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On the Classes of Analytic Functions Defined by Using Al - Oboudi Operator *

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Abstract

In this note, we define the new subclasses $\mathcal{N}_{m,n}(\alpha,\beta,\lambda)$ and $\tilde{\mathcal{N}}_{m,n}(\alpha,\beta,\lambda)$ of analytic functions using the Al - Oboudi operator. For functions belonging to these classes we determine coefficient inequalities, extreme points and integral means inequalities.

Keywords and Phrases: *Analytic functions, Al - Oboudi operator, Coefficient inequalities, Extreme points and Integral means.*

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1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Definition 1.1. *[1] Let $n \in \mathbb{N}$ and $\lambda \geq 0$, the Al - Oboudi operator $D_\lambda^n : \mathcal{A} \to \mathcal{A}$, is defined as $D_\lambda^0 f(z) = f(z)$, $D_\lambda^1 f(z) = (1-\lambda)f(z) + zf'(z) = D_\lambda f(z)$ and $D_\lambda^n f(z) = D_\lambda\left(D_\lambda^{n-1} f(z)\right)$.*

Further, if $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, then we have,

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\lambda]^n a_k z^k, \quad (n \in \mathbb{N}_0). \tag{1.2}$$

Remark 1.2. *It is easy to observe that for $\lambda = 1$, we get the Sălăgean operator [8].*

Definition 1.3. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{N}_{m,n}(\alpha, \beta, \lambda)$ if

$$\Re\left\{\frac{D_\lambda^m f(z)}{D_\lambda^n f(z)}\right\} > \beta\left|\frac{D_\lambda^m f(z)}{D_\lambda^n f(z)} - 1\right| + \alpha, \tag{1.3}$$

for some $0 \leq \alpha < 1$, $\beta \geq 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\lambda \geq 0$ and $z \in \mathcal{U}$.

The following are the special cases of the class $\mathcal{N}_{m,n}(\alpha, \beta, \lambda)$:

i. $\mathcal{N}_{m,n}(\alpha, \beta, 1) \equiv \mathcal{N}_{m,n}(\alpha, \beta)$, the class introduced by Eker and Owa [3].

ii. $\mathcal{N}_{1,0}(\alpha, \beta, 1) \equiv \mathcal{SD}(\alpha, \beta)$ and $\mathcal{N}_{2,1}(\alpha, \beta, 1) \equiv \mathcal{KD}(\alpha, \beta)$, the classes studied by Shams, Kulkarni and Jahangiri [9].

iii. $\mathcal{N}_{m,n}(\alpha, 0, 1) \equiv \mathcal{K}_{m,n}(\alpha)$, the class studied by Eker and Owa [4].

iv. $\mathcal{N}_{1,0}(\alpha, 0, 1) \equiv \mathcal{S}^\star(\alpha)$ and $\mathcal{N}_{2,1}(\alpha, 0, 1) \equiv \mathcal{K}(\alpha)$, the classes introduced by Robertson [7].

2. Coefficient Inequalities for the Class $\mathcal{N}_{m,n}(\alpha,\beta,\lambda)$

Theorem 2.1. *If $f \in \mathcal{A}$ satisfies,*

$$\sum_{k=2}^{\infty} \psi(\lambda,m,n,k,\alpha,\beta)\,|a_k| \leq 2(1-\alpha) \tag{2.1}$$

where,

$$\begin{aligned}\psi(\lambda,m,n,k,\alpha,\beta) \;&= |(1+\alpha)\,[1+(k-1)\lambda]^n - [1+(k-1)\lambda]^m| \\ &\quad + ((1-\alpha)\,[1+(k-1)\lambda]^n + [1+(k-1)\lambda]^m) \\ &\quad +2\beta\,|[1+(k-1)\lambda]^m - [1+(k-1)\lambda]^n|\end{aligned}$$

for some $\alpha\,(0 \leq \alpha < 1)$, $\beta \geq 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $\lambda \geq 0$, then $f \in \mathcal{N}_{m,n}(\alpha,\beta,\lambda)$.

