Tamsui Oxford Journal of Information and Mathematical Sciences **27(4)** (2011) 411-427
Aletheia University

Some Theorems on 3-dimensional Quasi-Sasakian Manifolds*

Abul Kalam Mondal†
Department of Mathematics, Dum Dum Motijheel Rabindra Mahavidyalaya 208/B/2, Dum Dum Road, Kolkata-700074, West Bengal, India

and

Avik De‡
Department of Pure Mathematics, University of Calcutta 35, B.C.Road, Kolkata-700019, West Bengal, India

Received January 15, 2010, Accepted February 21, 2011.

Abstract

The object of the present paper is to study $\phi-$Ricci symmetric and locally $\phi-$Ricci symmetric 3-dimensional quasi-Sasakian manifolds with structure function $\beta =$constant. Also we study $\eta-$parallel Ricci tensor and cyclic parallel Ricci tensor with $\beta =$constant. Applications of such manifold have been considered. The existence of 3-dimensional $\phi-$Ricci symmetric and locally $\phi-$Ricci symmetric quasi-Sasakian manifolds are also given by concrete examples.

Keywords and Phrases: *Quasi-Sasakians manifold, Structure function, $\phi-$symmetric, $\eta-$parallel Ricci tensor, Cyclic parallel Ricci tensor.*

---

*2000 *Mathematics Subject Classification*. Primary 53C15, 53C40.
†E-mail: kalam_ju@yahoo.co.in
‡Corresponding author. E-mail: de.math@gmail.com

1. Introduction

The notion of quasi-Sasakian structure was introduced by D. E. Blair [6] to unify Sasakian and cosymplectic structures. S. Tanno [22] also added some remarks on quasi-Sasakian structures. The properties of quasi-Sasakian manifolds have been studied by several authors, viz., J. C. Gonzalez and D. Chinea [11], S. Kanemaki [12], [13] and J. A. Oubina [20]. B. H. Kim [15] studied quasi-Sasakian manifolds and proved that fibred Riemannian spaces with invariant fibres normal to the structure vector field do not admit nearly Sasakian or contact structure but a quasi-Sasakian or cosymplectic structure. Recently, quasi-Sasakian manifolds have been the subject of growing interest in view of finding the significant applications to physics, in particular to super gravity and magnetic theory [1], [2]. Quasi-Sasakian structures have wide applications in the mathematical analysis of string theory [3], [9]. Motivated by the roles of curvature tensor and Ricci tensor of quasi-sasakian manifolds in string theory [3] we like to study $\phi-$Ricci symmetric quasi-Sasakian manifold and quasi-Sasakian manifold with $\eta-$parallel and cyclic parallel Ricci tensors in dimension three. On a 3-dimensional quasi-Sasakian manifold, the structure function β was defined by Z. Olszak[17] and with the help of this function he has obtained necessary and sufficient conditions for the manifold to be conformally flat[18]. Next he has proved that if the manifold is additionally conformally flat with $\beta =$ constant, then (a) the manifold is locally a product of R and a two-dimensional Kaehlerian space of constant Gauss curvature (the cosymplectic case), or, (b) the manifold is of constant positive curvature (the non-cosymplectic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure).

The present paper is to study the 3-dimensional quasi-Sasakian manifolds with $\beta =$ constant. After preliminaries in section 4 we prove that $\phi-$symmetry and $\phi-$ Ricci symmetry are equivalent on a 3-dimensional quasi-Sasakian manifold. Section 5 and Section 6 deal with the study of 3-dimensional quasi-Sasakian manifold with $\eta-$parallel Ricci tensor and cyclic Ricci tensor respectively. In section 7 we consider the applications of quasi-Sasakian manifolds. The last section contains some illustrative examples of a 3-dimensional non-cosympletic quasi-Sasakian manifold with constant scalar curvature.

2. Preliminaries

Let M be a (2n+1)-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ, ξ, η are tensor fields on M of types $(1,1)$, $(1,0)$, $(0,1)$ respectively, such that [4],[5], [23].

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Z), \quad X, Y \in T(M),$$

where $T(M)$ is the Lie algebra of vector fields of the manifold M.

