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Properties of Subclasses of Multivalent Functions Defined by a Multiplier Transformation*

H. E. Darwish†
Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt

and

T. Bulboacă‡
Faculty of Mathematics and Computer Science,
Babeş-Bolyai University, 400084 Cluj-Napoca, Romania

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Abstract

In this paper we introduced new classes of p–valent functions defined by the multiplier transformation, and certain inclusion relations are established. Also, we proved that a well-known class of integral operators preserve these subclasses of $A(p)$.

Keywords and Phrases: *Subordination, Multiplier transformation, p–valent starlike function, p–valent convex function, p–valent close-to-convex function.*

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†E-mail: darwish333@yahoo.co
‡Corresponding author. E-mail: bulboaca@math.ubbcluj.ro

1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k,\ (p\in\mathbb{N}=\{1,2,\dots\}),$$

which are analytic and p–valent in the open unit disc $\mathrm{U}=\{z\in\mathbb{C}:|z|<1\}$.

If f and g are analytic functions in U, we say that f is *subordinate* to g, written $f(z)\prec g(z)$, if there exists a *Schwarz function* w, which (by definition) is analytic in U with $w(0)=0$, and $|w(z)|<1$ for all $z\in\mathrm{U}$, such that $f(z)=g(w(z))$, for all $z\in\mathrm{U}$. Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0)\text{ and }f(\mathrm{U})\subset g(\mathrm{U}).$$

For $0\leq\eta<p$, we denote by $S_p^*(\eta)$, $K_p(\eta)$ and C_p the subclasses of $A(p)$ consisting of all analytic functions which are, respectively, *p–valent starlike of order η*, *p–valent convex of order η* and *close-to-convex* in U.

Let define the multiplier transformation $\mathrm{I}_{\lambda,p}^s:A(p)\to A(p)$ by

$$\mathrm{I}_{\lambda,p}^s f(z)=z^p+\sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^s a_kz^k,\quad(\lambda\geq0,\ s\in\mathbb{R}).$$

This operator is closely related to the Sălăgean derivative operators [13]. The special case $\mathrm{I}_{1,\lambda}^s$ was studied recently by Cho and Srivastava [4], and Cho and Kim [3], while $\mathrm{I}_{1,1}^s$ was studied by Uralegaddi and Somanatha [15]. An investigation of the $\mathrm{I}_{p;\lambda}^s$ operator was given by Aghalary et. al. [1]. We also mention the papers [2], [6], [7], [9], [11], [12] and [14], that are closely-related recent articles on the subject of the multiplier transformations investigated in our work.

Let $\mathcal{M}$ be the class of all functions φ which are analytic and univalent in U and for which $\varphi(\mathrm{U})$ is convex, with $\varphi(0)=1$ and $\operatorname{Re}\varphi(z)>0$ for all $z\in\mathrm{U}$.

Using the above subordination property between univalent functions, in order to generalize the previous subclasses $S_p^*(\eta)$, $K_p(\eta)$ and C_p, we define the following subclasses of $A(p)$:

$$S_p^*(\eta;\varphi)=\left\{f\in A(p):\frac{1}{p-\eta}\left(\frac{zf'(z)}{f(z)}-\eta\right)\prec\varphi(z)\right\},$$

$$K_p(\eta;\varphi)=\left\{f\in A(p):\frac{1}{p-\eta}\left(1+\frac{zf''(z)}{f'(z)}-\eta\right)\prec\varphi(z)\right\},$$

and

$$C_p(\eta,\delta;\varphi,\psi)=\left\{f\in A(p):\exists\, g\in S_p^*(\eta;\varphi),\frac{1}{p-\delta}\left(\frac{zf'(z)}{g(z)}-\delta\right)\prec\psi(z)\right\},$$

where $0\le\eta<p$, $0\le\delta<p$, and $\varphi,\psi\in\mathcal{M}$. It is easy to see that the next equivalence holds:

$$f\in K_p(\eta;\varphi)\Leftrightarrow\frac{zf'(z)}{p}\in S_p^*(\eta;\varphi). \tag{1.1}$$

Setting

$$f_{p;\lambda}^s(z)=z^p+\sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^s z^k,\quad (s\in\mathbb{R},\ \lambda\ge0),$$

we define a new function $f_{p;\lambda,\mu}^s$ in terms of the *Hadamard (or convolution) product*, by

$$f_{p;\lambda,\mu}^s(z)*f_{p;\lambda}^s(z)=\frac{z^p}{(1-z)^{\mu+p}},\quad (\mu>-p). \tag{1.2}$$

