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The Minimum Number of Dependent Arcs in C_{3k}^3 *

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Abstract

Let D be an acyclic orientation of a simple graph G. An arc is called *dependent* if its reversal creates a directed cycle. Let $d(D)$ denote the number of dependent arcs in D. Define $d_{\min}(G)$ to be the minimum number of $d(D)$ over all acyclic orientations D of G. Let C_n denote the cycle on n vertices. The cube C_n^3 is the graph defined on the same vertex set of C_n such that any two distinct vertices u and v are adjacent in C_n^3 if and only if their distance in C_n is at most 3. In this paper, we study the structure of C_{3k}^3 to determine its minimum number of dependent arcs.

Keywords and Phrases: *Acyclic orientation, Dependent arc, Cycle, Cube.*

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1. Introduction

Graphs considered in this paper are finite, without loops, or multiple edges. For a graph G, we denote its vertex set, edge set, and the degree of a vertex v by $V(G)$, $E(G)$, and $d(v)$, respectively. If V_1 is a nonempty set of vertices of G, then we use $G[V_1]$ to denote the induced subgraph of G with vertex set V_1. If E_1 is a set of edges of G, then we use $G - E_1$ to denote the spanning subgraph of G with edge set $E(G) - E_1$. An *orientation* D of G assigns a direction to each edge of G and it is called *acyclic* if there does not exist any directed cycle. An arc of D is called *dependent* if its reversal creates a directed cycle. Let $d(D)$ denote the number of dependent arcs of D. We use $d_{\min}(G)$ and $d_{\max}(G)$ to denote the minimum and maximum number of $d(D)$ over all acyclic orientations D of G, respectively. It is known ([2]) that $d_{\max}(G) = \|G\| - |G| + c$ for a graph G having c components.

A proper *k-coloring* of a graph G is a mapping f from $V(G)$ to the set $\{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ for each edge $xy \in E(G)$. The *chromatic number* $\chi(G)$ is the smallest integer k such that G has a k-coloring. The *girth* $g(G)$ is the minimum length of a cycle in a graph G if there is any, and is ∞ if G possesses no cycles.

An interpolation question asks whether G has an acyclic orientation with exactly k dependent arcs for each k satisfying $d_{\min}(G) \leqslant k \leqslant d_{\max}(G)$. The graph G is called *fully orientable* if its interpolation question has an affirmative answer.

West [7] showed that complete bipartite graphs are fully orientable. Fisher et al. [2] showed that G is fully orientable if $\chi(G) < g(G)$, and $d_{\min}(G) = 0$ in this case. Since it is well-known [3] that every planar graph G with $g(G) \geqslant 4$ is 3-colorable, planar graphs of girth at least 4 are fully orientable.

The full orientability for a few classes of special graphs has been recently investigated. Lih, Lin, and Tong [6] showed that outerplanar graphs are fully orientable. This has been generalized by Lai, Chang, and Lih [4] to 2-degenerate graphs. A graph G is called 2-*degenerate* if every subgraph H of G contains a vertex of degree at most 2 in H. Lai and Lih [5] gave further examples of fully orientable graphs, such as subdivisions of Halin graphs and graphs of maximum degree at most three. Let $K_{r(n)}$ denote the complete r-partite graph each of whose partite sets has n vertices. Chang, Lin, and Tong [1] proved that $K_{r(n)}$ is not fully orientable if $r \geqslant 3$ and $n \geqslant 2$. These are the only known graphs that are not fully orientable.

Suppose that G is a connected graph. For $m \geqslant 2$, the mth *power* of G, denoted G^m, is the graph defined on the same vertex set $V(G)$ such that two distinct vertices u and v are adjacent in G^m if and only if their distance in G is at most m. In particular, G^2 is called the *square* of G and G^3 is called the *cube* of G. Assume that C_n, $n \geqslant 3$, is the cycle $v_0, v_1, \ldots, v_{n-1}, v_0$. A problem posed in [8] states as follows. For a given integer $m \geqslant 2$, does there exist a smallest constant $c(m)$ such that C_n^m is fully orientable when $n \geqslant c(m)$? In this paper, we give a proof for the determination of $d_{\min}(C_{3k}^3)$. We have worked out a proof for the full orientability of C_n^3. However, it is too lengthy to be included here.

2. Results

Given an acyclic orientation D of G, we denote by $u \to v$ the arc with tail u and head v. If we do not know which of u, v is the head, we still use uv to denote the oriented version of uv in D. We make the convention that the script letter $\mathcal{G}$ is used to denote an acyclically oriented version of G if the orientation is tacitly understood. The *in-degree* $d_D^-(v)$ of a vertex v in D is the number of arcs with head v; the *out-degree* $d_D^+(v)$ of v in D is the number of arcs with tail v. We call v a *source* if $d_D^+(v) = d_G(v)$ and v a *sink* if $d_D^-(v) = d_G(v)$, where $d_G(v)$ is the degree of v in G. Let $R(D)$ denote the set of dependent arcs in D. If we reverse all arcs in D, we denote the new orientation by D^-. It is easy to see that D^- is also an acyclic orientation and an arc is dependent in D if and only if its reversal is dependent in D^-. If D' is a subdigraph of D, then we write $D' \subseteq D$. The complete graph on n vertices is denoted by K_n.