Then also

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi).$$

Let Φ be the fundamental 2-form of M defined by

$$\Phi(X, Y) = g(X, \phi Y) \quad X, Y \in T(M).$$

Then $\Phi(X, \xi) = 0, \quad X \in T(M)$. M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η) is normal and the fundamental 2-form Φ is closed, that is, for every $X, Y \in \mathcal{E}^{(2n+1)}$, where $\mathcal{E}^{(2n+1)}$ denotes the module of vector fields on M,

$$[\phi, \phi](X, Y) + d\eta(X, Y)\xi = 0,$$

$$d\Phi = 0, \quad \Phi(X, Y) = g(X, \phi Y).$$

This was first introduced by Blair [6]. There are many types of quasi-Sasakian structures ranging from the cosymplectic case, $d\eta = 0$(rank $\eta = 1$), to the Sasakian case, $\eta \wedge (d\eta)^n \neq 0$ (rank $\eta = 2n+1$, $\Phi = d\eta$). The $1-$form η has rank $r' = 2p$ if $d\eta^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$, and has rank $r' = 2p+1$ if $d\eta^p = 0$ and $\eta \wedge (d\eta)^p \neq 0$. We also say that r' is the rank of the quasi-Sasakian structure. Blair[6] also proved that there are no quasi-Sasakian structure of even rank. In order to study the properties of quasi-Sasakian manifolds Blair [6] proved some theorems regarding Kaehlerian manifolds and existence of quasi-Sasakian manifolds. S. Tanno [22] rectified some of these theorems. However, while Tanno studied locally product quasi-Sasakian manifolds he mentioned the following:

Let $M_1^{2p+1}(\phi_1, \xi_1, \eta_1, g_1)$ be a Sasakian manifold and let $M_2^{2q}(J_2, G_2)$ a Kaehlerian manifold. Then $M_1 \times M_2$ has a quasi-Sasakian structure (ϕ, ξ, η, g) of rank $2p+1$ such that

$$\phi X = (\phi_1 X_1, J_2 X_2), \quad \xi = (\xi_1, 0),$$

$$\eta(X) = \eta_1(X_1), \quad g(X,Y) = g_1(X_1, Y_1) + G_2(X_2, Y_2),$$

for the canonical decomposition $X = (X_1, X_2)$ of a vector field X on $M_1 \times M_2$ [6].

Conversely,

Theorem [22]: *Let $M(\phi, \xi, \eta, g)$ be a quasi-Sasakian manifold (more generally a normal almost contact Riemannian manifold) of rank $2p+1$. If g^* be defined by*

$$2g^*(X,Y) = -d\eta(X, \phi Y),$$

$X, Y \in \mathcal{E}^{2n+1}$, is positive definite on $\mathcal{E}^{2p}$ and $\overline{\nabla}\theta = 0$ with respect to the Riemannian metric $\overline{g}$ defined by

$$\overline{g}(X,Y) = \eta(X)\eta(Y) + g^*(\psi^2 X, \psi^2 Y) + g(\theta^2 X, \theta^2 Y),$$

where the $(1,1)$ tensors ψ and θ are given by

$$\begin{aligned} \psi(X) &= \phi(X) && \text{if } X \in \mathcal{E}^{2p}, \\ &= 0 && \text{if } X \in \mathcal{E}^{2q} \oplus \mathcal{E}^1, \end{aligned}$$

$$\begin{aligned} \theta(X) &= \phi(X) && \text{if } X \in \mathcal{E}^{2q}, \\ &= 0 && \text{if } X \in \mathcal{E}^{2p+1}, \end{aligned}$$

then $(\phi, \xi, \eta, \overline{g})$ is also a quasi-Sasakian structure of rank $2p+1$ and $M(\phi, \xi, \eta, \overline{g})$ is locally the product of Sasakian manifold and a Kaehler manifold.It is mentioned that $\mathcal{E}^{2p+1}$, $\mathcal{E}^{2q}$, $\mathcal{E}^1$ are submodules of $\mathcal{E}^{2n+1}$. S. Tanno [22] also gave an example of a 3-dimensional quasi-Sasakian manifold which is not Sasakian. For a quasi-Sasakian manifold we have the relation [7]

$$(\nabla_X \phi)Y = g(\nabla_{\phi X}\xi, Y)\xi - \eta(Y)\nabla_{\phi X}\xi,$$

which generalizes the well-known conditions $\nabla\phi = 0$ and $(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X$ characterizing respectively cosymplectic and Sasakian manifolds. The quasi-Sasakian condition also reflects in some properties of curvature and of *the vector field ξ. In fact we have the following results.*

Lemma[6], [19]: *Let $M(\phi, \xi, \eta, g)$ be a quasi-Sasakian manifold. Then*
(i) the vector field ξ is Killing and its integral curves are geodesics;
(ii) the Ricci curvature in the direction of ξ is given by $||\nabla\xi||^2$.