We now introduce the operator $\mathrm{I}_{p;\lambda,\mu}^s:A(p)\to A(p)$, defined by

$$\mathrm{I}_{p;\lambda,\mu}^s f(z)=f_{p;\lambda,\mu}^s(z)*f(z)=z^p+\sum_{k=p+1}^{\infty}\left(\frac{p+\lambda}{k+\lambda}\right)^s\frac{(p+\mu)_{k-p}}{(1)_{k-p}}a_kz^k, \tag{1.3}$$

where $s\in\mathbb{R}$, $\lambda\ge0$, $\mu>-p$, and $(d)_k$ denotes the *Pochhammer symbol*, i.e.

$$(d)_k=\begin{cases}1, & \text{if}\quad k=0,\ d\in\mathbb{C}\setminus\{0\},\\ d(d+1)\dots(d+k-1), & \text{if}\quad k\in\mathbb{N},\ d\in\mathbb{C}.\end{cases}$$

In particular, we note that

$$\mathrm{I}_{p;0,1-p}^0 f(z)=f(z)\quad\text{and}\quad \mathrm{I}_{p;0,2-p}^0=zf'(z)+(1-p)f(z).$$

In view of (1.2) and (1.3), we may easily obtain the following relations:

$$z\left(\mathrm{I}_{p;\lambda,\mu}^{s+1}f(z)\right)'=(\lambda+p)\,\mathrm{I}_{p;\lambda,\mu}^s f(z)-\lambda\,\mathrm{I}_{p;\lambda,\mu}^{s+1}f(z) \tag{1.4}$$

and

$$z\left(\mathrm{I}^s_{p;\lambda,\mu} f(z)\right)' = (\mu+p)\,\mathrm{I}^s_{p;\lambda,\mu+1} f(z) - \mu\,\mathrm{I}^s_{p;\lambda,\mu} f(z). \tag{1.5}$$

Now, by using the operator $\mathrm{I}^s_{p;\lambda,\mu}$, for $\varphi,\ \psi \in \mathcal{M}$, $s \in \mathbb{R}$, $\lambda \geq 0$, $\mu > -p$, $0 \leq \eta < p$, and $0 \leq \delta < p$, we will introduce the following subclasses of $A(p)$:

Definition 1.1. 1. Let denote by

$$S^s_{p;\lambda,\mu}(\eta;\varphi) = \left\{f \in A(p) : \mathrm{I}^s_{p;\lambda,\mu} f \in S^*_p(\eta;\varphi)\right\}$$

the class of p–valent generalized φ–starlike of order η.

2. Let

$$K^s_{p;\lambda,\mu}(\eta;\varphi) = \left\{f \in A(p) : \mathrm{I}^s_{p;\lambda,\mu} f \in K_p(\eta;\varphi)\right\}$$

be the class of p–valent generalized φ–convex of order η.

3. Let denote by

$$C^s_{p;\lambda,\mu}(\eta,\delta;\varphi,\psi) = \left\{f \in A(p) : \mathrm{I}^s_{p;\lambda,\mu} f \in C_p(\eta,\delta;\varphi,\psi)\right\}$$

the class of p–valent generalized φ–close-to-convex of order η.

Remark 1.1. If $f \in C^s_{p;\lambda,\mu}(\eta,\delta;\varphi,\psi)$, then $\mathrm{I}^s_{p;\lambda,\mu} f \in C_p(\eta,\delta;\varphi,\psi)$, hence there exists a function $g \in S^*_p\,(\eta;\varphi)$ such that

$$\frac{1}{p-\delta}\left(\frac{z\left(\mathrm{I}^s_{p;\lambda,\mu} f(z)\right)'}{\mathrm{I}^s_{p;\lambda,\mu} g(z)} - \delta\right) \prec \psi(z).$$

In this case we call that $f \in C^s_{p;\lambda,\mu}(\eta,\delta;\varphi,\psi)$, or f is a p–valent generalized φ–starlike of order η, related to the function $g \in S^*_p\,(\eta;\varphi)$.

Since $\mathrm{I}^0_{p;0,1-p} f = f$, these classes generalize the already defined $S^*_p(\eta;\varphi)$, $K_p(\eta;\varphi)$, and $C_p(\eta,\delta;\varphi,\psi)$ subclasses of $A(p)$. Also, it is easy to check that

$$\frac{z}{p}\left(\mathrm{I}^s_{p;\lambda,\mu} f(z)\right)' = \mathrm{I}^s_{p;\lambda,\mu}\left(\frac{zf'(z)}{p}\right), \tag{1.6}$$

and according to this formula, we have the next equivalence:

$$f \in K^s_{p;\lambda,\mu}(\eta;\varphi) \Leftrightarrow \frac{zf'(z)}{p} \in S^s_{p;\lambda,\mu}(\eta;\varphi). \tag{1.7}$$

In particular, we set

$$S^s_{p;\lambda,\mu}(\eta; A, B; \alpha) = S^s_{p;\lambda,\mu}\left(\eta; \left(\frac{1+Az}{1+Bz}\right)^{\alpha}\right),$$

and

$$K^s_{p;\lambda,\mu}(\eta; A, B; \alpha)) = K^s_{p;\lambda,\mu}\left(\eta; \left(\frac{1+Az}{1+Bz}\right)^{\alpha}\right),$$

where $0 < \alpha \le 1$, and $-1 \le B < A \le 1$.