The following two Lemmas are evident.

Lemma 1. *Let $D' \subseteq D$. If an arc is a dependent arc in D', then it is a dependent arc in D.*

Lemma 2. *For any acyclic orientation D of K_3, the number of dependent arcs is* 1. *Any vertex $v \in V(K_3)$ is a source or a sink in D if and only if v is incident with a dependent arc.*

Lemma 3. *For any acyclic orientation D of K_4, the number of dependent arcs is 3. Furthermore, every vertex of K_4 is incident with at least one dependent arc.*

Proof. It is well-known [7] that $d_{\min}(K_n) = d_{\max}(K_n) = (n-2)(n-1)/2$. Hence, $d_{\min}(K_4) = d_{\max}(K_4) = 3$. For any acyclic orientation D of the underlying graph K_4, $d^+_D(v) \geqslant 2$ or $d^-_D(v) \geqslant 2$. So v is a source or a sink in an acyclic orientation $D' \subseteq D$ of a certain subgraph K_3. By Lemma 2, v is incident with at least one dependent arc.

In C^3_n, any two subgraphs induced by the same number of vertices that are consecutive on C_n are isomorphic. In particular, every subgraph induced by three consecutive vertices is isomorphic to K_3, and every subgraph induced by four consecutive vertices is isomorphic to K_4. Denote by G_i, $i = 0, 1, \ldots, k-1$, the subgraph of C^3_{3k} induced by the vertex set $\{v_{3i}, v_{3i+1}, v_{3i+2}, v_{3i+3}, v_{3i+4}, v_{3i+5}\}$, where indices are taken modulo $3k$. These are k isomorphic subgraphs of C^3_{3k}. Let H denote G_0 for short. Let $H_1 = G[v_0, v_1, v_2]$, $H_2 = G[v_3, v_4, v_5]$, and $H_3 = H - (E(H_1) \cup E(H_2))$.

For an acyclic orientation D of C^3_{3k}, we use d_1, d_2, and d_3 to denote the number of dependent arcs of $E(H_1)$, $E(H_2)$, and $E(H_3)$ in D, respectively. According to the convention made at the beginning of this section, let $\mathcal{H}$ be an acyclically oriented version of H. Let $d'(\mathcal{H}) = d_3 + \frac{1}{2}(d_1 + d_2)$, and we always abbreviate $d'(\mathcal{H})$ to d'.

Remark. If $\mathcal{H} \subseteq \mathcal{G}$ and $\mathcal{G}$ is an acyclically oriented version of G, then $d'(\mathcal{H})$ evaluated in $\mathcal{G}$ is greater than or equal to $d'(\mathcal{H})$ evaluated in $\mathcal{H}$ by Lemma 1.

Lemma 4. *For any acyclic orientation D of H, $d'(\mathcal{H}) \geqslant 4$.*

Proof. Since H_1 and H_2 are triangles, $d_1 \geqslant 1$ and $d_2 \geqslant 1$ in any acyclic orientation D of H. Let $G[v_0, v_1, v_2, v_3]$ be the subgraph of G induced by the vertex set $\{v_0, v_1, v_2, v_3\}$. Since $G[v_0, v_1, v_2, v_3] \cong K_4$, at least one of v_0v_3, v_1v_3, v_2v_3 is dependent in D by Lemmas 1 and 3. So $d_3 \geqslant 1$.

Case 1. $d_3 = 1$.

Since $G[v_0, v_1, v_2, v_3] \cong G[v_2, v_3, v_4, v_5] \cong K_4$, at least one of v_0v_3, v_1v_3, v_2v_3 and at least one of v_2v_3, v_2v_4, v_2v_5 are dependent by Lemmas 1 and 3. Since $d_3 = 1$, v_2v_3 is the only dependent edge of $E(H_3)$.

We may assume that $v_2 \to v_3$.

By Lemma 2 and Lemma 3, we can determine the orientation of the following arcs: $v_2 \to v_0$, $v_2 \to v_1$, $v_2 \to v_4$, $v_4 \to v_1$, $v_4 \to v_3$, $v_5 \to v_3$, $v_1 \to v_3$, $v_3 \to v_0$, $v_1 \to v_0$, $v_5 \to v_2$, $v_5 \to v_4$. It follows that $v_2 \to v_0$, $v_2 \to v_1$, $v_1 \to v_0$,

$v_4 \to v_3$, $v_5 \to v_3$, and $v_5 \to v_4$ are dependent. So $d_1 = 3$ and $d_2 = 3$. Thus $d' = d_3 + \frac{1}{2}(d_1 + d_2) = 1 + \frac{1}{2}(3 + 3) = 4$.

Case 2. $d_3 = 2$.

Case 2.1. $d_1 = 1$. (By symmetry, the case $d_2 = 1$ is similar.)

