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Generalized Quasi-metric Spaces on L-semigroups^{*}

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Abstract

In this paper, we define generalized quasi-pseudo-metric spaces on an L-semigroup. Morever, we also attempt to extend Blumenthal's metric spaces in Boolean algebras to quasi-metric-spaces in Stone algebras.

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0. Introduction

Many generalizations and generalizing procedures of the notion of metric spaces come from and are permeated with an idea of symmetry are of crucial character.

A rejection of the condition of symmetry in the definition of metric (Wilson [11]) shows essential quality changes in such spaces.

1. Generalized Quasi-Pseudo-Metric Spaces

Definition 1.1. ([1]) An algebra $(G, +, \lor, e)$ is called an *L*-semigroup if (G, +, e) is an Abelian semigroup with the unit e and (G, \lor) is a \lor -semilattice and the following conditions hold: for all $a, b, c \in G$,

(1)
$$e \leq a$$

(2) $a + (b \lor c) = (a + b) \lor (a + c).$

Lemma 1.2. Let $(G, +, \lor, e)$ be an L-semigroup and let $a, b, c, d \in G$. Then the following conditions hold:

(1) If $a \le b$, then $a + c \le b + c$. (2) $(a + b) \lor (c + d) \le (a \lor c) + (b \lor d)$.

Proof. The condition $a \leq b$ means that $a \vee b = b$. Then, for any $c \in G$, we have, by (2) of Definition 1,

$$(b + c) = (a \lor b) + c = (a + c) \lor (b + c),$$

which proves (1).

To show (2), if we apply (1) and (2) of Definition 1,

$$(a \lor c) + (b \lor d) = (a + (b \lor d)) \lor (c + (b \lor d))$$
$$= ((a + b) \lor (a + d)) \lor ((c + b) \lor (c + d))$$
$$\ge (a + b) \lor (c + d),$$

which proves (2). This completes the proof.

Definition 1.2. An *L*-semigroup $(G, +, \lor, e)$ is called *conditionally complete* if the \lor -semilattice (G, \lor) is conditionally complete, that is, for each nonempty $A \subset G$ bounded from above, there exists a sup $A \in G$.

Lemma 1.2. Let $(G, +, \lor, e)$ be a conditionally complete L-semigroup. Then G is a lattice.

Proof. It suffices to show that, for all $a, b \in G$, there exists $a \wedge b \in G$. Indeed, put

$$a \wedge b = \wedge \{ c \in G : c \le a, c \le b \}.$$

The set $\{c \in G : c \leq a, c \leq b\}$ is nonempty by (1) of Definition 1.1 and bounded from above by $a \lor b$. Using Definition 1.2, we get the element $a \land b$.

Definition 1.3. By a generalized quasi-pseudo-metric space is meant an ordered triple (X, G, p), where X is a nonempty set, G is an L-semigroup and $p: X^2 \to G$ satisfies the following conditions: for all $x, y, z \in X$,

- (1) p(x,x) = e,
- (2) $p(x,y) \le p(x,z) + p(z,y).$

Lemma 1.3. Let p_1 and p_2 be generalized quasi-pseudo-metrics on a set X with values in a L-semigroup G. Then the functions $p_1 \vee p_2 : X^2 \to G$ and $p_1 + p_2 : X^2 \to G$ defined by, for all $x, y \in X$,

- (1) $(p_1 \lor p_2)(x, y) = p_1(x, y) \lor p_2(x, y),$
- (2) $(p_1 + p_2)(x, y) = p_1(x, y) + p_2(x, y),$

respectively, are generalized quasi-pseudo-metrics on X, too.

Proof. From (1) of Definition 1.3, we have

$$(p_1 \lor p_2)(x, x) = p_1(x, x) \lor p_2(x, x) = e \lor e = e,$$

 $(p_1 + p_2)(x, x) = p_1(x, x) + p_2(x, x) = e + e = e.$

The condition (2) of Definition 1.3 for $p_1 \vee p_2$ can be obtained by using (2) of Lemma 1.1. Indeed, we have

$$(p_1 \lor p_2)(x, y) = p_1(x, y) \lor p_2(x, y)$$

$$\leq (p_1(x, y) + p_1(z, y)) \lor (p_2(x, z) + p_2(z, y))$$

$$\leq (p_1(x, z) \lor p_2(x, z)) + (p_1(z, y) \lor p_2(z, y))$$

$$= (p_1 \lor p_2)(x, z) + (p_1 \lor p_2)(z, y).$$

The condition (2) of Definition 1.3 for $p_1 + p_2$ can be obtained by using (2) of Definition 1.1 and (1) of Lemma 1.1. Indeed, we have

$$(p_1 + p_2)(x, y) = p_1(x, y) + p_2(x, y)$$

= $(p_1(x, z) + p_1(z, y)) + (p_2(x, z) + p_2(z, y))$
= $(p_1(x, z) + p_2(x, z)) + (p_1(z, y) + p_2(z, y))$
= $(p_1 + p_2)(x, z) + (p_1 + p_2)(z, y).$

This completes the proof.

