorems for hybrid contractions, that is for multivalued and single-valued maps satisfying a general type of conditions. Several results obtained in [1], [18] and [25] are derived as special cases. As an application, we obtain a coincidence theorem for a hybrid pair of single-valued and multi-valued maps.

2. Preliminaries

Following Aamri et al. [1] Aamri and Moutawakil [2] Hicks and Rhoades [17], [18] Moutawakil [25] and Wilson [45], we will use the following notations and definitions. In all that follows, (X, d) will stand for a symmetric space and Y an arbitrary nonempty set. By a symmetric d (also called semi-metric, cf. Mihet,[24, p. 1413]) on a nonempty set X, as usual, we mean a nonnegative real-valued function d on $X \times X$ such that (i). $d(x, y) = 0$ if and only if $x = y$, and (ii). $d(x, y) = d(y, x)$ for all $x, y \in X$.

Definitions 2.1-2.6. Let (X, d) be a symmetric space. Then:

2.1. A nonempty subset P of X is d-closed if and only if $\bar{P}_d = P$, where $\bar{P}_d = \{x \in X : d(x, P) = 0\}$ and $d(x, P) = \inf\{d(x, p) : p \in P\}$.

2.2. A nonempty set P is called d-bounded if and only if $\delta_d(P) < \infty$ where $\delta_d(P) = \sup\{d(x, p) : x, p \in P\}$.

2.3. Space (X, d) is S-complete if for every d-Cauchy sequence $\{x_n\}$, there exists an x in X with $lim_{n\to\infty} d(x_n, x) = 0$.

2.4. Let (X, d) be a d-bounded symmetric space and let $CB(X)$ the set of all nonempty d-closed subsets of X. The Hausdorff symmetric H induced by the symmetric d is defined in the usual way:
$H(A, B) = \max\{\sup_{b\in B} d(b, A); \sup_{a\in A} d(a, B)\}$ for all $A, B \in CB(X)$.

2.5. The maps $f : X \to X$ and $T : X \to CB(X)$ are compatible if and only if $fTx \in CB(X)$ for each $x \in X$ and $lim_{n\to\infty} H(fTx_n, Tfx_n) =$

0 whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty}Tx_n = M \in CB(X), lim_{n\to\infty}fx_n = t \in M$ (also see [40]).

2.6. The maps $f : X \to X$ and $T : X \to CB(X)$ are weakly compatible if they commute at their coincidence points, i.e., $fTx = Tfx$, whenever $fx \in Tx$.

It is well known that the commuting maps T, f (that is, when $fTx = Tfx, x \in X$) are weakly commuting (that is, $H(fTx, Tfx) \leq d(Tx, fx), x \in X$), weakly commuting maps are compatible, and compatible maps T, f are weakly compatible but there are examples in literature to show that the reverse implication is not true (see, for instance, [38] - [40]).

Definition 2.7. Maps $f : X \to X$ and $T : X \to CB(X)$ are (IT)-commuting at a point $x \in X$ if $fTx \subset Tfx$ (Itoh and Takahashi [19]).

Definition 2.8. Maps $f : X \to X$ and $T : X \to CB(X)$ are reciprocally continuous on X (resp. at $t \in X$) if and only if $fTx \in CB(X)$ for each $x \in X$ (resp. $fTt \in CB(X)$) and $\lim_{n\to\infty} fTx_n = fM, \lim_{n\to\infty} Tfx_n = Tt$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = M \in CB(X)$, $\lim_{n\to\infty} fx_n = t \in M$ ([40, p. 628]).

The following example shows that (IT)-commutativity of a hybrid pair T and f at a coincidence point $x \in X$ is more general than its compatibility and weak compatibility at the same point (also see, Example 1, [39]).

Example 2.1. Let $X = [0, \infty)$ with usual metric and $fx = 2x, Tx = [x + 2, \infty), x \in X$. Then $f2 \in T2, fT2 = [8, \infty) \subset [6, \infty) = Tf2$ and T, f are (IT)-commuting at the coincidence point $x = 2$. We see that f and T are not weakly compatible since $fT2 \neq Tf2$.

Remark 2.1. Nonvacuous compatibility of T and f implies the existence of at least a sequence $\{x_n\}$ in X such that $lim_{n\to\infty}Tx_n = M \in CB(X)$, $lim_{n\to\infty}fx_n = t \in M$ ([40]).