At least three arcs in $\mathcal{G}[v_0, v_1, v_2, v_3]$ are dependent by Lemma 3. Since $d_3 = 2$ and $d_1 = 1$, two arcs of v_0v_3, v_1v_3, and v_2v_3 in D are dependent and none of v_1v_4, v_2v_4, and v_2v_5 is dependent. So from $\mathcal{G}[v_1, v_2, v_4]$, neither v_0v_1 nor v_0v_2 is dependent since v_1v_2 is dependent.

We may assume that $v_2 \to v_3$.

Then $v_5 \to v_3, v_4 \to v_3, v_1 \to v_3$, $v_0 \to v_3$.

Case 2.1.1. Assume $v_1 \to v_2$.

Then $v_1 \to v_0$, $v_0 \to v_2$, $v_1 \to v_4$, $v_4 \to v_2$, $v_2 \to v_5$, $v_4 \to v_5$. It follows that $v_0 \to v_3$, $v_1 \to v_3$, and $v_2 \to v_3$ are dependent, contradicting the assumption that $d_3 = 2$.

Case 2.1.2. Assume $v_2 \to v_1$.

By Lemma 2 and Lemma 3, $v_0 \to v_1$, $v_2 \to v_0$, $v_4 \to v_1$, $v_2 \to v_4$, $v_5 \to v_2$, $v_5 \to v_4$. It follows that $v_5 \to v_4$, $v_5 \to v_3$, and $v_4 \to v_3$ are dependent, i.e., $d_2 = 3$. So $d' = d_3 + \frac{1}{2}(d_1 + d_2) = 2 + \frac{1}{2}(1 + 3) = 4$.

Case 2.2. $d_1 \geqslant 2$ and $d_2 \geqslant 2$.

In this subcase $d' = d_3 + \frac{1}{2}(d_1 + d_2) \geqslant 2 + \frac{1}{2}(2 + 2) = 4$.

Case 3. $d_3 \geqslant 3$.

Since $d_1 \geqslant 1$ and $d_2 \geqslant 1$, $d' = d_3 + \frac{1}{2}(d_1 + d_2) \geqslant 3 + \frac{1}{2}(1 + 1) = 4$.

By Lemma 4, we know that $d' \geqslant 4$ in any acyclic orientations D of H. Now we are going to determine all the cases for which $d' = 4$.

Lemma 5. *Let D be an acyclic orientation of H. If we suppose that $v_2 \to v_3$ in D, then there are only 12 possible cases for D for which $d'(\mathcal{H}) = 4$.*

Proof. We again use d' to abbreviate $d'(\mathcal{H})$ in the following proof.

Case 1. $d_3 = 1$.

Since $d' = 4$, we have $d_1 = 3$ and $d_2 = 3$. By Case 1 of Lemma 4, we have $v_2 \to v_0$, $v_2 \to v_1$, $v_2 \to v_4$, $v_4 \to v_1$, $v_4 \to v_3$, $v_5 \to v_3$, $v_1 \to v_3$, $v_3 \to v_0$, $v_1 \to v_0$, $v_5 \to v_2$, $v_5 \to v_4$.

We denote by Q_1 this directed version of H.

Case 2. $d_3 = 2$.

Case 2.1. $d_1 = 1$.

Since $d' = 4$, we have $d_2 = 3$. By Case 2.1 of Lemma 4, we have $v_5 \to v_3, v_4 \to v_3, v_1 \to v_3$, $v_0 \to v_3$, $v_2 \to v_1$, $v_0 \to v_1$, $v_2 \to v_0$, $v_4 \to v_1$, $v_2 \to v_4$, $v_5 \to v_2$, $v_5 \to v_4$.

We denote by Q_2 this directed version of H.

Case 2.2. $d_1 = 2$.

Since $d' = 4$, we have $d_2 = 2$.

Case 2.2.1. $v_2 \to v_3$ is dependent.

Case 2.2.1.1. v_0v_3 is dependent.

Since $d_3 = 2$, none of v_1v_3, v_1v_4, v_2v_4, and v_2v_5 is dependent. From $\mathcal{G}[v_1, v_2, v_3, v_4]$, v_1v_2 and v_3v_4 are dependent by Lemma 3. From $\mathcal{G}[v_2, v_4, v_5]$, v_4v_5 is dependent by Lemma 2. Since $d_2 = 2$, v_3v_5 is not dependent. Then $v_2 \to v_5$, $v_5 \to v_3$, $v_4 \to v_2$, $v_4 \to v_5$, $v_4 \to v_3$, $v_1 \to v_4$, $v_1 \to v_2$, $v_1 \to v_3$. It follows that $v_1 \to v_3$ is dependent, and hence $d_3 \geqslant 3$, a contradiction.

Case 2.2.1.2. v_2v_5 is dependent.

By symmetry, this case is similar to the case 2.2.1.1, and there does not exist any acyclic orientation to satisfy the conditions of this case.

Case 2.2.1.3. v_1v_3 is dependent.