3. 3-dimensional Quasi-Sasakian Manifold

An almost contact metric manifold M is a 3-dimensional quasi-Sasakian manifold if and only if [17]

$$\nabla_X \xi = -\beta\phi X, \quad X \in T(M), \tag{3.1}$$

for a certain function β on M, such that $\xi\beta = 0$, ∇ being the operator of the covariant differentiation with respect to the Levi-Civita connection of M. Clearly, such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. Here we have shown that the assumption $\xi\beta = 0$ is not necessary.

As a consequence of (3.1), we have[17]

$$(\nabla_X \phi)(Y) = \beta(g(X, Y)\xi - \eta(Y)X), \quad X, Y \in T(M). \tag{3.2}$$

Because of (3.1) and (3.2), we find

$$\nabla_X(\nabla_Y \xi) = -(X\beta)\phi Y - \beta^2\{g(X, Y)\xi - \eta(Y)X\} - \beta\phi\nabla_X Y$$

which implies that

$$R(X, Y)\xi = -(X\beta)\phi Y + (Y\beta)\phi X + \beta^2\{\eta(Y)X - \eta(X)Y\}. \tag{3.3}$$

Thus we get from (3.3)

$$\begin{aligned} R(X, Y, Z, \xi) &= (X\beta)g(\phi Y, Z) - (Y\beta)g(\phi X, Z) \\ &- \beta^2\{\eta(Y)g(X, Z) - \eta(X)g(Y, Z)\}, \end{aligned} \tag{3.4}$$

where $R(X, Y, Z, W) = g(R(X, Y, Z), W)$.
Putting $X = \xi$, in (3.4) we obtain

$$R(\xi, Y, Z, \xi) = \beta^2\{g(Y, Z) - \eta(Y)\eta(Z)\} + g(\phi Y, Z)\xi\beta. \tag{3.5}$$

Interchanging Y and Z of (3.5) yields

$$R(\xi, Z, Y, \xi) = \beta^2\{g(Y, Z) - \eta(Y)\eta(Z)\} + g(\phi Z, Y)\xi\beta. \tag{3.6}$$

Since $R(\xi, Y, Z, \xi) = R(Z, \xi, \xi, Y) = R(\xi, Z, Y, \xi)$, from (3.5) and (3.6) we have

$$\{g(\phi Y, Z) - g(\phi Z, Y)\}\xi\beta = 0.$$

Therefore, we can easily verify that $\xi\beta = 0$.

In a 3-dimensional Riemannian manifold, we always have

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X \\ &- S(X, Z)Y - \frac{r}{2}(g(Y, Z)X - g(X, Z)Y), \end{aligned} \tag{3.7}$$

where Q is the Ricci operator, that is, g(QX,Y)=S(X,Y) and r is the scalar curvature of the manifold.

Throughout this paper we consider β as a constant. Let M be a 3-dimensional quasi-Sasakian manifold. Since β is a constant the Ricci tensor S of M is given in [18] takes the form

$$S(Y, Z) = (\frac{r}{2} - \beta^2)g(Y, Z) + (3\beta^2 - \frac{r}{2})\eta(Y)\eta(Z), \tag{3.8}$$

where r is the scalar curvature of M.