For details of topological preliminaries, one may refer to [15]-[17], [21], [22], [42], [43] and [45]. We shall use the following properties essentially due to W. A. Wilson [45] (see also [1], [2], [18] and [25]).

(W.3) Given $\{x_n\}$, x and y in $X, \lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.

(W.4) Given $\{x_n\}, \{y_n\}$ and x in X, $lim_{n\to\infty} d(x_n, x) = 0$ and $lim_{n\to\infty} d(x_n, y_n) = 0$ imply that $\lim_{n\to\infty} d(y_n, x) = 0$.

The following property is due to Mihet [24].

(W) Given $\{x_n\}, \{y_n\}$ and $\{z_n\}$ in $X, d(x_n, y_n) \to 0, d(y_n, z_n) \to 0 \Rightarrow d(x_n, z_n) \to 0$.

Mihet has observed the implication $(W) \Rightarrow (W.4) \Rightarrow (W.3)$.

Unless stated otherwise, let N, R and R^+ denote the set of natural numbers, set of real numbers and set of nonnegative real numbers respectively. Let $f, g : Y \to X; S, T : Y \to CB(X)$. Consider the following conditions for all $x, y \in Y$ and some $k \in (0, 1)$:

(C.1) $H(Tx, Ty) \leq kd(fx, fy)$,

(C.2)
$$\begin{aligned} H(Tx, Ty) \leq\ & k \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), \\ & \sqrt{d(fx, fy).d(fx, Tx)}, \sqrt{d(fx, fy).d(fy, Ty)}, \\ & \sqrt{d(fx, Tx).d(fy, Ty)}\} \end{aligned}$$

(C.3)
$$\begin{aligned} H^2(Sx, Ty) &\leq k.M(x, y), where \\ M(x, y) &= \max\{d^2(fx, gy), d^2(fx, Sx), d^2(gy, Ty), d(fx, gy). \\ & \quad d(fx, Sx), d(fx, gy).d(gy, Ty), d(fx, Sx). \\ & \quad d(gy, Ty), d(fx, Ty).d(gy, Sx)\}, \end{aligned}$$

(C.4)
$$\begin{aligned} H^2(Sx, Ty) \leq\ & k. \max\{d^2(fx, gy), d^2(fx, Sx), d^2(gy, Ty), d(fx, Ty). \\ & d(gy, Sx)\}, \end{aligned}$$

(C.5) $H(Tx, T_i y) < M_i(x, y), i > 1$, where
$$M_i(x, y) = \max\{d(fx, Tx), d(gy, T_i y), d(fx, T_i y), d(gy, Tx), d(fx, gy)\}.$$

Let $\phi : R^+ \to R^+$ a nondecreasing nonnegative and upper semi continuous function such that $\phi(t) < kt$ for each $t > 0$.

(C.6) $H^2(Sx, Ty) \leq \phi(M(x, y))$, for all $x, y \in Y$.

Remark 2.2. The condition (C.1) with $Y = X$ and $f = id$, the identity map of X, is the multivalued contraction (mvc) studied by Moutawakil [25, Theorem 2.2.1].

Remark 2.3. Moutawakil's mvc in symmetric spaces was essentially introduced and studied by Nadler Jr. [26] in metric spaces. Nadler's mvc has been studied, generalized, and used extensively in nonlinear multivalued analysis (see, for instance, [3], [4], [8], [10], [13], [14], [19], [20], [23], [27], [28], [35]-[40]). The condition (C.1) with $Y = X$ was first studied in metric spaces by Singh and Kulshrestha [36]. For a historical development of hybrid contractions in metric spaces, refer to [39] (see also, [4], [5], [7], [8], [11], [12], [19], [26], [29]-[32], [34] and [36]-[38]).

Remark 2.4. The condition (C.1) $\Rightarrow$ (C.2) but there are examples in literature to show that the reverse implication is not true.

Remark 2.5. The condition (C.3) is motivated by the conditions of Liu et al. [22], Tan et al. [43] and Singh and Arora [42] in metric spaces. It can be observed that the condition (C.3) reduces to (C.2) when we substitute $S = T$ and $f = g$ in (C.3). Notice that the condition (C.3) is equivalent to condition (C.4).