Since $d_3 = 2$, none of v_0v_3, v_1v_4, v_2v_4, and v_2v_5 is dependent. From $\mathcal{G}[v_1, v_2, v_3, v_4]$, v_1v_2 and v_3v_4 are dependent. From $\mathcal{G}[v_2, v_4, v_5]$, v_4v_5 is dependent. So v_3v_5 is not dependent. Then $v_2 \to v_5$, $v_5 \to v_3$, $v_4 \to v_2$, $v_4 \to v_5$, $v_4 \to v_3$, $v_1 \to v_4$, $v_1 \to v_2$, $v_1 \to v_3$, $v_2 \to v_0, v_1 \to v_0$. It follows that $v_1 \to v_0$ is dependent, and hence $v_2 \to v_0$ is not dependent. Then $v_0 \to v_3$.

We denote by Q_3 this directed version of H.

Case 2.2.1.4. v_2v_4 is dependent.

By symmetry, this case is similar to the case 2.2.1.3.

Then $v_2 \to v_0$, $v_0 \to v_3$, $v_3 \to v_1$, $v_0 \to v_1$, $v_2 \to v_1$, $v_1 \to v_4$, $v_3 \to v_4$, $v_2 \to v_4$, $v_5 \to v_3, v_5 \to v_4$, $v_2 \to v_5$.

We denote by Q_4 this directed version of H.

Case 2.2.1.5. v_1v_4 is dependent.

Since $d_3 = 2$, none of v_0v_3, v_1v_3, v_2v_4, and v_2v_5 is dependent. From $\mathcal{G}[v_0, v_1, v_2]$, v_0v_1 is dependent. From $\mathcal{G}[v_2, v_4, v_5]$, v_4v_5 is dependent.

Case 2.2.1.5.1. v_0v_2 is dependent.

Since $d_1 = 2$, v_1v_2 is not dependent. From $\mathcal{G}[v_1, v_2, v_3, v_4]$, v_3v_4 is dependent. Then $v_2 \to v_1$, $v_1 \to v_3$, $v_3 \to v_0$, $v_2 \to v_0$, $v_1 \to v_0$, $v_4 \to v_2$, $v_4 \to v_1$, $v_4 \to v_3$, $v_2 \to v_5$, $v_4 \to v_5$, $v_5 \to v_3$.

We denote by Q_5 this directed version of H.

Case 2.2.1.5.2. v_0v_2 is not dependent.

Since $d_1 = 2$, v_1v_2 is dependent. Then $v_2 \to v_0$, $v_0 \to v_3$, $v_3 \to v_1$, $v_0 \to v_1$, $v_2 \to v_1$, $v_4 \to v_1$, $v_4 \to v_3$, $v_5 \to v_3$. If $v_4 \to v_5$ in this case, then $v_4 \to v_2$, $v_2 \to v_5$. We denote by Q_6 this directed version of H. If $v_5 \to v_4$ in this case, then $v_5 \to v_2$, $v_2 \to v_4$. We denote by Q_7 this directed version of H.

Case 2.2.2. $v_2 \to v_3$ is not dependent.

From $\mathcal{G}[v_0, v_1, v_2, v_3]$ and $\mathcal{G}[v_2, v_3, v_4, v_5]$, at least one of v_0v_3 and v_1v_3 and at least one of v_2v_4 and v_2v_5 are dependent by Lemma 3. Since $d_3 = 2$, only one of v_0v_3 and v_1v_3 is dependent, only one of v_2v_4, v_2v_5 is dependent and v_1v_4 is not dependent.

From $\mathcal{G}[v_1, v_2, v_3, v_4]$, at least one of v_1v_3 and v_2v_4 is dependent.

Case 2.2.2.1. Only one of v_1v_3 and v_2v_4 is dependent.

From $\mathcal{G}[v_1, v_2, v_3, v_4]$, both v_1v_2 and v_3v_4 are dependent.

Case 2.2.2.1.1. v_1v_3 is dependent.

So v_2v_4 is not dependent. v_2v_5 is dependent and v_0v_3 is not dependent. Then $v_4 \to v_2$, $v_4 \to v_3$, $v_1 \to v_4$, $v_1 \to v_2$, $v_3 \to v_0$, $v_2 \to v_0$, $v_1 \to v_0$. It follows that $v_1 \to v_0, v_2 \to v_0$ and $v_1 \to v_2$ are dependent, contradicting the assumption that $d_1 = 2$.

Case 2.2.2.1.2. v_1v_3 is not dependent.

So v_2v_4 is dependent. By symmetry, this case is similar to the case 2.2.2.1.1, and there does not exist any acyclic orientation to satisfy the conditions of this case.

Case 2.2.2.2. Both v_1v_3 and v_2v_4 are dependent.

Since $d_3 = 2$, neither v_0v_3 nor v_2v_5 is dependent. From $\mathcal{G}[v_0, v_2, v_3]$, v_0v_2 is dependent. From $\mathcal{G}[v_2, v_3, v_5]$, v_3v_5 is dependent.

Case 2.2.2.2.1. v_0v_1 is dependent.

So v_1v_2 is not dependent. Then $v_3 \to v_0$, $v_2 \to v_0$, $v_1 \to v_2$, $v_1 \to v_3$, $v_1 \to v_0$, $v_4 \to v_1$, $v_4 \to v_2$, $v_4 \to v_3$. It follows that $v_4 \to v_3$ is dependent, and hence v_4v_5 is not dependent. Then $v_4 \to v_5$, $v_5 \to v_2$, and $v_5 \to v_3$.