In the first part of this paper we investigate several inclusion properties of the classes $S^s_{p;\lambda,\mu}(\eta;\varphi)$, $K^s_{p;\lambda,\mu}(\eta;\varphi)$ and $C^s_{p;\lambda,\mu}(\eta,\delta;\varphi,\psi)$, associated with the operator $\mathrm{I}^s_{p;\lambda,\mu}$, while in the second paper we will prove that a well-known class of integral operators preserve these subclasses of $A(p)$. Some applications involving these and other classes of integral operators are also considered.

2. Inclusion Properties Involving the Operator $\mathrm{I}^s_{p;\lambda,\mu}$

The following results will be required in our investigation.

Lemma 2.1. *[8] Let φ be convex (univalent) in* U*, with $\varphi(0) = 1$ and* $\mathrm{Re}\,(\beta\varphi(z) + \gamma) > 0$ *for all $z \in$ U, where $\beta, \gamma \in \mathbb{C}$. If the function q is analytic in* U*, with $q(0) = 1$, then*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec \varphi(z),$$

implies that

$$q(z) \prec \varphi(z).$$

Lemma 2.2. *[10] Let φ be convex (univalent) in* U*, and let w be analytic in* U*, with* $\mathrm{Re}\, w(z) \ge 0$ *for all $z \in$ U. If the function q is analytic in* U*, with $q(0) = \varphi(0)$, then*

$$q(z) + w(z)zq'(z) \prec \varphi(z)$$

implies that

$$q(z) \prec \varphi(z).$$

With the help of Lemma 2.1, we obtain the next inclusions:

Theorem 2.1. *Let $s \in \mathbb{R}$, $\lambda \geq 0$, $\mu > -p$, $0 \leq \eta < p$, and $\mu + \eta \geq 0$.*

1. If $f \in S^s_{p;\lambda,\mu+1}(\eta;\varphi)$, then $f \in S^s_{p;\lambda,\mu}(\eta;\varphi)$, whenever $\mathrm{I}^s_{p;\lambda,\mu} f(z) \neq 0$, for all $z \in \dot{\mathrm{U}} = \mathrm{U} \setminus \{0\}$.

2. If $f \in S^s_{p;\lambda,\mu}(\eta;\varphi)$, then $f \in S^{s+1}_{p;\lambda,\mu}(\eta;\varphi)$, whenever $\mathrm{I}^{s+1}_{p;\lambda,\mu} f(z) \neq 0$, for all $z \in \dot{\mathrm{U}}$.

Proof. Let $f \in S^s_{p;\lambda,\mu+1}(\eta;\varphi)$, and set

$$q(z) = \frac{1}{p-\eta}\left(\frac{z(\mathrm{I}^s_{p;\lambda,\mu} f(z))'}{\mathrm{I}^s_{p;\lambda,\mu} f(z)} - \eta\right). \tag{2.1}$$

From the assumption, the function q is analytic in U, with $q(0) = 1$. According to (2.1) and using the relation (1.5), we obtain

$$(p+\mu)\frac{\mathrm{I}^s_{p;\lambda,\mu+1} f(z)}{\mathrm{I}^s_{p;\lambda,\mu} f(z)} = (p-\eta)q(z) + \eta + \mu. \tag{2.2}$$

Taking the logarithmic differential on both sides of (2.2), and multiplying then by z, we have

$$q(z) + \frac{zq'(z)}{(p-\eta)q(z)+\eta+\mu} = \frac{1}{p-\eta}\left(\frac{z(\mathrm{I}^s_{p;\lambda,\mu+1} f(z))'}{\mathrm{I}^s_{p;\lambda,\mu+1} f(z)} - \eta\right) \prec \varphi(z). \tag{2.3}$$

Since $\varphi \in \mathcal{M}$, then $\operatorname{Re}\varphi(z) > 0$ for all $z \in \mathrm{U}$, and from the assumptions $\mu > -p$ and $\mu + \eta \geq 0$ we get

$$\operatorname{Re}\left((p-\eta)\varphi(z) + \eta + \mu\right) > 0,\ z \in \mathrm{U}. \tag{2.4}$$

Now, by applying Lemma 2.1 for the subordination (2.3), it follows that $q(z) \prec \varphi(z)$, i.e. $f \in S^s_{p;\lambda,\mu}(\eta;\varphi)$.