In this paper, our main existence results (Theorem 3.1 and 3.2) are obtained under the condition (C.4). In the sequel, we shall need the following results:

Lemma 2.1. ([9], [25]). *Let (X, d) be a d-bounded symmetric space. Let $A, B \in CB(X)$ and $\mu > 1$. Then, for each $a \in A$, there exists an element b in B such that: $d(a, b) \leq \mu H(A, B)$.*

Lemma 2.2. *Let (X, d) be a d-bounded symmetric space satisfying condition (W). Let $\{y_n\}$ be a sequence in X such that $d(y_j, y_{j+1}) \leq qd(y_{j-1}, y_j), j = 1, 2, 3, ...,$ where $0 \leq q < 1$, then $\{y_n\}$ is a d-Cauchy sequence.*

Lemma 2.2 is stated in [18, p.339] without the condition (W). However, Mihet [24] has observed that the condition (W) is needed in the above result.

Lemma 2.3. ([33]). *Let $A, B \in CB(X)$. Then for $x \in A$ and for some $k \in (0,1)$ there exists a $y \in B$ such that $d^2(x,y) \leq k^{-1/2} H^2(A,B)$.*

This is essentially a modified version of Ciric's result [9].

Aamri and Moutawakil [2] obtained a fixed point result for a pair of weakly compatible self mappings of X. They exactly proved the following:

Theorem 2.1 [2]. *Let (X,d) be a d-bounded symmetric space that satisfies $(W.3)$. Let A and B be two weakly compatible self mappings of X such that*

(i) $d(Ax, Ay) \leq \phi(d(Bx, By)), \quad \forall x, y \in X,$

(ii) $AX \subseteq BX.$

If the range of A or B is a S-complete subspace of X, then A and B have a unique fixed point.

Moutawakil [25] obtained a generalization of the Nadler's fixed point theorem [26] in the settings of symmetric spaces in the following manner:

Theorem 2.2 [25]. *Let (X,d) be a d-bounded and S-complete symmetric space satisfying $(W.4)$ and $T : X \to CB(X)$ be a multi-valued mapping such that:*
$H(Tx,Ty) \leq kd(x,y), k \in [0,1), \ \forall x, y \in X$. Then there exists $u \in X$ such that $u \in Tu$.

Further, Aamri et al [1] proved a common fixed point theorem as a generalization of the results of Hicks and Rhoades [18]. Motivated by the works in [1], [2], [18] and [25], we are now in a position to present our main results in the following section.

3. Main Results

Theorem 3.1. *Let Y be an arbitrary nonempty set and (X,d) a d-bounded symmetric space satisfying condition (W). Let $f, g : Y \to X$ and $S, T : Y \to CB(X)$ satisfy $(C.4)$ and*

(3.1.1) *$S(Y) \subseteq g(Y)$ and $T(Y) \subseteq f(Y)$.*

If one of $S(Y), T(Y), g(Y)$ or $f(Y)$ is an S-complete symmetric subspace of X, then

(I) *S and f have a coincidence, i.e., there exists a $v \in Y$ such that $fv \in Sv$;*

(II) *T and g have a coincidence, i.e., there exists a, $w \in Y$ such that $gw \in Tw$.*

Further, if $Y = X$, then

(III) *S and f have a common fixed point fv provided that $ffv = fv$ and S and f are (IT)-commuting at v;*

(IV) *T and g have a common fixed point gw provided $ggw = gw$ and T and g are (IT)-commuting at w;*

(V) *S, T, f and g have a common fixed point provided (III) and (IV) both are true.*

Proof. Pick $x_0 \in Y$. Construct sequences $\{x_n\}$ and $\{y_n\}$ in the following manner. Choose $x_1 \in Y$ such that $y_1 = gx_1 \in Sx_0$. We may do so since $S(Y) \subseteq g(Y)$. Similarly we choose a point $x_2 \in Y$ such that $y_2 = fx_2 \in Tx_1$ and $d^2(y_1, y_2) = d^2(gx_1, fx_2) \leq k^{-1/2} H^2(Sx_0, Tx_1)$. Such a choice is justified (cf. Lemma 2.3). Continuing this process, in general, we have
$y_{2n} = fx_{2n} \in Tx_{2n-1}$ and $y_{2n+1} = gx_{2n+1} \in Sx_{2n}$
such that

$$d^2\left(y_{2n}, y_{2n+1}\right) = d^2\left(fx_{2n}, gx_{2n+1}\right) \leq k^{-1/2} H^2\left(Tx_{2n-1}, Sx_{2n}\right)$$

and

$$d^2\left(y_{2n+1}, y_{2n+2}\right) = d^2\left(gx_{2n+1}, fx_{2n+2}\right) \leq k^{-1/2} H^2\left(Sx_{2n}, Tx_{2n+1}\right)$$