We denote by Q_8 this directed version of H.

Case 2.2.2.2.2. v_0v_1 is not dependent.

So v_1v_2 is dependent. Then $v_3 \to v_0$, $v_2 \to v_0$, $v_0 \to v_1$, $v_3 \to v_1$, $v_2 \to v_1$, $v_5 \to v_2$, $v_5 \to v_3$, $v_2 \to v_4$, $v_5 \to v_4$. It follows that $v_5 \to v_4$ is dependent, and hence v_3v_4 is not dependent. Then $v_3 \to v_4$ and $v_4 \to v_1$.

We denote by Q_9 this directed version of H.

Case 2.3. $d_1 = 3$.

Since $d' = 4$, $d_2 = 1$. Then $v_2 \to v_5$, $v_2 \to v_4$, $v_4 \to v_5$, $v_5 \to v_3$, $v_4 \to v_3$, $v_4 \to v_1$, $v_1 \to v_3$, $v_3 \to v_0$, $v_1 \to v_0$, $v_2 \to v_1$, $v_2 \to v_0$.

We denote by Q_{10} this directed version of H.

Case 3. $d_3 = 3$.

Since $d' = 4$, $d_1 = 1$, and $d_2 = 1$. So at least two of v_0v_3, v_1v_3, and v_2v_3 are dependent and at least two of v_2v_3, v_2v_4, and v_2v_5 are dependent. Since $d_3 = 3$, v_2v_3 is dependent. Only one of v_0v_3 and v_1v_3 is dependent, only one of v_2v_4 and v_2v_5 is dependent, and v_1v_4 is not dependent.

Case 3.1. v_0v_3 is dependent.

So v_1v_3 is not dependent. From $\mathcal{G}[v_1, v_3, v_4]$, v_3v_4 is dependent. Since $d_2 = 1$, neither v_3v_5 nor v_4v_5 is dependent. From $\mathcal{G}[v_2, v_3, v_4, v_5]$, v_2v_5 is dependent. So v_2v_4 is not dependent. From $\mathcal{G}[v_1, v_2, v_4]$, v_1v_2 is dependent. Since $d_1 = 1$, neither v_0v_1 nor v_0v_2 is dependent. Then $v_2 \to v_1, v_2 \to v_4, v_4 \to v_1, v_2 \to v_0, v_0 \to v_1, v_0 \to v_3, v_1 \to v_3, v_4 \to v_3$, $v_4 \to v_5, v_5 \to v_3, v_2 \to v_5$.

We denote by Q_{11} this directed version of H.

Case 3.2. v_0v_3 is not dependent.

So v_1v_3 is dependent. From $\mathcal{G}[v_0, v_1, v_2, v_3]$, at least one of v_0v_1 and v_0v_2 is dependent. Since $d_1 = 1$, v_1v_2 is not dependent. From $\mathcal{G}[v_1, v_2, v_4]$, v_2v_4 is dependent. So v_2v_5 is not dependent. From $\mathcal{G}[v_2, v_3, v_4, v_5]$, at least one of v_3v_5 and v_4v_5 is dependent. Since $d_2 = 1$, v_3v_4 is not dependent. Then

$v_2 \to v_0, v_1 \to v_3, v_1 \to v_0, v_1 \to v_4, v_4 \to v_3, v_2 \to v_1, v_2 \to v_4, v_5 \to v_3$, $v_5 \to v_4$. It follows that $v_5 \to v_3$ and $v_2 \to v_0$ are dependent, and hence $v_5 \to v_4$ and $v_1 \to v_0$ are not dependent. Then $v_0 \to v_3, v_2 \to v_5$.

We denote by Q_{12} this directed version of H.

Remark. By Lemma 5 and assuming $v_2 \to v_3$ in D, there are only 12 possible choices for D to make $d'(\mathcal{H}) = 4$. If we assume that $v_3 \to v_2$ in D, there are another 12 possible choices for D to make $d'(\mathcal{H}) = 4$ and they are actually Q_i^- for $i = 1, 2, \ldots, 12$. Since the structures of Q_i and Q_i^- are essentially the same, we use Q_i to represent Q_i and Q_i^- unless otherwise stated.

It is easy to see that $C_n^3 \cong K_n$ for $3 \leqslant n \leqslant 7$. Thus, $d_{\min}(C_{3k}^3) = 1$ when $k = 1$ and $d_{\min}(C_{3k}^3) = 10$ when $k = 2$.

Theorem 6. *If $k \geqslant 3$ then $d_{\min}(C_{3k}^3) = 4k + 1$.*

Proof. We first prove that $d_{\min}(C_{3k}^3) \geqslant 4k + 1$. Suppose to the contrary that $d_{\min}(C_{3k}^3) < 4k + 1$. In the paragraph after Lemma 3, we defined G_i to be the subgraph of C_{3k}^3 induced by the vertex set $\{v_{3i}, v_{3i+1}, v_{3i+2}, v_{3i+3}, v_{3i+4}, v_{3i+5}\}$ for $i = 0, 1, \ldots, k-1$ and H to be G_0. Any two of these G_i's are isomorphic.