To prove the second part, let $f \in S^s_{p;\lambda,\mu}(\eta;\varphi)$ and put

$$h(z) = \frac{1}{p-\eta}\left(\frac{z(\mathrm{I}^{s+1}_{p;\lambda,\mu+1} f(z))'}{\mathrm{I}^{s+1}_{p;\lambda,\mu} f(z)} - \eta\right).$$

From the assumption, we have that h is analytic in U, and $h(0) = 1$. Then, by using similar arguments to those detailed above, together with the relation (1.4), it follows that

$$h(z) + \frac{zh'(z)}{(p-\eta)h(z)+\eta+\mu} = \frac{1}{p-\eta}\left(\frac{z(\mathrm{I}^s_{p;\lambda,\mu} f(z))'}{\mathrm{I}^s_{p;\lambda,\mu} f(z)} - \eta\right) \prec \varphi(z). \tag{2.5}$$

Like in the first part of the proof, the inequality (2.4) holds, and then by applying Lemma 2.1 for the subordination (2.5), it follows that $h(z) \prec \varphi(z)$, i.e. $f \in S^{s+1}_{p;\lambda,\mu}(\eta;\varphi)$.

Theorem 2.2. *Let $s \in \mathbb{R}$, $\lambda \geq 0$, $\mu > -p$, $0 \leq \eta < p$, and $\mu + \eta \geq 0$.*

1. If $f \in K^s_{p;\lambda,\mu+1}(\eta;\varphi)$, then $f \in K^s_{p;\lambda,\mu}(\eta;\varphi)$, whenever $\mathrm{I}^s_{p;\lambda,\mu}\left(\frac{zf'(z)}{p}\right) \neq 0$, for all $z \in \dot{\mathrm{U}}$.

2. If $f \in K^s_{p;\lambda,\mu}(\eta;\varphi)$, then $f \in K^{s+1}_{p;\lambda,\mu}(\eta;\varphi)$, whenever $\mathrm{I}^{s+1}_{p;\lambda,\mu}\left(\frac{zf'(z)}{p}\right) \neq 0$, for all $z \in \dot{\mathrm{U}}$.

Proof. If $f \in K^s_{p;\lambda,\mu+1}(\eta;\varphi)$, by definition we have $\mathrm{I}^s_{p;\lambda,\mu+1} f \in K_p(\eta;\varphi)$. According to (1.7) and (1.6), this last relation is equivalent to

$$\mathrm{I}^s_{p;\lambda,\mu+1}\left(\frac{zf'(z)}{p}\right) = \frac{z}{p}\left(\mathrm{I}^s_{p;\lambda,\mu+1} f(z)\right)' \in S^*_p(\eta;\varphi),$$

i.e. $\frac{zf'(z)}{p} \in S^s_{p;\lambda,\mu+1}(\eta;\varphi)$. By using the first part of Theorem 2.1 together with (1.6), it follows that $\frac{zf'(z)}{p} \in S^s_{p;\lambda,\mu}(\eta;\varphi)$, or

$$\frac{z}{p}\left(\mathrm{I}^s_{p;\lambda,\mu} f(z)\right)' = \mathrm{I}^s_{p;\lambda,\mu}\left(\frac{zf'(z)}{p}\right) \in S^*_p(\eta;\varphi).$$

Using (1.7), this is equivalent to $\mathrm{I}^s_{p;\lambda,\mu} f \in K_p(\eta;\varphi)$, i.e. $f \in K^s_{p;\lambda,\mu}(\eta;\varphi)$.

For the second part of the theorem, let $f \in K^s_{p;\lambda,\mu}(\eta;\varphi)$. Form (1.7), that means $\frac{zf'(z)}{p} \in S^s_{p;\lambda,\mu}(\eta;\varphi)$, and by using the second part of Theorem 2.1 together with (1.6), it follows that $\frac{zf'(z)}{p} \in S^{s+1}_{p;\lambda,\mu}(\eta;\varphi)$, or

$$\mathrm{I}^{s+1}_{p;\lambda,\mu}\left(\frac{zf'(z)}{p}\right) = \frac{z}{p}\left(\mathrm{I}^{s+1}_{p;\lambda,\mu} f(z)\right)' \in S^*_p(\eta;\varphi).$$

From (1.7), this is equivalent to $\mathrm{I}^{s+1}_{p;\lambda,\mu} f \in K_p(\eta;\varphi)$, i.e. $f \in K^{s+1}_{p;\lambda,\mu}(\eta;\varphi)$.