By (C.4),

$$\begin{aligned} d^2\left(y_{2n}, y_{2n+1}\right) &\leq k^{1/2} \max\{d^2\left(fx_{2n}, gx_{2n-1}\right), d^2\left(fx_{2n}, Sx_{2n}\right), \\ &\quad d^2\left(gx_{2n-1}, Tx_{2n-1}\right), d(fx_{2n}, Tx_{2n-1}).d(gx_{2n-1}, Sx_{2n})\}. \\ &\leq k^{1/2} \max\{d^2(fx_{2n}, gx_{2n-1}), d^2(fx_{2n}, gx_{2n+1}), \\ &\quad d^2(gx_{2n-1}, fx_{2n}), d(fx_{2n}, fx_{2n}).d(gx_{2n-1}, gx_{2n+1})\} \\ &= k^{1/2} \max\{d^2(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n-1}).d(y_{2n}, y_{2n+1}), 0\}. \end{aligned}$$

Thus $d(y_{2n}, y_{2n+1}) \leq pd(y_{2n-1}, y_{2n})$, where $p = k^{1/4} < 1$.
Also

$$\begin{aligned} d^2(y_{2n+1}, y_{2n+2}) &\leq k^{1/2} \max\{d^2(fx_{2n}, gx_{2n+1}), \\ &\quad d^2(fx_{2n}, Sx_{2n}), d^2(gx_{2n+1}, Tx_{2n+1}), \\ &\quad d(fx_{2n}, Tx_{2n+1}).d(gx_{2n+1}, Sx_{2n})\}, \\ &\leq k^{1/2} \max\{d^2(fx_{2n}, gx_{2n+1}), d^2(fx_{2n}, gx_{2n+1}), \\ &\quad d^2(gx_{2n+1}, fx_{2n+2}), d(fx_{2n}, fx_{2n+2}).d(gx_{2n-1}, gx_{2n+1})\}. \\ &= k^{1/2} \max\{d^2(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}).d(y_{2n+1}, y_{2n+2}), 0\} \end{aligned}$$

Thus $d(y_{2n+1}, y_{2n+2}) \leq pd(y_{2n}, y_{2n+1})$. In general, we have
$d(y_n, y_{n+1}) \leq pd(y_{n-1}, y_n), n = 1, 2, 3, \ldots$
So, by Lemma 2.2, the sequence $\{y_n\}$ is a d-Cauchy sequence.
Now let $f(Y)$ is complete. Then the subsequence $\{y_{2n}\}$ has a limit in $f(Y)$, call it u. Then there exists an element $v \in Y$ such that $fv = u$. From $(C.4)$, we have

$$\begin{aligned} d^2(Sv, y_{2n}) &\leq k^{-1/2} H^2(Sv, Tx_{2n-1}) \\ &\leq k^{1/2} \max\{d^2(fv, gx_{2n-1}), d^2(fv, Sv), d^2(gx_{2n-1}, Tx_{2n-1}), \\ &\quad d(fv, Tx_{2n-1}).d(gx_{2n-1}, Sv)\} \\ &\leq k^{1/2} \max\{d^2(fv, gx_{2n-1}), d^2(fv, Sv), d^2(gx_{2n-1}, fx_{2n}), \\ &\quad d(fv, fx_{2n}).d(gx_{2n-1}, Sv)\}. \\ &= k^{1/2} \max\{d^2(fv, y_{2n-1}), d^2(fv, Sv), d^2(y_{2n-1}, y_{2n}), \\ &\quad d(fv, y_{2n}).d(y_{2n-1}, Sv)\}. \end{aligned}$$

Making $n \to \infty$ and noting that the subsequence $\{y_{2n-1}\}$ also converges to u, we obtain

$$\begin{aligned} d^2(Sv, u) &\leq k^{1/2} \max\{d^2(fv, u), d^2(fv, Sv), d^2(u, u), d(fv, u).d(u, Sv)\} \\ &= k^{1/2} d^2(fv, Sv), \end{aligned}$$

yielding $fv = u \in Sv$. This proves (I).