As a consequence of (3.8), we get for the Ricci operator Q

$$QX = (\frac{r}{2} - \beta^2)X + (3\beta^2 - \frac{r}{2})\eta(X)\xi. \tag{3.9}$$

From (3.8) we have

$$S(X, \xi) = 2\beta^2\eta(X). \tag{3.10}$$

Moreover, as a consequence of (3.7)-(3.10), we find

$$R(X, Y)\xi = \beta^2(\eta(Y)X - \eta(X)Y), \quad X, Y \in T(M). \tag{3.11}$$

As a consequence of (3.1) we also have [17]

$$(\nabla_X\eta)(Y) = g(\nabla_X\xi, Y) = -\beta g(\phi X, Y). \tag{3.12}$$

Also from (3.8) it follows that

$$S(\phi X, \phi Z) = S(X, Z) - 2\beta^2 \eta(X)\eta(Z). \tag{3.13}$$

4. ϕ-Ricci Symmetric 3-dimensional Quasi-Sasakian Manifold

Definition 4.1 A quasi-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be $\phi-$symmetric if the curvature tensor R satisfies

$$\phi^2(\nabla_W R)(X, Y)Z = 0,$$

for all vector fields $X, Y, Z, W \in T(M)$.

If X, Y, Z, W are orthogonal to ξ, then the manifold is said to be locally $\phi-$ symmetric. The notion of locally $\phi-$symmetric on a Sasakian manifold was introduced by Takahashi [21].

Definition 4.2 A quasi-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be $\phi-$Ricci symmetric if the Ricci operator Q satisfies

$$\phi^2(\nabla_X Q)(Y) = 0,$$

for all vector fields $X, Y \in T(M)$ and $S(X, Y) = g(QX, Y)$.

If X, Y are orthogonal to ξ, then the manifold is said to be locally $\phi-$Ricci symmetric.

From the definition it follows that $\phi-$symmetric implies $\phi-$Ricci symmetric, but the converse, is not, in general true. $\phi-$Ricci symmetric Sasakian manifolds have been studied by De and Sarkar [8].

Let us suppose that the manifold is $\phi-$Ricci symmetric. Then by definition

$$\phi^2(\nabla_X Q)(Y) = 0.$$

Using (2.1) we have from above

$$-(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = 0. \tag{4.1}$$

From (4.1) it follows that

$$-g(\nabla_X Q(Y), Z) + S(\nabla_X Y, Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0. \tag{4.2}$$

Putting $Y = \xi$, we get from (4.2)

$$-g(\nabla_X Q(\xi), Z) + S(\nabla_X \xi, Z) + \eta((\nabla_X Q)\xi)\eta(Z) = 0. \tag{4.3}$$

In view of (3.1) and (3.9), we get from (4.3)

$$-g(2\beta^2 \nabla_X \xi, Z) + S(\nabla_X \xi, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{4.4}$$

Putting ϕZ instead of Z in (4.4) yields

$$2\beta^2 g(\phi X, \phi Z) = S(\phi X, \phi Z), \tag{4.5}$$

since M is non-cosymplectic.

Using (3.13), (4.5) yields

$$S(X, Z) = 2\beta^2 g(X, Z). \tag{4.6}$$

Using (4.6), in (3.7) we have

$$R(X, Y)Z = \beta^2\{g(Y, Z)X - g(X, Z)Y\}.$$

Then clearly,

$$\phi^2(\nabla_W R)(X, Y)Z = 0.$$

This helps us to conclude the following:

Theorem 4.1. *On a 3-dimensional non-cosymplectic quasi-Sasakian manifold $\phi-$Ricci symmetry and $\phi-$symmetry are equivalent provided β is a constant.*

Differentiating (3.9) covariantly along W we obtain

$$\begin{aligned}(\nabla_W Q)(X) &= \frac{1}{2}\{dr(W)X - dr(W)\eta(X)\xi\} + (3\beta^2 - \frac{r}{2})(\nabla_W \eta)(X)\xi \\ &+ (3\beta^2 - \frac{r}{2})\eta(X)(\nabla_W \xi).\end{aligned} \tag{4.7}$$

Applying ϕ^2 on both side of (4.7) and using (2.1) we have

$$\begin{aligned}\phi^2(\nabla_W Q)(X) &= \frac{1}{2}\{dr(W)(-X + \eta(X)\xi) \\ &+ (6\beta^2 - r)\eta(X)\phi^2(\nabla_W \xi)\}.\end{aligned} \tag{4.8}$$

Now if X is orthogonal to ξ, (4.8) gives

$$\phi^2(\nabla_W Q)(X) = -\frac{1}{2}dr(W)X.$$

From above expression we can state

Theorem 4.2. *A 3-dimensional non-cosympletic quasi-Sasakian manifold with $\beta =$ constant, is locally ϕ-Ricci symmetric if and only if the scalar curvature is constant.*

5. η-Parallel Ricci Tensor

Definition 5.1. The Ricci tensor S of a quasi-Sasakian manifold is called η-parallel if it satisfies

$$(\nabla_X S)(\phi Y, \phi Z) = 0,$$

for all vector fields X, Y, Z. The notion of η-parallelity for Sasakian manifold was introduced by Kon[16].