Taking

$$\varphi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha} \quad (-1 \leq B < A \leq 1,\ 0 < \alpha \leq 1)$$

in Theorem 2.1 and Theorem 2.2, we have the following special cases:

Corollary 2.1. *Let $s \in \mathbb{R}$, $\lambda \geq 0$, $\mu > -p$, $0 \leq \eta < p$, and $\mu + \eta \geq 0$. Suppose that $-1 \leq B < A \leq 1$, and $0 < \alpha \leq 1$.*

1. If $f \in S^s_{p;\lambda,\mu+1}(\eta; A, B; \alpha)$, then $f \in S^s_{p;\lambda,\mu}(\eta; A, B; \alpha)$, whenever $\mathrm{I}^s_{p;\lambda,\mu} f(z) \neq 0$, for all $z \in \dot{\mathrm{U}}$.

2. If $f \in S^s_{p;\lambda,\mu}(\eta; A, B; \alpha)$, then $f \in S^{s+1}_{p;\lambda,\mu}(\eta; A, B; \alpha)$, whenever $\mathrm{I}^{s+1}_{p;\lambda,\mu} f(z) \neq 0$, for all $z \in \dot{\mathrm{U}}$.

3. If $f \in K^s_{p;\lambda,\mu+1}(\eta; A, B; \alpha))$, then $f \in K^s_{p;\lambda,\mu}(\eta; A, B; \alpha))$, whenever $\mathrm{I}^s_{p;\lambda,\mu}\left(\dfrac{zf'(z)}{p}\right) \neq 0$, for all $z \in \dot{\mathrm{U}}$.

4. If $f \in K^s_{p;\lambda,\mu}(\eta; A, B; \alpha))$, then $f \in K^{s+1}_{p;\lambda,\mu}(\eta; A, B; \alpha))$, whenever $\mathrm{I}^{s+1}_{p;\lambda,\mu}\left(\dfrac{zf'(z)}{p}\right) \neq 0$, for all $z \in \dot{\mathrm{U}}$.

Theorem 2.3. *Let $s \in \mathbb{R}$, $\lambda \geq 0$, $\mu > -p$, $0 \leq \eta < p$, and $\mu + \eta \geq 0$.*

1. If $f \in C^s_{p;\lambda,\mu+1}(\eta, \delta; \varphi, \psi)$ related to $g \in S^s_{p;\lambda,\mu+1}(\eta; \varphi)$, then $f \in C^s_{p;\lambda,\mu}(\eta, \delta; \varphi, \psi)$ related to $g \in S^s_{p;\lambda,\mu}(\eta; \varphi)$, whenever $\mathrm{I}^s_{p;\lambda,\mu} g(z) \neq 0$, for all $z \in \dot{\mathrm{U}}$.

2. If $f \in C^s_{p;\lambda,\mu}(\eta, \delta; \varphi, \psi)$ related to $g \in S^s_{p;\lambda,\mu}(\eta; \varphi)$, then $f \in S^{s+1}_{p;\lambda,\mu}(\eta, \delta; \varphi, \psi)$ related to $g \in S^{s+1}_{p;\lambda,\mu}(\eta; \varphi)$, whenever $\mathrm{I}^{s+1}_{p;\lambda,\mu} g(z) \neq 0$, for all $z \in \dot{\mathrm{U}}$.

Proof. If $f \in C^s_{p;\lambda,\mu+1}(\eta, \delta; \varphi, \psi)$ related to $g \in S^s_{p;\lambda,\mu+1}(\eta; \varphi)$, according to the definition of these classes, we have

$$\frac{1}{p-\delta}\left(\frac{z\left(\mathrm{I}^s_{p;\lambda,\mu+1} f(z)\right)'}{\mathrm{I}^s_{p;\lambda,\mu+1} g(z)} - \delta\right) \prec \psi(z). \tag{2.6}$$

Now, if we let

$$q(z) = \frac{1}{p-\delta}\left(\frac{z(\mathrm{I}^s_{p;\lambda,\mu} f(z))'}{\mathrm{I}^s_{p;\lambda,\mu} g(z)} - \delta\right),$$

then q is analytic in U, with $q(0) = 1$. Using (1.5), we obtain

$$[(p-\delta)q(z) + \delta]\, \mathrm{I}^s_{p;\lambda,\mu} g(z) + \mu\, \mathrm{I}^s_{p;\lambda,\mu} f(z) = (p+\mu)\, \mathrm{I}^s_{p;\lambda,\mu+1} f(z). \tag{2.7}$$

Differentiating (2.7) and multiplying by z, we have

$$\begin{aligned}&(p+\mu)z\left(\mathrm{I}^s_{p;\lambda,\mu+1} f(z)\right)' = \mu z\left(\mathrm{I}^s_{p;\lambda,\mu} f(z)\right)' + \\ &(p-\delta)zq'(z)\, \mathrm{I}^s_{p;\lambda,\mu} g(z) + [(p-\delta)q(z) + \delta]\, z\left(\mathrm{I}^s_{p;\lambda,\mu} g(z)\right)'.\end{aligned} \tag{2.8}$$