Proof. Let, the expression (2.1) be true for $0 \leq \alpha < 1$, $\beta \geq 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $\lambda \geq 0$. Hence it suffices to show that,

$$\left|(1-\alpha)D_\lambda^n f(z) + D_\lambda^m f(z) - \beta e^{i\theta}\,|D_\lambda^m f(z) - D_\lambda^n f(z)|\right|$$

$$-\left|(1+\alpha)D_\lambda^n f(z) - D_\lambda^m f(z) + \beta e^{i\theta}\,|D_\lambda^m f(z) - D_\lambda^n f(z)|\right| > 0.$$

So, we have

$$\begin{aligned}
&\left|(1-\alpha)D_\lambda^n f(z)+D_\lambda^m f(z)-\beta e^{i\theta}\left|D_\lambda^m f(z)-D_\lambda^n f(z)\right|\right|\\
&\qquad-\left|(1+\alpha)D_\lambda^n f(z)-D_\lambda^m f(z)+\beta e^{i\theta}\left|D_\lambda^m f(z)-D_\lambda^n f(z)\right|\right|\\
&=\left|(2-\alpha)z+\sum_{k=2}^{\infty}\left\{(1-\alpha)\left[1+(k-1)\lambda\right]^n+\left[1+(k-1)\lambda\right]^m\right\}a_k z^k\right.\\
&\qquad\left.-\beta e^{i\theta}\left|\sum_{k=2}^{\infty}\left\{\left[1+(k-1)\lambda\right]^m-\left[1+(k-1)\lambda\right]^n\right\}a_k z^k\right|\right|\\
&\qquad-\left|\alpha z+\sum_{k=2}^{\infty}\left\{(1+\alpha)\left[1+(k-1)\lambda\right]^n-\left[1+(k-1)\lambda\right]^m\right\}a_k z^k\right.\\
&\qquad\left.+\beta e^{i\theta}\left|\sum_{k=2}^{\infty}\left\{\left[1+(k-1)\lambda\right]^m-\left[1+(k-1)\lambda\right]^n\right\}a_k z^k\right|\right|\\
&\geq(2-\alpha)|z|-\sum_{k=2}^{\infty}\left|(1-\alpha)\left[1+(k-1)\lambda\right]^n+\left[1+(k-1)\lambda\right]^m\right||a_k||z|^k\\
&\qquad-\beta\left|e^{i\theta}\right|\sum_{k=2}^{\infty}\left|\left[1+(k-1)\lambda\right]^m-\left[1+(k-1)\lambda\right]^n\right||a_k||z|^k\\
&\qquad-\alpha||z|-\sum_{k=2}^{\infty}\left|(1+\alpha)\left[1+(k-1)\lambda\right]^n-\left[1+(k-1)\lambda\right]^m\right||a_k||z|^k\\
&\qquad-\beta\left|e^{i\theta}\right|\sum_{k=2}^{\infty}\left|\left[1+(k-1)\lambda\right]^m-\left[1+(k-1)\lambda\right]^n\right||a_k||z|^k\\
&\geq 2(1-\alpha)-\sum_{k=2}^{\infty}\left\{\left|(1+\alpha)\left[1+(k-1)\lambda\right]^n-\left[1+(k-1)\lambda\right]^m\right|\right.\\
&\qquad+\left((1-\alpha)\left[1+(k-1)\lambda\right]^n+\left[1+(k-1)\lambda\right]^m\right)\\
&\qquad\left.+2\beta\left|\left[1+(k-1)\lambda\right]^m-\left[1+(k-1)\lambda\right]^n\right|\right\}|a_k|\geq 0.
\end{aligned}$$