The following is an immediate corollary from the above lemma:

Theorem 1.1. Let M be the family of all generalized quasi-pseudo-metrics on a set X with values in an L-semigroup G. Then the algebra $(M, +, \lor, e)$ is an L-semigroup too, where the operations + and \lor are given by the formulae (2), (1) of Lemma 1.3, respectively, and the function $e: X^2 \to G$ is defined by e(x, y) = e for all $x, y \in X$.

Corollary 1.1. If G is a conditionally complete L-semigroup, then the algebra $(M, +, \lor, e)$ is a lattice.

Proof. This result follows immediately from Lemma 1.1.

Definition 1.4. Let (X, G, p) be a generalized quasi-pseudo-metric space. Then (X, G, q) is also a generalized quasi-pseudo-metric space, where the function $q: X^2 \to G$ is given by the formula: for all $x, y \in X$,

$$(A) q(x,y) = p(y,x).$$

Then the generalized quasi-pseudo-metrics p and q are called the *conjugate* each other. A structure generated by p is denoted by (X, G, p, q).

If p and q are two conjugate of generalized quasi-pseudo-metrics on a set X, then the generalized quasi-pseudo-metric $p \lor q$ satisfies the symmetry condition by (1) of Lemma 1.3 and (A) and hence it is a generalized pseudo-metric.

Theorem 1.2. Let (X, G, p) be a generalized quasi-pseudo-metric space. Then the relation \leq on a set X^2 defined by the formula:

$$x \le y$$
 if and only if $p(x, y) = e$

is a quasi-order on X, that is, it is reflexive and transitive. If p, additionally, satisfies the condition:

(B)
$$p(x,y) \neq e \text{ or } p(y,x) \neq e \text{ whenever } x \neq y,$$

then \leq is a partial order.

Proof. The reflexivity and transitivity of the relation follow, respectively, from the conditions (1), (2) of Definition 1.3. The second part of the theorem follows from the fact that, whenever $x \leq y$ and $y \leq x$, then we simultaneously have p(x, y) = e and p(y, x) = e. By (B), we infer that x = y.

Remark 1.1. If, in Theorem 1.2, we replace the condition (B) by the following:

If $x \neq y$, then exactly one of p(x, y) and p(y, x) is equal to e,

then the relation \leq defined in (B) is a linear order on X.

Theorem 1.3. For each quasi-order \leq on X, there exists a generalized quasipseudo-metric which generates this order.

Proof. Let $G = \{0, 1\}$ be the two-point lattice and let $p : X^2 \to G$ be a function defined by

$$p(x,y) = \begin{cases} 0, & \text{if } x \leq y, \\ 1, & \text{if } x \not\leq y. \end{cases}$$

Then (X, p) is a generalized quasi-pseudo-metric space satisfying the conditions of the theorem.

Example 1.1. ([6], [8]) If $G = (\mathbb{R}^+, +, 0, \leq)$ is an *L*-semigroup, then (X, G, p) is a usual quasi-pseudo-metric space.

A distance distribution function (briefly, d.d.f.) is a nondecreasing function $F: [-\infty, +\infty] \rightarrow [0, 1]$ which is left-continuous on $(-\infty, +\infty)$ and F(0) = 0 and $F(+\infty) = 1$.

We denote the set of all d.d.f's by Δ^+ . Among the elements of Δ^+ , there is a unit step-function u_a defined by

$$u_a = 1_{(a,+\infty)}$$
 for all $a \in R_+ = [0,+\infty]$.

The elements of Δ^+ are partially ordered by

 $F \leq G$ if and only if $F(x) \leq G(x)$ for all $x \in X$.