Since $G_i \cong H$, $d'(\mathcal{G}_i) \geqslant 4$ for all i by Lemma 4. So for any acyclic orientation D of C_{3k}^3, $d(D) = \sum_{i=0}^{k-1} d'(\mathcal{G}_i) \geqslant 4k$. The assumption that $d_{\min}(C_{3k}^3) < 4k + 1$ implies that $d_{\min}(C_{3k}^3) = 4k$. Hence, there exists an acyclic orientation D of C_{3k}^3 such that $d(D) = 4k$ and $d'(\mathcal{G}_i) = 4$ for all i.

If $\mathcal{G}_0$ is Q_j for some $j \in \{7, 10, 11\}$, then $d'(\mathcal{G}_1) > 4$ by Lemma 5, a contradiction. Since $G_i \cong G_0$ for all i, every $\mathcal{G}_i$ is different from Q_j for $j \in \{7, 10, 11\}$.

If $\mathcal{G}_0$ is Q_j for some $j \in \{2, 5\}$, then $d'(\mathcal{G}_{k-1}) > 4$ by Lemma 5, a contradiction. Hence, all $\mathcal{G}_i$ must be different from Q_j for $j \in \{2, 5\}$.

If $\mathcal{G}_0$ is Q_6, then, for $i = 1, 2, \ldots, k-2$, $d'(\mathcal{G}_i) = 4$ only when $\mathcal{G}_i$ is Q_3 by Lemma 5. But then $d'(\mathcal{G}_{k-1}) > 4$, a contradiction. Hence, all $\mathcal{G}_i$ must be different from Q_6.

If $\mathcal{G}_0$ is Q_1, then, for $i = 1, 2, \ldots, k-1$, $d'(\mathcal{G}_i) = 4$ only when $\mathcal{G}_i$ is Q_1 by Lemma 5. Assume that $v_2 \to v_3$ in $\mathcal{G}_0$. Then a directed cycle $v_0 \to v_{n-3} \to v_{n-6} \cdots \to v_3 \to v_0$ is produced, contradicting to the acyclicity of D. Hence, all $\mathcal{G}_i$ must be different from Q_1.

If $\mathcal{G}_0$ is Q_3, then, for $i = 1, 2, \ldots, k-1$, $d'(\mathcal{G}_i) = 4$ only when $\mathcal{G}_i$ is Q_3 by Lemma 5. Assume that $v_2 \to v_3$ in $\mathcal{G}_0$. Then a directed cycle $v_0 \to v_3 \to v_6 \cdots \to v_{n-3} \to v_0$ is produced, contradicting to the acyclicity of D. Hence, all $\mathcal{G}_i$ must be different from Q_3.

If $\mathcal{G}_0$ is Q_4, then, for $i = 1, 2, \ldots, k-1$, $d'(\mathcal{G}_i) = 4$ only when $\mathcal{G}_i$ is Q_4 by Lemma 5. Assume that $v_2 \to v_3$ in $\mathcal{G}_0$. Then a directed cycle $v_0 \to v_3 \to v_6 \cdots \to v_{n-3} \to v_0$ is produced, contradicting to the acyclicity of D. Hence, all $\mathcal{G}_i$ must be different from Q_4.

If $\mathcal{G}_0$ is Q_8, then, for $i = 1, 2, \ldots, k-1$, $d'(\mathcal{G}_i) = 4$ only when $\mathcal{G}_i$ is Q_8 by Lemma 5. Assume that $v_2 \to v_3$ in $\mathcal{G}_0$. Then a directed cycle $v_0 \to v_{n-3} \to v_{n-6} \cdots \to v_3 \to v_0$ is produced, contradicting to the acyclicity of D. Hence, all $\mathcal{G}_i$ must be different from Q_8.

If $\mathcal{G}_0$ is Q_9, then, for $i = 1, 2, \ldots, k-1$, $d'(\mathcal{G}_i) = 4$ only when $\mathcal{G}_i$ is Q_9 by Lemma 5. Assume that $v_2 \to v_3$ in $\mathcal{G}_0$. Then a directed cycle $v_0 \to v_{n-3} \to v_{n-6} \cdots \to v_3 \to v_0$ is produced, contradicting to the acyclicity of D. Hence, all $\mathcal{G}_i$ must be different from Q_9.

If $\mathcal{G}_0$ is Q_{12}, then, for $i = 1, 2, \ldots, k-1$, $d'(\mathcal{G}_i) = 4$ only when $\mathcal{G}_i$ is Q_{12} by Lemma 5. Assume that $v_2 \to v_3$ in $\mathcal{G}_0$. Then a directed cycle $v_0 \to v_3 \to v_6 \cdots \to v_{n-3} \to v_0$ is produced, contradicting to the acyclicity of D. Hence, all $\mathcal{G}_i$ must be different from Q_{12}.

In summary, for any orientation D, there exist $i_0 \in \{0, 1, 2 \cdots k-1\}$ such that $d'(\mathcal{G}_{i_0}) > 4$. Hence, $d_{\min}(C_{3k}^3) \geqslant 4k+1$.