Since the subsequence $\{y_{2n+1}\}$ converges to u, and $S(Y) \subseteq g(Y)$, there exists a $w \in Y$ such that $gw = u$.

$$\begin{aligned} d^2(Tw, y_{2n+1}) &\leq k^{-1/2} H^2(Sx_{2n}, Tw) \\ &\leq k^{1/2} \max\{d^2(fx_{2n}, gw), d^2(fx_{2n}, Sx_{2n}), d^2(gw, Tw), \\ &\quad d(fx_{2n}, Tw).d(gw, Sx_{2n})\}. \end{aligned}$$

Making $n \to \infty$, we obtain

$$\begin{aligned} d^2(Tw, u) &\leq k^{1/2} \max\{d^2(u, gw), d^2(u, u), d^2(gw, Tw), d(u, Tw).d(gw, u)\} \\ &= k^{1/2} d^2(u, Tw), \end{aligned}$$

implying $gw = u \in Tw$. This proves (II).

If $g(Y)$ is complete, an analogous argument establishes (I) and (II). The other cases are evident. If $S(Y)$ (respectively $T(Y)$) is complete, then $u \in S(Y) \subseteq g(Y)$ (respectively $u \in T(Y) \subseteq f(Y)$) and the above arguments prove (I) and (II).

If $Y = X$, then since $fv \in Sv$ and $u = fv$,
this imply $fu = u$.
Since f and S are (IT)-commuting at v, then we obtain,
$u = fv = ffv \in fSv \subset Sfv = Su$.

This proves (III) and analogously (IV). Now (V) is immediate from (III) and (IV).

Remark 3.1. When we substitute $Y = X$, $f = g = id$, the identity map and $S = T$, above result improves partially the result of Moutawakil [25, Th. 2.2.1]. Further, if we take $f = g$, $Y = X$ and S and T single-valued on X, we obtain the improved version of the result of Hicks and Rhoades as in their results both the maps are commuting [18, Th. 1].

Remark 3.2. Theorem 3.1 with $Y = X$, $S = T$ and $f = g$ significantly improves a result of Aamri et al [1, Cor. 2.1], as in their result both the maps are weakly compatible.

The maps f and g in Theorem 3.1 may be replaced by a sequence of single-valued maps. Now we do this.

Theorem 3.2. *Let Y be an arbitrary nonempty set and (X, d) a d-bounded symmetric space satisfying condition (W). Let $f_n : Y \to X$ and $S, T : Y \to CB(X)$ satisfy (C.4) and*

(3.2.1) $S(Y) \subseteq f_{2n-1}(Y)$ *and* $T(Y) \subseteq f_{2n}(Y)$.

If one of $S(Y), T(Y), f_{2n-1}(Y)$ or $f_{2n}(Y)$ is an S-complete symmetric subspace of X, then

(I) *S and f_{2i} have a coincidence point for $i \in N$,*

(II) *T and f_{2i-1} have a coincidence point for $i \in N$.*

Further, if $Y = X$, then

(III) *S and f_{2i} have a common fixed point $f_{2i}v$ provided $f_{2i}(f_{2i}v) = f_{2i}v$ and S and f_{2i} are (IT)-commuting at v such that $f_{2i}v \in Sv$ for $i \in N$,*

(IV) *T and f_{2i-1} have a common fixed point $f_{2i-1}w$ provided $f_{2i-1}(f_{2i-1}w) = f_{2i-1}w$ and T and f_{2i-1} are (IT)-commuting at w such that $f_{2i-1}w \in Tw$ for $i \in N$,*

(V) *S, T and $f_n (n \in N)$ have a common fixed point provided (III) and (IV) both are true.*

Proof. The proof may be completed following Singh and Mishra [38] and the proof of the Theorem 3.1.

Remark 3.3. The coincidence points u and v guaranteed by Theorem 3.1 are different in many cases.

Example 3.1. Let $X = [0, \infty)$ be endowed with usual metric. Define $f, g, S, T : X \to X$ such that $fx = 5x^2, gx = 5x^4, Sx = x^2 + 4/25$ and $Tx = x^4 + 4/25$ for all $x \in X$. Then, for any $x, y \in X, d(Sx, Ty) = 1/5d(fx, gy)$. Obviously, condition $(C.4)$ is satisfied with $k = 1/5$ and $S(X) = T(X) = [4/25, \infty) \subset X = f(X) = g(X)$. Also, we have $f(1/5) = S(1/5)$ and $g(1/\sqrt{5}) = T(1/\sqrt{5})$, that is S and f have a coincidence at $x = 1/5$ and T, g have a coincidence at $x = 1/\sqrt{5}$.

If $f = g$ in Theorem 3.1, then we have a slight improved version.