For any $F, G \in \Delta^+$ and $h \in (0, 1]$, let (G, F : h) denote the condition:

$$G(x) \le F(x+h) + h$$
 for all $x \in (0, h^{-1})$

and let

$$d_L(F,G) = \inf\{h : \text{ both } (F,G:h) \text{ and } (G,F:h) \text{ hold } \}.$$

As shown by Sibley [10], the function d_L is a metric in Δ^+ which is a modified form of the well-known Levy metric for distribution functions and the metric space (Δ^+, d_L) is compact and hence complete (see [9, p. 45-49]).

Definition 1.5. ([9]) A binary operation $* : \Delta^+ \times \Delta^+ \to \Delta^+$ is a triangle function if $(\Delta^+, *)$ is an Abelian monoid with identity u_0 in Δ^+ such that, for any $F, F', G, G' \in \Delta^+$,

$$F * G \leq F' * G'$$
 whenever $F \leq F', G \leq G'$.

Note that a triangle function * is continuous if it is continuous with respect to the metric topology induced by d_L .

Example 1.2. ([4]) Let $G = (\Delta^+, *, u_0, \geq)$ be an *L*-semigroup. Then (X, G, p) is a probabilistic quasi-pseudo-metric space defined on X.

Let L be a distributive lattice with the universal bounds 0 and 1. We say that $a^* \in L$ is a *pseudo-complement* of $a \in L$ if, for every $x \in L$ such that $a \wedge x = 0$, one has $x \leq a^*$. A *Stone algebra* is a distributive lattice L with the condition (see Grätzer [5], p. 152]):

$$a^* \vee a^{**} = 1.$$

Some properties of a Stone algebra are given as follows:

Lemma 1.4. Let L be a Stone algebra and let $x, y, z \in L$. Then the following hold:

(1)
$$(x \wedge y)^* = x^* \vee y^*$$
.

- (2) $(x \wedge y)^{**} = x^{**} \wedge y^{**}.$
- (3) If $x \le y$, then $x^{**} \land y^* = 0$.
- (4) $x^{**} \wedge y^* \leq (x^{**} \wedge z^*) \vee (z^{**} \wedge y^*).$

Proof. (1) Note that $x \wedge y \leq x$ and hence $x^* \leq (x \wedge y)^*$ and, analogously, $y^* \leq (x \wedge y)^*$. Thus $x^* \vee y^* \geq (x \wedge y)^*$. On the other hand, by the distributivity of L, we have $(x \wedge y) \wedge (x^* \vee y^*) = 0$ and hence $x^* \vee y^* \leq (x \wedge y)^*$.

(2) Note that $x \wedge y \leq x$ implies $(x \wedge y)^{**} \leq x^{**}$ and $(x \wedge y)^{**} \leq y^{**}$. Hence it follows that $(x \wedge y)^{**} \leq x^{**} \wedge y^{**}$. On the other hand, by (1) and the distributivity of L, we have

$$(x \wedge y)^* \wedge (x^{**} \wedge y^{**}) = (x^* \vee y^*) \wedge (x^{**} \vee y^{**})$$

= $(x^* \wedge x^{**} \wedge y^{**}) \vee (y^* \wedge x^{**} \wedge y^{**})$
= 0.

This means that $x^{**} \wedge y^{**} \leq (x \wedge y)^{**}$.

(3) Let $x \leq y$. Then $x \wedge y = x$ and, by (2), we have $x^{**} \wedge y^* = (x^{**} \wedge y^{**}) \wedge y^* = 0$.

(4) Consider the following:

$$\begin{aligned} x^{**} \wedge z^{*}) &\lor (z^{**} \wedge y^{*}) \\ &= ((x^{**} \wedge z^{*}) \lor z^{**}) \wedge ((x^{**} \wedge z^{*}) \lor y^{*}) \\ &= ((x^{**} \lor z^{**}) \wedge (z^{*} \lor z^{**})) \wedge ((z^{**} \wedge z^{*}) \lor y^{*}) \\ &= (x^{**} \wedge z^{**} \wedge z^{*}) \lor (z^{**} \wedge x^{**} \wedge z^{*}) \lor (x^{**} \wedge y^{*}) \lor (z^{**} \wedge y^{*}) \\ &= (x^{**} \wedge z^{*}) \lor (z^{**} \wedge y^{*}) \lor (x^{**} \wedge y^{*}), \end{aligned}$$

which implies

$$x^{**} \wedge y^* \leq (x^{**} \wedge z^*) \vee (z^{**} \wedge y^*).$$

This completes the proof.