In the second part, we are going to prove that $d_{\min}(C_{3k}^3) \leqslant 4k+1$. In fact, an acyclic orientation D_0 of G will be constructed so that $d(D_0) = 4k+1$.

Let D_0 be defined as follows.

$v_3 \to v_1 \to v_0,\ v_1 \to v_2 \to v_0,\ v_3 \to v_0, v_{3k-1} \to v_1 \to v_{3k-2} \to v_0 \to v_{3k-3}$, $v_{3k-1} \to v_0$, $v_{3k-1} \to v_2$, $v_1 \to v_4$, $v_3 \to v_2 \to v_4$, $v_5 \to v_2$, $v_3 \to v_5 \to v_4$, $v_3 \to v_4$, $v_3 \to v_6$;

$v_{3i} \to v_{3i-3}$ for each $i = 3, 4, \ldots, k-1$;

$v_{3i+2} \to v_{3i-1} \to v_{3i+1} \to v_{3i-2} \to v_{3i}$ for each $i = 2, 3, \ldots, k-1$;

$v_{3i+2} \to v_{3i+1} \to v_{3i}$ for each $i = 2, 3, \ldots, k-1$;

$v_{3i-1} \to v_{3i}$ and $v_{3i+2} \to v_{3i}$ for each $i = 2, 3, \ldots, k-1$.

Clearly, D_0 is an acyclic orientation of C_{3k}^3 such that the set of dependent arcs is as follows.

$R(D_0) = \{v_{3k-1} \to v_0, v_{3k-1} \to v_2, v_1 \to v_0, v_3 \to v_0, v_1 \to v_4, v_3 \to v_2, v_3 \to v_4, v_5 \to v_4, v_3 \to v_6\} \cup \{v_{3i+2} \to v_{3i+1}, v_{3i+1} \to v_{3i}, v_{3i+2} \to v_{3i}, v_{3i-1} \to v_{3i} \mid i = 2, \ldots, k-1\}$.

Therefore, $d(D_0) = |R(D_0)| = 4k+1$. This completes the proof of the theorem.

In this paper, we have only determined the minimum number of dependent arcs of C_{3k}^3. A complete proof for the determination of the minimum number of dependent arcs and the full orientability of C_n^3 is too lengthy to be included here. However, the proof methods used in the present paper fully illustrate the techniques that would be employed in a complete proof.

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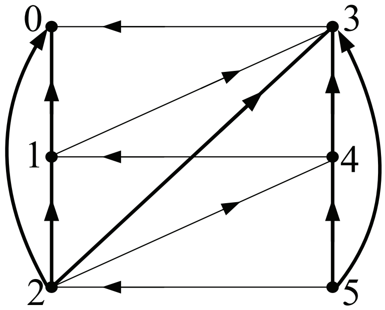
Appendix

Note 1. Let G_0 and G_1 be the induced subgraphs defined in Theorem 6. Suppose that $\mathcal{G}_0$ is Q_i and $\mathcal{G}_1$ is Q_j. We say that Q_i and Q_j can be *pasted together* if there exists an acyclic orientation D of $G_0 \cup G_1$ such that (i) $Q_i, Q_j \subset D$; (ii) an edge of $G_0 \cap G_1$ is a dependent edge in D if and only if it is a dependent edge in Q_i and Q_j. In Table 1, a tick in the (i, j) cell represents that Q_i and Q_j can be pasted together.

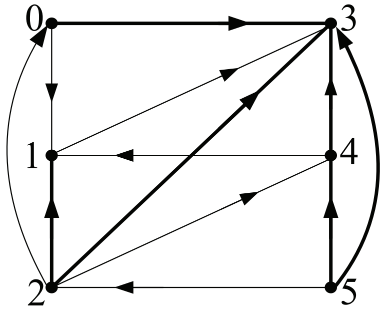
Note 2. All digraphs Q_1 to Q_{12} are depicted at the end of this appendix. Under each Q_i, the three rows of pairs (i, j) represent all the dependent arcs of $E(H_1)$, $E(H_2)$, and $E(H_3)$ in Q_i, respectively.

Table 1: Whether Q_i and Q_j can be pasted together.

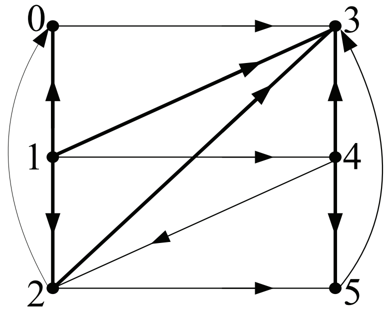
	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6	Q_7	Q_8	Q_9	Q_{10}	Q_{11}	Q_{12}
Q_1	$\surd$									$\surd$		
Q_2	$\surd$									$\surd$		
Q_3			$\surd$									
Q_4				$\surd$		$\surd$	$\surd$					
Q_5			$\surd$									
Q_6			$\surd$									
Q_7												
Q_8								$\surd$				
Q_9									$\surd$			
Q_{10}												
Q_{11}												
Q_{12}												$\surd$