Corollary 3.1. *Let Y be an arbitrary nonempty set, (X, d) a d-bounded symmetric space satisfying condition (W), $f : Y \to X$ and $S, T : Y \to CB(X)$ satisfy*

(3.1.1a)
$$H(Sx, Ty) \leq k \max\{d(fx, fy), d(fx, Sx), d(fy, Ty), \sqrt{d(fx, fy).d(fx, Sx)}, \sqrt{d(fx, fy).d(fy, Ty)}, \sqrt{d(fx, Sx).d(fy, Ty)}, \sqrt{d(fx, Ty).d(fy, Sx)}\}$$

(3.1.1b) $S(Y) \cup T(Y) \subseteq f(Y)$.

If one of $S(Y)$ or $T(Y)$ or $f(Y)$ is an S-complete symmetric subspace of X, then

(I) *S, T and f have a coincidence.*

Further, if $Y = X$, then

(II) *S, T and f have a common fixed point fv provided $ffv = fv$ and f is (IT)-commuting with each of S and T at v such that $fv \in Sv \cap Tv$.*

Proof. The proof may be completed following the proof of the Theorem 3.1 as condition (3.1.1a) is contained in (C.4).

Theorem 3.3. *Let Y be an arbitrary nonempty set and (X, d) a d-bounded symmetric space satisfying condition (W). Let $f, g : Y \to X$ and $S, T : Y \to CB(X)$ satisfy $(C.5)$. If one of $f(Y)$ or $S(Y)$ or $T(Y)$ or $g(Y)$ is an S-complete symmetric subspace of X, then*

(I) *S and f have a coincidence, i. e., there exists a $v \in Y$ such that $fv \in Sv$;*

(II) *T and g have a coincidence, i. e., there exists a $w \in Y$ such that $gw \in Tw$.*

Futher if $Y = X$, then

(III) *S and f have a common fixed point fv provided $ffv = fv$ and S and f are (IT)-commuting at v;.*

(IV) *T and g have a common fixed point gw provided $ggw = gw$ and T and g are (IT)-commuting at w.*

(V) *S, T, f and g have a common fixed point provided (III) and (IV) both are true.*

Proof. The proof may be completed following Singh and Mishra ([39], Theorem1) and the proof of the Theorem 3.1.

Theorem 3.4. *Let (X, d) be a symmetric space and $f, g : X \to X; T, T_i : X \to CB(X), i \in N$, satisfy $(C.5)$ and*

(3.4.1) *$T(X) \subseteq g(X)$ and the maps T, f are reciprocally continuous and nonvacuously compatible.*

Then,

(I) *T_i and f have a coincidence, i. e., there exists a $z \in X$ such that $fz \in T_i z$.*

(II) *T_i and g have a coincidence, i. e., there exists a $y \in X$ such that $gy \in T_i y$.*

Further,

(III) *f and T have a common fixed point fu provided that fu is a fixed point of f and $fu \in Tu$.*

(IV) *g and T_i have a common fixed point gw provided that gw is a fixed point of g and g and T_i are (IT)-commmuting at w where $gw \in T_i w$.*

(V) *f, g, T and T_i have a common fixed point provided (III) and (IV) both are true.*

Proof. The pair (T, f) is nonvacuously compatible. Therefore, there exists a sequence $\{x_n\}$ in X such that
$\lim_{n\to\infty} Tx_n = M \in CB(X)$ and $\lim_{n\to\infty} fx_n = t \in M$ and
$\lim_{n\to\infty} H(Tfx_n, fTx_n) = 0$.
As maps T and f are reciprocally continuous then
$\lim_{n\to\infty} fTx_n = fM$ and $\lim_{n\to\infty} Tfx_n = Tt$. So, $H(Tt, fM) = 0$. $t \in M$ implies $ft \in Tt$.
Since $T(X) \subseteq g(X)$, there exists a point $x \in X$ such that $ft = gx \in Tt$.
If $gx \notin T_i x$, then
$$\begin{aligned} d(gx, T_i x) &\leq H(Tt, T_i x) \\ &< \max\{d(ft, Tt), d(gx, T_i x), d(ft, T_i x), d(gx, Tt), \\ &\qquad d(ft, gx)\}. \\ &= d(gx, T_i x). \end{aligned}$$
Hence $gx \in T_i x$. This proves (II).

Further, (IT)-commutativity of f with T at t implies that $fTt \subseteq Tft$.

Therefore $ft = fft \in fTt \subseteq Tft$.

This establishes (III). Similar arguments yield (IV).