Theorem 1.4. Let L be a Stone algebra and let $p : L^2 \to L$ be a function defined by the following formula:

$$p(x,y) = x^{**} \wedge y^*.$$

Then the function p has the following properties:

(1)
$$p(x, x) = 0,$$

(2) $p(x, y) \le p(x, z) \lor p(z, y).$

Proof. The proof follows directly from the conditions (3), (4) of Lemma 1.4.

Corollary 1.2. The pair (L, p) is a generalized quasi-pseudo-metric space defined on L with values in a Stone algebra considered as an L-semigroup (with the group operation + being identified with the lattice operation \lor).

Corollary 1.3. Let (L, p) be a generalized quasi-pseudo-metric space. Then (L, q) is a generalized quasi-pseudo-metric space which is a conjugate of (L, p), where q is defined by (A). Also, $(L, p \lor q)$ is a generalized pseudo-metric space.

Corollary 1.4. ([2]) Let L be a Boolean algebra. Then $(L, p \lor q)$ is a generalized metric space.

Remark 1.2. The question of generalized metric spaces is dealt with in the paper by Blumenthal and Menger ([2], Chap. III).

Remark 1.3. Consider (X, G, p, q) of Definition 1.3 generated by a generalized quasi-pseudo-metric p. Then we note that the condition of an "internal" symmetry dropped in the definition of p has been structurally imposed on the pair (p and q) of mutually conjugate quasi-pseudo-metrics, because the operation $i: X^2 \to X^2$ defined by $(x, y) \to (y, x)$, mapping p to q, is an involution. Thus it seems that, by dropping the symmetry, one destroys the conjugation.

Thus we have the following:

Definition 1.6. By a *G*-bimetric space we mean an ordered 4-tuple (X, G, p_1, p_2) , where X is a nonempty set, G is an L-semigroup and p_1, p_2 are arbitrary generalized quasi-pseudo-metrics.

Remark 1.4. We have taken the name a "bimetric space" to make a distinction from the concept of a 2-metric space introduced by Gähler ([6]) as a generalization of the area function in \mathbb{R}^2 .

Remark 1.5. By Lemma 1.3 with a *G*-bimetric space, there are associated two generalized quasi-pseudo-metric spaces $(X, G, p_1 \vee p_2)$ and $(X, G, p_1 + p_2)$. For p_1 being a conjugate of p_2 , a *G*-bimetric space is provided by the structure of Definition 1.3. On the other hand, if $p_1 = p_2$, then the *G*-bimetric space reduces itself to a (·)-metric space.

2. Hemimetric Spaces

In this section, we present the definition of the notion of Hemimetric space as a structural generalization of the notion of ordering metric and quasi-metric space ([6], [11]), which was dealt with in the paper by Blumenthal and Menger [2].

The essence of this procedure consists in introducing a disjunctive "triangle condition". We give several example of such spaces, and at the some time, we define their topologies. We also give condition for hemimetrizability of these spaces.

Definition 2.1. An ordered triple (X, h, G) is called a *hemimetric space*, where X is a nonempty set, G is an L-semigroup and the function $h : X^2 \to G$ fulfills the conditions:

for all $x \in X$, where $e \in G$ is a unit in group G and the disjunction of the following inequalities hold: for all $x, y, z \in X$,

(Q-1) $h(x, y) \le h(x, z) + h(z, y),$

(M-1) $h(x,y) \le h(y,z) + h(z,x),$

(M-2) $h(x,y) \le h(x,z) + h(y,z),$

(M-3) $h(x,y) \le h(z,x) + h(z,y).$

Each of the conditions (Q-1), (M-1), (M-2), (M-3) is called a *weak triangle condition* or a *disjunctive triangle condition*.

Remark 2.1. If the condition (Q-1) is fulfilled, then the function h is a generalized quasi-pseudo-metric. On the other hand, if one of the conditions (M-i) for i = 1, 2, 3 is strictly fulfilled, then the function h is a generalized pseudo-metric. Note that the symmetry condition follows directly from (C) and any condition (M-i) for i = 1, 2, 3. Indeed, to exemplify, take the condition (M-1)

$$h(x,y) \le h(y,x) + h(x,x) = h(y,x).$$

Analogously, we get $h(y, x) \leq h(x, y)$ and hence the equality h(x, y) = h(y, x).

Definition 2.2. Let (X, h, G) be a hemimetric space. Then the function $g: X^2 \to G$ defined by

$$(D) g(x,y) = h(y,x)$$

for all $x, y \in X$ is a hemimetric given in X. An ordered triple (X, g, G) is called a *hemimetric space conjugate* with (X, h, G).