Definition 5.2. A 3-dimensional quasi-Sasakian manifold is said to be an $\eta-$Einstein manifold if the Ricci tensor is of the form

$$S = ag + b\eta \otimes \eta,$$

where a and b are smooth functions on M.

From (3.8) we get

$$(\nabla_X S)(\phi Y, \phi Z) = \frac{1}{2}dr(X)g(\phi Y, \phi Z). \tag{5.1}$$

If the Ricci tensor is $\eta-$parallel, then from (5.1)

$$\frac{1}{2}dr(X)g(\phi Y, \phi Z) = 0. \tag{5.2}$$

from which it follows that

$$dr(X) = 0.$$

Hence we find that the scalar curvature is constant. Moreover, β is constant. Thus in view of (3.8) a 3-dimensional non-cosymplectic quasi-Sasakian manifold M with $\eta-$parallel Ricci tensor is an $\eta-$Einstein manifold.

Conversely, if the quasi-Sasakian manifold M is $\eta-$Einstein, then

$$(\nabla_X S)(\phi Y, \phi Z) = 0.$$

Thus we can state the following:

Theorem 5.1. *In a 3-dimensional non-cosymplectic quasi-Sasakian manifold with β = constant, the Ricci tensor is $\eta-$parallel if and only if M is $\eta-$Einstein.*

Again the equation (5.1) yields

Theorem 5.2. *In a 3-dimensional non-cosymplectic quasi-Sasakian manifold with β = constant, the Ricci tensor is $\eta-$parallel if and only if the scalar curvature is constant .*

From Theorem 4.2 and Theorem 5.2 we can state the following:

Corollary 5.1. *In a 3-dimensional non-cosymplectic quasi-Sasakian manifold with β = constant, the Ricci tensor is $\eta-$parallel if and only if the manifold is locally $\phi-$Ricci symmetric.*

6. Cyclic Parallel Ricci Tensor

A. Gray [10] introduced two classes of Riemannian manifold determined by covariant derivative of Ricci tensor. The class $\mathcal{A}$ consisting of all Riemannian manifold whose Ricci tensor S is a Codazzi tensor, that is,

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z).$$

The class $\mathcal{B}$ consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, that is,

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) = 0. \tag{6.1}$$

A Riemannian manifold is said to satisfy cyclic parallel Ricci tensor if the Ricci tensor is non-zero and satisfies the condition (6.1). It is known [14] that Cartan hypersurface are manifolds with non-parallel Ricci tensor satisfying the condition (6.1).

From (6.1), it follows that $r = constant$.

Differentiating (3.8) covariantly along X, using (3.12) we have

$$\begin{aligned}(\nabla_X S)(Y,Z) &= \frac{1}{2}dr(X)(g(Y,Z)-\eta(Y)\eta(Z)) + (3\beta^2\\ &- \frac{r}{2})((\nabla_X\eta)(Y)\eta(Z) + (\nabla_X\eta)(Z)\eta(Y))\\ &= -\beta(3\beta^2 - \frac{r}{2})\{g(\phi X,Y)\eta(Z) - g(\phi X,Z)\eta(Y)\}. \quad (6.2)\end{aligned}$$

Using (6.2), clearly

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y) = 0.$$

Thus we are in a position to state the following:

Theorem 6.1. *A 3-dimensional non-cosymplectic quasi-Sasakian manifold with $\beta = constant$, satisfies cyclic parallel Ricci tensor if and only if the scalar curvature is constant.*

From Theorem 4.2 and Theorem 6.1 we have the following:

Corollary 6.1. *A 3-dimensional non-cosymplectic quasi-Sasakian manifold with $\beta = constant$, satisfies cyclic parallel Ricci tensor if and only if it is locally $\phi-$Ricci symmetric.*

From Corollary 5.1 and Corollary 6.1 we have the following:

Theorem 6.2. *A 3-dimensional non-cosymplectic quasi-Sasakian manifold with $\beta = constant$, satisfies cyclic parallel Ricci tensor if and only if it satisfies $\eta-$parallel Ricci tensor.*

7. Example of 3-dimensional Quasi-Sasakian Manifolds

Example 1. It is known [17] that a conformally flat 3-dimensional quasi-Sasakian manifold is of positive constant curvature, hence it is an Einstein manifold and therefore the manifold is ϕ-Ricci symmetric.