(V) is immediate from (III) and (IV).

Remark 3.4. In view of the observations made by Mihet [24], the requirement of nonvacous compatibility (cf. (3.4.1)) is essential. Indeed, as pointed out by Mihet [op. cit.], the routine way of producing a Cauchy sequence for four maps satisfying (C.5) in symmetric spaces does not seem possible (see [24]).

Corollary 3.2. *Let* (X, d) *be a symmetric space and* $f, g : X \to X; S, T : X \to CB(X)$ *satisfy*

(3.2.1a) $T(X) \subseteq g(X)$ *and the maps* T, f *are reciprocally continuous and nonvacuously compatible,*

(3.2.1b) $H(Sx, Ty) < \max\{d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx), d(fx, gy)\}.$

Then,

(I) f *and* S *have a coincidence, i. e., there exists a* $z \in X$ *such that* $fz \in Sz$.

(II) g *and* T *have a coincidence, i. e., there exists a* $y \in X$ *such that* $gy \in Ty$.

Further,

(Ia) f *and* S *have a common fixed point* fu *provided that* fu *is a fixed point of* f *and* $fu \in Su$.

(IIa) g *and* T *have a common fixed point* gw *provided that* gw *is a fixed point of* g *and* g *and* T *are (IT)-commuting at* w *where* $gw \in Tw$.

(IIIa) f, g, S *and* T *have a common fixed point provided (III) and (IV) both are true.*

Proof. It follows from the proof of Theorem 3.4.

4. Applications

In this section we follow the notations and definitions of Hicks and Rhoades [18], Moutawakil [25] and Singh and Prasad [41].

Definition 4.1. A function $F : R \to [0, 1]$ is said to be a distribution function if
(i) F is non-decreasing, (ii) F is left continuous, and (iii) $\inf_{x\in R} F(x) = 0$ and $\sup_{x\in R} F(x) = 1$.

Definition 4.2. Let X be a set and $\Im$ a function defined on $X \times X$ such that $\Im(x, y) = F(x, y)$ is a distribution function. Consider the following conditions:

(iv) $F(x, y, 0) = 0$ for all $x, y \in X$.

(v) $F(x, y) = f$ if and only if $x = y$, where f is the distribution function defined by $f(x) = 0$ if $x \le 0$, and $f(x) = 1$ if $x > 0$.

(vi) $F(x, y) = F(y, x)$ for all $x, y \in X$.

(vii) If $F(x, y, \alpha) = 1$ and $F(y, z, \beta) = 1$ then $F(x, z, \alpha + \beta) = 1$, for all $x, y, z \in X$.

If $\Im$ satisfies (iv) and (v), then it is called a PPM-structure on X and the pair $(X, \Im)$ is called a PPM-space and $\Im$ satisfying (vi) is said to be symmetric. A symmetric PPM-structure $\Im$ satisfying (vii) is a probabilistic metric structure and the pair $(X, \Im)$ is a probabilistic metric space.

Let $(X, \Im)$ be a symmetric PPM-space. For $\alpha, \gamma > 0$ and $x \in X$, let $N_x(\alpha, \gamma) = \{y \in X : F(x, y, \alpha) > 1 - \gamma\}$. A T_1 topology $t(\Im)$ on X is defined as follows:
$t(\Im) = \{U \subseteq X$: for each x in U, there exists $\alpha > 0$, such that $N_x(\alpha, \alpha) \subseteq U\}$.

Definition 4.3. Let $(X, \Im)$ be a symmetric PPM-space. A sequence $\{x_n\}$ in X is called a fundamental sequence if $\lim_{n,m\to\infty} F(x_n, x_m, t) = 1$ for all $t > 0$. The space is called F-complete if for every fundamental sequence $\{x_n\}$ in X, there exists an $x \in X$ such that $\lim_{n\to\infty} F(x_n, x, t) = 1$ for all $t > 0$.

In the space $(X, \Im)$, the condition $(W.4)$ is equivalent to the following:

(P.4) $\lim_{n\to\infty} F(x_n, x, t) = 1$ and $\lim_{n\to\infty} F(x_n, y_n, t) = 1$ imply $\lim_{n\to\infty} F(y_n, x, t) = 1$ for all $t > 0$.

Definition 4.4. Let $(X, \Im)$ be a symmetric PPM-space. A nonempty subset P of X is called $\Im$-closed if and only if $\overline{P}_{\Im} = P$, where

$\overline{P}_{\Im} = \{x \in X : \sup_{a\in P} F(x, a, t) = 1 \text{ for all } t > 0\}$.