Remark 2.2. Note that the conditions (M-2) and (M-3) are mutually conjugate in the sense that, at the moment of transition from the function h to g, (M-2) becomes (M-3) and, conversely. On the other hand, the conditions (Q-1) and (M-1) are self-conjugate in the sense of the above notion.

Example 2.1. Let X be an Euclidean plane with the pointwise partial order induced from a natural order of real numbers R and let the function $h: X^2 \to X^2$

 R^+ be given by the following formula:

$$h(A,B) = \begin{cases} 0, & \text{if } A \leq B, A, B \in X \text{ or } A, B \in OY, \\ |AB|, & \text{otherwise,} \end{cases}$$

for all $A = (x_1, y_1)$ and $B = (x_2, y_2) \in X$. The symbol |AB| stands for the distance between the points A and B. Then the ordered triple (X, h, R^+) is a hemimetric space for the sake of the L-semigroup $(R^+, +, 0, \leq)$.

Note that, for the points A = (1, 0), B = (0, 1) and C = (0, 0), the condition (M-2) holds. Respectively, for the points A = (-1, 0), B = (0, -1) and C = (0, 0), the condition (M-3) holds. On the other hand, for the points A = (-1, 0), B = (0, 1) and C = (0, 0), the condition (M-1) holds. Finally, for the points A = (1, 0), B = (0, -1) and C = (0, 0), the condition (Q-1) holds.

Example 2.3. Let $X = R \times (R^+ \cup \{0\})$ and let a function $h : X^2 \to R^+$ for all $A, B \in X$ be defined by

$$h(A,B) = \begin{cases} 0, & \text{if } A = B, \\ 1, & \text{if } A \neq B \in OX \text{ or } A \in OX \text{ and } B \in OX, \\ \min\{1,r\}, & \text{if } A \in OX \text{ and } B \notin OX, \\ \min\{1,|AB|\}, & \text{if } A \notin OX, \end{cases}$$

where r stands for the radius of the circle with the center in point $S(A_x, r)$ and tangent to the line OX in the point A. Then the pair (X, h) is a hemimetric space. Note at the same time that the family of all the neighborhoods of the points from X given by the formula

$$U_A^h(\varepsilon) = \{ B \in X : h(A, B) < \varepsilon \}$$

generates the topology T_h identical with the topology of Niemytzki space (see V. Niemytzki [8]).

Remark 2.3. This example demonstrates that Niemytzki space is a hemimetrizable space.

Example 2.3. Let $X = I \vee \{P\}$ for $P \notin I = [0, 1]$. On the other hand, the

function $h: X^2 \to R^+$ is given as follows: for all $x, y \in X$,

$$h(x,y) = \begin{cases} 0, & \text{for } x \in I \text{ and } y \in P \\ & \text{or } (x \ge y \text{ and } x, y \in I) \text{ and } x = y, \\ x - y, & \text{if } x, y \in I \text{ and } x < y, \\ 1, & \text{if } x = P \text{ and } y \in I. \end{cases}$$

Then (X, h) is a hemimetric space.

According to (D), the function g(x, y) = h(y, x) is also a hemimetric. The families of neighborhoods $\{U_x^h(\varepsilon)\}_{x \in X}$ and $\{U_x^g(\varepsilon)\}_{x \in X}$ generate the topologies T_h and T_g , that is, they determine a bitopological space (X, T_h, T_g) , which was introduced by J. C. Kelly [6].

Note that the neighborhoods of the points from X are of the form $U_x^h(\varepsilon) = [0, x + \varepsilon) \cup \{P\}$ if $x \neq P$ and $U_p^h(\varepsilon) = \{P\}$. Analogously, for the function g, we have $U_x^g(\varepsilon) = (x - \varepsilon, 1] \cup \{P\}$ if $x \neq P$ and $U_p^g(\varepsilon) = \{P\}$.

The topologies T_h and T_g are T_o , while the bitopological space (X, T_h, T_g) is mutually T_1 , yet it does not fulfill the condition $T_{1\frac{1}{2}}$ and therefore it is not semi-Hausdorff space.

Remark 2.4. The question of separation axioms in the bitopological space (X, T_h, T_g) is dealt with in the papers by J. C. Kelly [6], M. G. Murdeshwar and S. A. Naimpally [7].

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