Since $g \in S^s_{p;\lambda,\mu+1}(\eta;\varphi)$, by the first part of Theorem 2.1 we have $g \in S^s_{p;\lambda,\mu}(\eta;\varphi)$. Letting

$$Q(z) = \frac{1}{p-\eta}\left(\frac{z\left(\mathrm{I}^s_{p;\lambda,\mu}\, g(z)\right)'}{\mathrm{I}^s_{p;\lambda,\mu}\, g(z)} - \eta\right),$$

then $Q(z) \prec \varphi(z)$, and using (1.5) once again, we have

$$(\mu+p)\frac{\mathrm{I}^s_{p;\lambda,\mu+1}\, g(z)}{\mathrm{I}^s_{p;\lambda,\mu}\, g(z)} = (p-\eta)Q(z) + \eta + \mu. \tag{2.9}$$

From (2.8) and (2.9), we obtain

$$\frac{1}{p-\delta}\left(\frac{z(\mathrm{I}^s_{p;\lambda,\mu+1}\, f(z))'}{\mathrm{I}^s_{p;\lambda,\mu+1}\, g(z)} - \delta\right) = q(z) + \frac{zq'(z)}{(p-\eta)Q(z)+\mu+\eta},$$

and combining with (2.6) we deduce that

$$q(z) + w(z)zq'(z) \prec \psi(z), \text{ where } w(z) = \frac{1}{(p-\eta)Q(z)+\mu+\eta}. \tag{2.10}$$

Since $p > \eta$, $\mu+\eta \geq 0$ and $Q(z) \prec \varphi(z) \in \mathcal{M}$, then $\operatorname{Re} w(z) > 0$ for all $z \in \mathrm{U}$. According to Lemma 2.2, the subordination (2.10) yields that $q(z) \prec \psi(z)$, where $g \in S^s_{p;\lambda,\mu}(\eta;\varphi)$, i.e. $f \in C^s_{p;\lambda,\mu}(\eta,\delta;\varphi,\psi)$.

Since for the second part of the theorem we used similar arguments to those detailed above together with the identity (1.4), we will omit this proof.

3. The Subclasses Images by the Integral Operator $F_{p;c}$

Let consider the integral operator $F_{p;c} : A(p) \to A(p)$, defined by

$$F_{p;c}(f)(z) = \frac{p+c}{z^c}\int_0^z t^{c-1} f(t)\,\mathrm{d}\,t, \quad (c > -p). \tag{3.1}$$

In this section we will prove that this operator preserves the classes $S^s_{p;\lambda,\mu}(\eta;\varphi)$, $K^s_{p;\lambda,\mu}(\eta;\varphi)$ and $C^s_{p;\lambda,\mu}(\eta,\delta;\varphi,\psi)$.

Theorem 3.1. *Let $s \in \mathbb{R}$, $\lambda \geq 0$, $c > -p$, and $c + \eta \geq 0$. If $f \in S^s_{p;\lambda,\mu}(\eta;\varphi)$, then $F_{p;c}(f) \in S^s_{p;\lambda,\mu}(\eta;\varphi)$, whenever $\mathrm{I}^s_{p;\lambda,\mu} F_{p;c}(f)(z) \neq 0$ for all $z \in \dot{\mathrm{U}}$.*

Proof. If we let $f \in S^s_{p;\lambda,\mu}(\eta;\varphi)$, and

$$q(z) = \frac{1}{p-\eta}\left(\frac{z(\mathrm{I}^s_{p;\lambda,\mu} F_{p;c}(f)(z))'}{\mathrm{I}^s_{p;\lambda,\mu} F_{p;c}(f)(z)} - \eta\right), \tag{3.2}$$

then q is analytic in U, with $q(0) = 1$.

From (3.1), according to (1.6), we have

$$z\left(\mathrm{I}^s_{p;\lambda,\mu} F_{p;c}(f)(z)\right)' = (c+p)\,\mathrm{I}^s_{p;\lambda,\mu} f(z) - c\,\mathrm{I}^s_{p,\lambda,\mu} F_{p;c}(f)(z), \tag{3.3}$$

and then, by using (3.2) and (3.3), we obtain

$$(c+p)\frac{\mathrm{I}^s_{p;\lambda,\mu} f(z)}{\mathrm{I}^s_{p;\lambda,\mu} F_{p;c}(f)(z)} = (p-\eta)q(z) + c + \eta. \tag{3.4}$$