For the details of the topological preliminaries, one may refer to [18] and [21]. In all that follows we denote the set of all nonempty $\Im$-closed subsets of X by $CB_{\Im}(X)$.

The following is a slightly modified version of Moutawakil [25, Prop. 2.3.1].

Proposition 4.1 ([25]). *Let $(X, \Im)$ be a symmetric PPM-space. Let p be a compatible symmetric function for $t(\Im)$. For $A, B \in CB(X)$, set*

$E(A, B, \epsilon) = \min\{\inf_{a\in A} \sup_{b\in B} F(a, b, \epsilon); \inf_{b\in B} \sup_{a\in A} F(a, b, \epsilon)\}, \epsilon > 0,$
and
$P(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} p(a, b); \sup_{b\in B} \inf_{a\in A} p(a, b)\}.$

Let $f : X \to X$ and $S, T : X \to CB(X)$ and if $F(fx, fy, t) > 1 - t$ implies $E(Tx, Ty, kt) > 1 - kt, 0 \le k < 1$, for all $t > 0$ and all $x, y \in X$. Then, $P(Sx, Ty) \le kp(fx, fy)$.

In a symmetric PPM space $(X, \Im)$, if p is a compatible symmetric function on $t(\Im)$ then $CB_{\Im}(X) = CB(X)$, where $CB(X)$ is the set of all nonempty p-closed subsets of (X, p).

Hicks and Rhoades [18] obtained the following result showing that each symmetric PPM-space admits a compatible symmetric function.

Theorem 4.1 ([18]). *Let $(X, \Im)$ be a symmetric PPM-space. Let $p : X \times X \to R^+$ be a function defined as follows:*

$$p(x, y) = \begin{cases} 0 & \text{if} \quad y \in N_x(t, t) \quad \forall t > 0 \\ \sup\{t : y \notin N_x(t, t), 0 < t < 1\} & \text{otherwise} \end{cases}$$

Then

(viii) *$p(x, y) < t$ if and only if $F(x, y, t) > 1 - t$;*

(ix) *p is a compatible symmetric for $t(\Im)$;*

(x) *$(X, \Im)$ is F-complete if and only if (X, p) is F-complete.*

Now we present the following result in a symmetric PPM-space.

Theorem 4.2. *Let $(X, \Im)$ be an F-complete symmetric PPM-space that satisfied $(P.4)$ and p a compatible symmetric function for $t(\Im)$. Let $f : Y \to X$ and $S, T : Y \to C_{\Im}(X)$ such that $F(fx, fy, t) > 1 - t$ implies $E(Sx, Ty, kt) > 1 - kt, 0 \leq k < 1$, for all $x, y \in Y$.*
If one of $f(Y)$ or $S(Y)$ or $T(Y)$ is an F-complete symmetric subspace of X, then there exists a $z \in Y$ such that $fz \in Sz \cap Tz$.
Further, if $Y = X$ and $ffz = fz$ then f, S and T have a common fixed point provided that f is (IT)-commuting with each of S and T.

Proof. Clearly (X, p) is a bounded and S-complete space and we have $p(fx, fy) < t$ if and only if $F(fx, fy, t) > 1 - t$.
Given $\epsilon > 0$, put $t = p(fx, fy) + \epsilon$.

Then, $F(fx, fy, t) > 1 - t$ implies $E(Sx, Ty, kt) > 1 - kt$, for all $x, y \in Y$.

From Proposition 4.1, we obtain

$P(Sx, Ty) \leq kt = kp(fx, fy) + k\epsilon$
As $\epsilon > 0$ is arbitrary, on letting ϵ tend to 0, we get
$P(Sx, Ty) \leq kp(fx, fy)$.

An application of Corollary 3.1 completes the proof.

Corollary 4.1. *Let $(X, \Im)$ be an F-complete symmetric PPM-space that satisfied $(P.4)$ and p a compatible symmetric function for $t(\Im)$. Let $f : Y \to X$ and $S, T : X \to C_{\Im}(X)$ such that*

$F(x, y, t) > 1 - t$ implies $E(Sx, Ty, kt) > 1 - kt, 0 \leq k < 1$, for all $x, y \in Y$. Then S and T have a common fixed point.

Acknowledgement. The authors thank the referee for his appreciation and valuable suggestions to improve upon the original typescript.

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