3. Relation for $\tilde{\mathcal{N}}_{m,n}(\alpha, \beta, \lambda)$

By Theorem 2.1, we introduce the class $\tilde{\mathcal{N}}_{m,n}(\alpha, \beta, \lambda)$ as the subclass of $\mathcal{N}_{m,n}(\alpha, \beta, \lambda)$ consisting of f satisfying

$$\sum_{k=2}^{\infty} \psi(\lambda, m, n, k, \alpha, \beta) \, |a_k| \le 2(1-\alpha) \tag{3.1}$$

where,

$$\begin{aligned} \psi(\lambda, m, n, k, \alpha, \beta) &= |(1+\alpha)\,[1+(k-1)\lambda]^n - [1+(k-1)\lambda]^m| \\ &+ ((1-\alpha)\,[1+(k-1)\lambda]^n + [1+(k-1)\lambda]^m) \\ &+ 2\beta\,|[1+(k-1)\lambda]^m - [1+(k-1)\lambda]^n| \end{aligned}$$

for some $\alpha\,(0 \le \alpha < 1)$, $\beta \ge 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $\lambda \ge 0$.

Theorem 3.1. *If $f \in \mathcal{A}$, then $\tilde{\mathcal{N}}_{m,n}(\alpha, \beta_2, \lambda) \subset \tilde{\mathcal{N}}_{m,n}(\alpha, \beta_1, \lambda)$ for some β_1 and β_2, such that*
$0 \le \beta_1 \le \beta_2$.

Proof. For $0 \le \beta_1 \le \beta_2$, we have

$$\sum_{k=2}^{\infty} \psi(\lambda, m, n, k, \alpha, \beta_1) \, |a_k| \le \sum_{k=2}^{\infty} \psi(\lambda, m, n, k, \alpha, \beta_2) \, |a_k| \, .$$

Therefore, if $f \in \tilde{\mathcal{N}}_{m,n}(\alpha, \beta_2, \lambda)$, then $f \in \tilde{\mathcal{N}}_{m,n}(\alpha, \beta_1, \lambda)$.

4. Extreme Points

The determination of the extreme points of a family $\mathcal{F}$ of univalent functions enables us to solve many external problems for $\mathcal{F}$.

Theorem 4.1. *Let $f_1(z) = z$ and*

$$f_k(z) = z + \frac{2(1-\alpha)\varepsilon_k}{\psi(\lambda, m, n, k, \alpha, \beta)} \, z^k, \quad (k = 1, 2, ...; \, |\varepsilon_k| = 1).$$

Then, $f \in \tilde{\mathcal{N}}_{m,n}(\alpha,\beta,\lambda)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z),$$

where, $\lambda_k \geq 0$ *and* $\sum_{k=1}^{\infty} \lambda_k = 1.$

Proof. Let $f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$, $\lambda_k \geq 0$, $k = 1, 2, ...,$ with $\sum_{k=1}^{\infty} \lambda_k = 1$. Then, we have

$$\begin{aligned} f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z) &= \lambda_1 z + \sum_{k=2}^{\infty} \lambda_k \left(z + \frac{2(1-\alpha)\varepsilon_k}{\psi(\lambda,m,n,k,\alpha,\beta)}\, z^k \right) \\ &= z + \sum_{k=2}^{\infty} \lambda_k \frac{2(1-\alpha)\varepsilon_k}{\psi(\lambda,m,n,k,\alpha,\beta)}\, z^k. \end{aligned}$$

That is,

$$\begin{aligned} \sum_{k=2}^{\infty} \psi(\lambda,m,n,k,\alpha,\beta) \left| \frac{2(1-\alpha)\varepsilon_k}{\psi(\lambda,m,n,k,\alpha,\beta)}\, \lambda_k \right| &= \sum_{k=2}^{\infty} 2(1-\alpha)\lambda_k \\ &= 2(1-\alpha)(1-\lambda_1) \leq 2(1-\alpha), \end{aligned}$$

which is the condition (3.1) for $f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$. Thus, $f \in \tilde{\mathcal{N}}_{m,n}(\alpha,\beta,\lambda)$. Conversely, let $f \in \tilde{\mathcal{N}}_{m,n}(\alpha,\beta,\lambda)$. Since

$$|a_k| \leq \frac{2(1-\alpha)}{\psi(\lambda,m,n,k,\alpha,\beta)}, \quad (k = 2, 3, ...)$$