Now, taking the logarithmic differentiation on both sides of (3.4) and multiplying by z, we have

$$q(z) + \frac{zq'(z)}{(p-\eta)q(z) + c + \eta} = \frac{1}{p-\eta}\left(\frac{z(\mathrm{I}^s_{p;\lambda,\mu} f(z))'}{\mathrm{I}^s_{p;\lambda,\mu} f(z)} - \eta\right) \prec \varphi(z). \tag{3.5}$$

Since $\varphi \in \mathcal{M}$, then $\operatorname{Re}\varphi(z) > 0$ for all $z \in \mathrm{U}$, and from the assumptions $\mu > -p$ and $c + \eta \geq 0$ we get

$$\operatorname{Re}\left((p-\eta)\varphi(z) + c + \eta\right) > 0,\ z \in \mathrm{U}.$$

Hence, by virtue of Lemma 2.1, the subordination (3.5) implies that $q(z) \prec \varphi(z)$, i.e. $F_{p;c}(f) \in S^s_{p;\lambda,\mu}(\eta;\varphi)$.

Next we derive an inclusion property involving the images of the subclasses $K^s_{p;\lambda,\mu}(\eta;\varphi)$ via the operator $F_{p;c}$, which is given by the following result:

Theorem 3.2. *Let $s \in \mathbb{R}$, $\lambda \geq 0$, $c > -p$, and $c + \eta \geq 0$. If $f \in K^s_{p;\lambda,\mu}(\eta;\varphi)$, then $F_{p;c}(f) \in K^s_{p;\lambda,\mu}(\eta;\varphi)$, whenever $\mathrm{I}^s_{p;\lambda,\mu} F_{p;c}\left(\dfrac{zf'(z)}{p}\right) \neq 0$ for all $z \in \dot{\mathrm{U}}$.*

Proof. If $f \in K^s_{p;\lambda,\mu}(\eta;\varphi)$, by definition we have $\mathrm{I}^s_{p;\lambda,\mu} f \in K_p(\eta;\varphi)$, and from (1.7) and (1.6), this is equivalent to

$$\mathrm{I}^s_{p;\lambda,\mu}\left(\frac{zf'(z)}{p}\right) = \frac{z}{p}\left(\mathrm{I}^s_{p;\lambda,\mu} f(z)\right)' \in S^*_p(\eta;\varphi),$$

i.e. $\dfrac{zf'(z)}{p} \in S^s_{p;\lambda,\mu}(\eta;\varphi)$. By using Theorem 3.1 together with (1.6), it follows that $F_{p;c}\left(\dfrac{zf'(z)}{p}\right) \in S^s_{p;\lambda,\mu}(\eta;\varphi)$, or

$$F_{p;c}\left(\frac{zf'(z)}{p}\right) = \frac{z}{p}\left(F_{p;c}f(z)\right)' \in S^s_p(\eta;\varphi),$$

that is

$$\mathrm{I}^s_{p;\lambda,\mu}\left(\frac{z\left(F_{p;c}f(z)\right)'}{p}\right) = \frac{z}{p}\left(\mathrm{I}^s_{p;\lambda,\mu} F_{p;c}f(z)\right)' \in S^*_p(\eta;\varphi).$$

Using (1.1), this is equivalent to $\mathrm{I}^s_{p;\lambda,\mu} F_{p;c}(f) \in K_p(\eta;\varphi)$, i.e. $F_{p;c}(f) \in K^s_{p;\lambda,\mu}(\eta;\varphi)$, which proves the theorem.

From Theorem 3.1 and Theorem 3.2, we have the following:

Corollary 3.1. *Let $s \in \mathbb{R}$, $\lambda \geq 0$, $c > -p$, and $c + \eta \geq 0$. Suppose that $-1 \leq B < A \leq 1$, and $0 < \alpha \leq 1$.*

1. If $f \in S^s_{p;\lambda,\mu}(\eta;A,B;\alpha)$, then $F_{p;c}(f) \in S^s_{p;\lambda,\mu}(\eta;A,B;\alpha)$, whenever $\mathrm{I}^s_{p;\lambda,\mu} F_{p;c}(f)(z) \neq 0$ for all $z \in \dot{\mathrm{U}}$.

2. If $f \in K^s_{p;\lambda,\mu}(\eta;A,B;\alpha)$, then $F_{p;c}(f) \in K^s_{p;\lambda,\mu}(\eta;A,B;\alpha)$, whenever $\mathrm{I}^s_{p;\lambda,\mu} F_{p;c}\left(\dfrac{zf'(z)}{p}\right) \neq 0$, for all $z \in \dot{\mathrm{U}}$.

Finally, we will prove that the operator $F_{p;c}$ preserves the classes $C^s_{p;\lambda,\mu}(\eta,\delta;\varphi,\psi)$ of p–valent generalized φ–starlike of order η.