Example 2. We consider the three-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3, (x,y,z) \neq 0\}$, where (x,y,z) are standard co-ordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = \frac{\partial}{\partial z} - y\frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = 2\frac{\partial}{\partial x}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1,e_3) = g(e_1,e_2) = g(e_2,e_3) = 0$$

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1$$

Let η be the 1-form defined by $\eta(Z) = g(Z,e_3)$ for any $Z \in T(M)$.

Let ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = 1,$$

$$\phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z,W) - \eta(Z)\eta(W),$$

for any $Z, W \in T(M)$.

Thus for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$\begin{aligned}[e_1, e_2] &= e_1e_2 - e_2e_1 \\ &= (\frac{\partial}{\partial z} - y\frac{\partial}{\partial x})\frac{\partial}{\partial y} - \frac{\partial}{\partial y}(\frac{\partial}{\partial z} - y\frac{\partial}{\partial x}) \\ &= \frac{\partial}{\partial x} \\ &= \frac{1}{2}e_3.\end{aligned}$$

Similarly,

$$[e_1, e_3] = 0 \quad and \quad [e_2, e_3] = 0.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned}2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y])\end{aligned} \tag{7.1}$$

which is known as Koszul's formula.

Using (7.1) we have

$$2g(\nabla_{e_1}e_3, e_1) = 0 = 2g(\frac{1}{4}e_2, e_1). \tag{7.2}$$

Again by (7.1)

$$2g(\nabla_{e_1}e_3, e_2) = -g(-\frac{1}{2}e_3, e_3) = 2g(\frac{1}{4}e_2, e_2) \tag{7.3}$$

and

$$2g(\nabla_{e_1}e_3, e_3) = 0 = 2g(\frac{1}{4}e_2, e_3). \tag{7.4}$$

From (7.2), (7.3) and (7.4) we obtain

$$2g(\nabla_{e_1}e_3, X) = 2g(\frac{1}{4}e_2, X),$$

for all $X \in T(M)$.

Thus

$$\nabla_{e_1} e_3 = \frac{1}{4} e_2.$$

(7.1) further yields

$$\begin{aligned}
&\nabla_{e_1} e_3 = \frac{1}{4} e_2, \quad \nabla_{e_1} e_2 = -\frac{1}{4} e_3, \quad \nabla_{e_1} e_1 = 0, \\
&\nabla_{e_2} e_3 = -\frac{1}{4} e_1, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_1 = \frac{1}{4} e_3, \\
&\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = -\frac{1}{4} e_1, \quad \nabla_{e_3} e_1 = \frac{1}{4} e_2. \qquad (7.5)
\end{aligned}$$

We see that the structure (ϕ, ξ, η, g) satisfies the formula $\nabla_X \xi = -\beta\phi X$ for $\beta = \frac{1}{4}$. Hence the manifold is a 3-dimensional quasi-Sasakian manifold with the constant structure function $\beta = \frac{1}{4}$.

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

With the help of the above formula and using (7.5) it can be easily verified that

$$\begin{aligned}
&R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = \frac{1}{16} e_2, \quad R(e_1, e_3)e_3 = \frac{1}{16} e_1, \\
&R(e_1, e_2)e_2 = -\frac{3}{16} e_1, \quad R(e_2, e_3)e_2 = -\frac{1}{16} e_3, \quad R(e_1, e_3)e_2 = 0, \\
&R(e_1, e_2)e_1 = \frac{3}{16} e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = -\frac{1}{16} e_3.
\end{aligned}$$

From the above expression of the curvature tensor we obtain

$$\begin{aligned}
S(e_1, e_1) &= g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) \\
&= -\frac{1}{8}.
\end{aligned}$$

Similarly we have

$$S(e_2, e_2) = -\frac{1}{8} \quad \textit{and} \quad S(e_3, e_3) = \frac{1}{8}.$$

Now clearly

$$\phi^2(\nabla_X Q)(Y) = 0,$$

for all X and $Y \in T(M)$.