we put

$$\lambda_k = \frac{\psi(\lambda,m,n,k,\alpha,\beta)}{2(1-\alpha)\varepsilon_k} a_k, \quad (|\varepsilon_k| = 1)$$

and

$$\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k.$$

Then,

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z).$$

Corollary 4.2. *The extreme points of* $\tilde{\mathcal{N}}_{m,n}(\alpha, \beta, \lambda)$ *are the functions* $f_1(z) = z$ *and*

$$f_k(z) = z + \frac{2(1-\alpha)\varepsilon_k}{\psi(\lambda, m, n, k, \alpha, \beta)}\, z^k, \quad (k = 2, 3, ...;\ |\varepsilon_k| = 1).$$

5. Integral Means Inequalities

For any two functions f and g analytic in $\mathcal{U}$, f is said to be subordinate to g in $\mathcal{U}$, denoted by $f \prec g$ if there exists an analytic function ω defined $\mathcal{U}$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z)),\ z \in \mathcal{U}$.
In particular, if the function g is univalent in $\mathcal{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. In 1925, Littlewood [6] proved the following Subordination Theorem.

Theorem 5.1. *[6] : If f and g are any two functions, analytic in $\mathcal{U}$, with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}$, $(0 < r < 1)$,*

$$\int_0^{2\pi} |f(z)|^{\mu}\, d\theta \le \int_0^{2\pi} |g(z)|^{\mu}\, d\theta.$$

Theorem 5.2. *Let* $f \in \tilde{\mathcal{N}}_{m,n}(\alpha, \beta, \lambda)$ *and* f_k *be defined by*

$$f_k(z) = z + \frac{2(1-\alpha)\varepsilon_k}{\psi(\lambda, m, n, k, \alpha, \beta)}\, z^k, \quad (k = 2, 3, ...;\ |\varepsilon_k| = 1).$$

If there exists an analytic function $\omega(z)$ *given by*

$$[\omega(z)]^{k-1} = \frac{\psi(\lambda, m, n, k, \alpha, \beta)}{2(1-\alpha)\varepsilon_k} \sum_{k=2}^{\infty} a_k z^{k-1},$$

then for $z = re^{i\theta}$ *and* $0 < r < 1$,

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\mu} d\theta \le \int_0^{2\pi} \left|f_k(re^{i\theta})\right|^{\mu} d\theta, \quad (\mu > 0).$$

Proof. We have to prove that

$$\int_0^{2\pi} \left| 1 + \sum_{k=2}^{\infty} a_k z^{k-1} \right|^{\mu} d\theta \leq \int_0^{2\pi} \left| 1 + \frac{2(1-\alpha)\varepsilon_k}{\psi(\lambda, m, n, k, \alpha, \beta)} \, z^{k-1} \right|^{\mu} d\theta.$$

By Theorem 5.1, it suffices to show that

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 + \frac{2|b|(1-\alpha)\varepsilon_k}{\psi(\lambda, m, n, k, \alpha, \beta)} \, z^{k-1}.$$

By taking

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} = 1 + \frac{2(1-\alpha)\varepsilon_k}{\psi(\lambda, m, n, k, \alpha, \beta)} \, [\omega(z)]^{k-1}$$

we get

$$[\omega(z)]^{k-1} = \frac{\psi(\lambda, m, n, k, \alpha, \beta)}{2(1-\alpha)\varepsilon_k} \sum_{k=2}^{\infty} a_k z^{k-1}.$$

Clearly, $\omega(0) = 0$. By (3.1), we have

$$\begin{aligned} |[\omega(z)]|^{k-1} &= \left| \frac{\psi(\lambda, m, n, k, \alpha, \beta)}{2(1-\alpha)\varepsilon_k} \sum_{k=2}^{\infty} a_k z^{k-1} \right| \\ &\leq \frac{\psi(\lambda, m, n, k, \alpha, \beta)}{2(1-\alpha)\, |\varepsilon_k|} \sum_{k=2}^{\infty} |a_k| \, |z|^{k-1} \\ &\leq |z| < 1. \end{aligned}$$

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