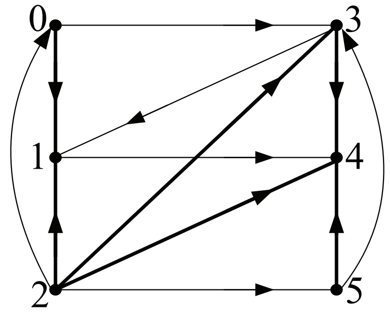
Q_1 (1,0) (2,0) (2,1)
(4,3) (5,3) (5,4)
(2,3)



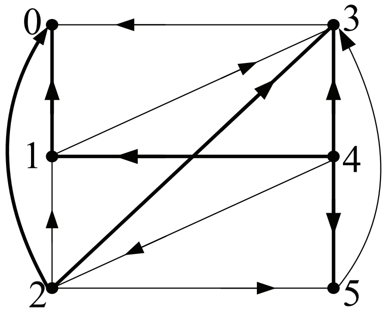
Q_2 (2,1)
(4,3) (5,3) (5,4)
(0,3) (2,3)



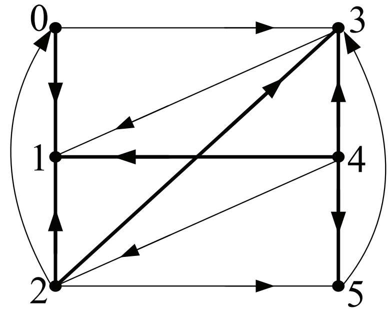
Q_3 (1,0) (1,2)
(4,3) (4,5)
(1,3) (2,3)



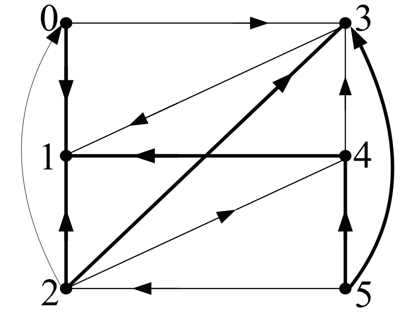
Q_4 (0,1) (2,1)
(3,4) (5,4)
(2,3) (2,4)



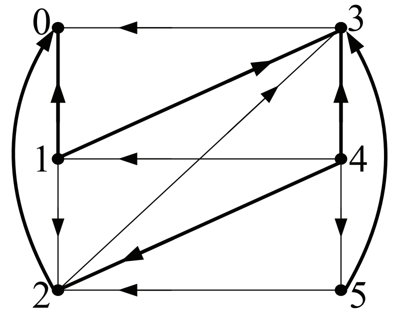
Q_5 (1,0) (2,0)
(4,3) (4,5)
(2,3) (4,1)



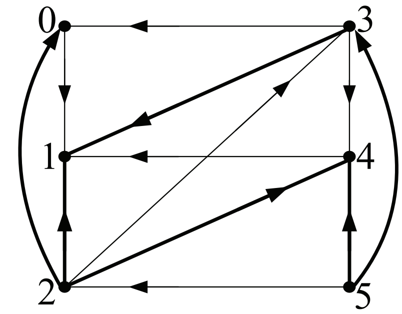
Q_6 (0,1) (2,1)
(4,3) (4,5)
(2,3) (4,1)



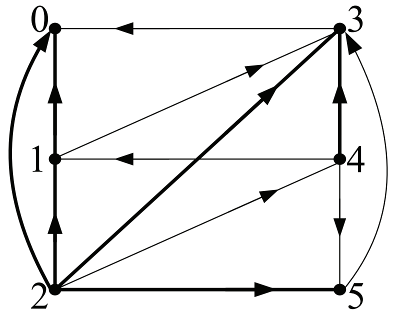
Q_7 (0,1) (2,1)
(5,3) (5,4)
(2,3) (4,1)



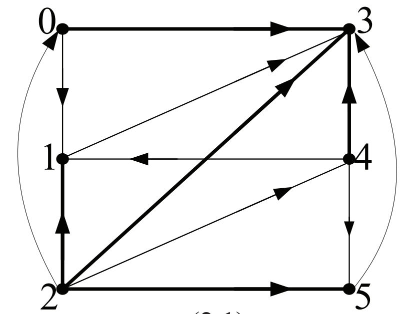
Q_8 (1,0) (2,0)
(4,3) (5,3)
(1,3) (4,2)



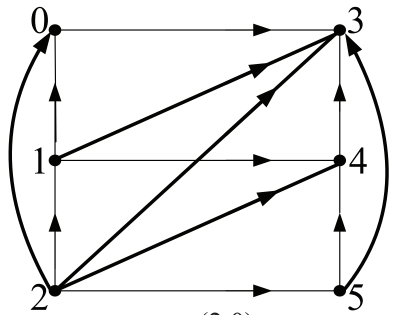
Q_9 (2,0) (2,1)
(5,3) (5,4)
(3,1) (2,4)



Q_{10} (1,0) (2,0) (2,1)
(4,3)
(2,3) (2,5)



Q_{11} (2,1)
(4,3)
(0,3) (2,3) (2,5)



Q_{12} (2,0)
(5,3)
(1,3) (2,3) (2,4)