Hence M is locally $\phi-$Ricci symmetric.

Also,

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -\frac{1}{8}.$$

Therefore the scalar curvature r is constant. So Theorem 4.2 is verified. It is straight forword to verify that the Ricci tensor of M is $\eta-$parallel, cyclic parallel and $\eta-$Einstein.

Acknowledgement. The authors are thankful to the referee for his valuable suggestions towards the improvement of the paper.

References

[1] B. S. Acharya, Figurea A-O'Farrell, C. M. Hull, and B. J. Spence, Branes at Canonical singularities and holography, *Adv. Theor. Math. Phys.*, **2**(1999), 1249-1286.

[2] I. Agricola and T. Friedrich, Killing spinors in supergravity with 4-fluxes, *Class. Quant. Grav.*, **20**(2003), 4707-4717.

[3] I. Agricola, T. Friedrich, P. A. Nagy, and C. Puhle, On the Ricci tensor in the common sector of type II, string theoty, *Class. Quantum Grav.*, **22**(2005), 2569-2577.

[4] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Note in Mathematics, **509**, Springer-Verlag, Berlin-New York, 1976.

[5] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Mathematic, **203**, Birkhauser Boston , Inc., Boston, 2002.

[6] D. E. Blair, The theory of quasi-Sasakian structure, *J. Differential Geom.*, **1**(1967), 331-345.

[7] Calin,C., Foliation and complemented framed structures, *Bull. of the Belgian Math. Soc. Simon Stevin*, **17**(2010), 499-512.

[8] U. C. De and A. Sarkar, $\phi-$Ricci symmetric Sasakian manifolds, *Proce. Jangjeon Math. Society*, **11**(2008), 47-52.

[9] T. Friedrich and S. Ivanov, Almost contact manifolds, connections with torsion and parallel spinors, *J. Reine Angew. Math.*, **559**, 217-236.

[10] A. Gray, Two class of Riemannian manifolds, *Geom. Dedicata*, **7**(1978), 259-280.

[11] J. C. Gonzalez and D. Chinea, Quasi-Sasakian homogeneous structures on the generalized Heisenberg group $H(p, 1)$, *Proc. Amer. Math. Soc*, **105**(1989), 173-184.

[12] S. Kanemaki, Quasi-Sasakian manifolds, *Tohoku Math. J.*, **29**(1977), 227-233.

[13] S. Kanemaki, On quas-Sasakian manifolds, *Differential Geometry Banach center publications*, **12**(1984), 95-125.

[14] U. H. Ki and H. Nakagawa, A characterization of the Cartan hypersurface in a sphere, *Tohoku Math. J.*, **39**(1987), 27-40.

[15] B. H. Kim, Fibred Riemannian spaces with quasi-Sasakian structure, *Hiroshima Math. J.*, **20**(1990), 477-513.

[16] M. Kon, Invariat submanifolds in Sasakian manifolds, *Math. Ann.*, **219**(1976), 277-290.

[17] Z. Olszak, Normal almost contact metric manifolds of dimension three, *Ann. Polon. Math.*, **47**(1986), 41-50.

[18] Z. Olszak,, On three-dimensional conformally flat quasi-Sasakian manifolds, Period, *Math. Hungar.*, **33** no.2 (1996), 105-113.

[19] Z. Olszak, Curvature properties of quasi-Sasakian manifolds, *Tensor*, **38**(1982), 19-28.

[20] J. A. Oubina, New classes of almost contact metric structures, *Publ. Math. Debreceen*, **32**(1985), 187-193.

[21] T. Takahashi, Sasakian $\phi-$symmetric spaces, *Tohoku math. J.*, **29**(1977), 91-113.

[22] S. Tanno, Quasi-Sasakian structure of rank $2p+1$, *J. Differential Geom.*, **5**(1971), 317-324.

[23] K. Yano and M. Kon, *Structure on manifolds*, World scientific, 1984.