Theorem 3.3. *Let $s \in \mathbb{R}$, $\lambda \geq 0$, $c > -p$, and $c + \eta \geq 0$. If $f \in C^s_{p;\lambda,\mu}(\eta,\delta;\varphi,\psi)$ related to $g \in S^s_{p;\lambda,\mu}(\eta;\varphi)$, then $F_{p;c}(f) \in C^s_{p;\lambda,\mu}(\eta,\delta;\varphi,\psi)$ related to $F_{p;c}(g) \in S^s_{p;\lambda,\mu}(\eta;\varphi)$, whenever $\mathrm{I}^s_{p;\lambda,\mu} F_{p;c}(g)(z) \neq 0$, for all $z \in \dot{\mathrm{U}}$.*

Proof. If $f \in C^s_{p;\lambda,\mu}(\eta,\delta;\varphi,\psi)$, then in view of the Definition 1.1, there exists a function $g \in S^s_{p;\lambda,\mu}(\eta;\varphi)$ such that

$$\frac{1}{p-\delta}\left(\frac{z(\mathrm{I}^s_{p;\lambda,\mu} F_{p;c}f(z))'}{\mathrm{I}^s_{p;\lambda,\mu} g(z)} - \delta\right) \prec \psi(z). \tag{3.6}$$

Thus, if we set

$$q(z)=\frac{1}{p-\delta}\left(\frac{z(\mathrm{I}^s_{p;\lambda,\mu}F_{p;c}(f)(z))'}{\mathrm{I}^s_{p;\lambda,\mu}F_{p;c}(g)(z)}-\delta\right),$$

then q is analytic in U, with $q(0)=1$. Using (3.3), we have

$$(c+p)\,\mathrm{I}^s_{p;\lambda,\mu}f(z)=[(p-\delta)q(z)+\delta]\,\mathrm{I}^s_{p;\lambda,\mu}F_{p;c}(g)(z)+c\,\mathrm{I}^s_{p,\lambda,\mu}F_{p;c}(f)(z),$$

and taking the logarithmic derivative of this identity and multiplying by z, we have

$$(c+p)z\left(\mathrm{I}^s_{p;\lambda,\mu}f(z)\right)'=(p-\delta)zq'(z)\,\mathrm{I}^s_{p;\lambda,\mu}F_{p;c}(g)(z)+$$
$$[(p-\delta)q(z)+\delta]\,z\left(\mathrm{I}^s_{p;\lambda,\mu}F_{p;c}(g)(z)\right)'+cz\left(\mathrm{I}^s_{p;\lambda,\mu}F_{p;c}(f)(z)\right)'. \quad (3.7)$$

Letting

$$Q(z)=\frac{1}{p-\eta}\left(\frac{z(\mathrm{I}^s_{p;\lambda,\mu}F_{p;c}(g)(z))'}{\mathrm{I}^s_{p;\lambda,\mu}F_{p;c}(g)(z)}-\eta\right),$$

since $g\in S^s_{p;\lambda,\mu}(\eta;\varphi)$, from Theorem 3.1 we have that $F_{p;c}(g)\in S^s_{p;\lambda,\mu}(\eta;\varphi)$, hence $Q(z)\prec\varphi(z)$.

Using again (3.3), we obtain

$$(c+p)\,\mathrm{I}^s_{p;\lambda,\mu}g(z)=[(p-\eta)Q(z)+c+\eta]\,\mathrm{I}^s_{p;\lambda,\mu}F_{p;c}(g)(z), \quad (3.8)$$

and then, from (3.7) and (3.8), we deduce that

$$q(z)+\frac{zq'(z)}{(p-\eta)Q(z)+c+\eta}=\frac{1}{p-\delta}\left(\frac{z(\mathrm{I}^s_{p;\lambda,\mu}F_{p;c}f(z))'}{\mathrm{I}^s_{p;\lambda,\mu}g(z)}-\delta\right).$$

Combining this last identity together with the subordination (3.6), we deduce that

$$q(z)+w(z)zq'(z)\prec\psi(z),\ \text{where}\ w(z)=\frac{1}{(p-\eta)Q(z)+c+\eta}. \quad (3.9)$$

Since $p>\eta$, $c+\eta\geq 0$ and $Q(z)\prec\varphi(z)\in\mathcal{M}$, then $\operatorname{Re}w(z)>0$ for all $z\in$ U. Using Lemma 2.2, the subordination (3.9) implies that $q(z)\prec\psi(z)$, where $F_{p;c}(g)\in S^s_{p;\lambda,\mu}(\eta;\varphi)$, i.e. $F_{p;c}(f)\in C^s_{p;\lambda,\mu}(\eta,\delta;\varphi,\psi)$.

Remark 3.1. Putting $p=1$ in the above results we obtain the results of [5].

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