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On Implicit Sum-difference Equations of Fredholm Type *

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Abstract

In the present paper we study some fundamental properties of solutions of certain first order implicit sum-difference equations of Fredholm type. A variant of a certain finite difference inequality with explicit estimate is obtained and used to establish the results.

Keywords and Phrases: *Implicit sum-difference equations, Fredholm type, Finite difference inequality, Explicit estimate, Qualitative properties, Continuous dependence.*

1. Introduction

Sum-difference equations occur frequently in numerous settings and forms, both in mathematics itself and its applications. A simple way of solving numerically, various types of differential and integral equations is to write down the equations for a set of equidistant points and to approximate the integral terms by appropriate quadrature formulas. In the present paper we consider

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the initial value problem (IVP, for short) for implicit sum-difference equations of Fredholm type

$$\Delta u(n) = f(n, u(n), \Delta u(n), Hu(n)), \ u(\alpha) = u_0, \tag{1.1}$$

where

$$Hu(n) := \sum_{\tau=\alpha}^{\beta} h(n, \tau, u(\tau), \Delta u(\tau)), \tag{1.2}$$

$\Delta u(n) = u(n+1) - u(n)$; f, h are given functions and u is the unknown function to be found. The origin of equations like (1.1) can be traced back to the study of discrete analogue of the well known Clairaut's differential equation, see [1, p. 117] and [6].

In the general case, solving (1.1) is highly nontrivial problem and handling the study of its qualitative properties need a fresh outlook. The problem of existence of solutions for equations like (1.1) can be dealt with the method employed in [3] (see also [2, 4-6]). In the present work, we focus our attention to study some basic qualitative properties of solutions of IVP (1.1) by using a variant of a certain finite difference inequality with explicit estimate given in [8]. A particular feature of our approach here is that it present conditions under which we can offer simple, unified and concise proofs of some of the important qualitative properties of solutions of IVP (1.1).

2. A Basic Finite Difference Inequality

Let R^m denote the real m-dimensional Euclidean space with appropriate norm $|.|$. Let $R_+ = [0, \infty), N_0 = \{0, 1, 2, ...\}$, $N_{\alpha,\beta} = \{\alpha, \alpha+1, ..., \ \alpha+n = \beta\}$ $(\alpha \in N_0, n \in N)$ be the given subsets of R, the set of real numbers. Let $D(A, B)$ denote the class of discrete functions from the set A to the set B and for $w \in D(N_0, R^m)$, we define the operator Δ by $\Delta w(n) = w(n+1) - w(n)$. Throughout, we assume that $f \in D(N_{\alpha,\beta+1} \times R^m \times R^m \times R^m, R^m)$, $h \in D\left(N^2_{\alpha,\beta+1} \times R^m \times R^m, R^m\right)$ and use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. By a solution of IVP (1.1) we mean a function $u \in D(N_{\alpha,\beta+1}, R^m)$ for which $\Delta u(n)$ exists and satisfies the IVP (1.1). It is easy to observe that the solution $u(n)$ of IVP (1.1) satisfies the following sum-difference equation

$$u(n) = u_0 + \sum_{s=\alpha}^{n-1} f(s, u(s), \Delta u(s), Hu(s)),$$

for $n \in N_{\alpha,\beta+1}$.

We require the following variant of the finite difference inequality established by the present author in [8, Theorem 4.5.1 Part (a_2), p. 224]. We shall state and prove it in the following theorem for completeness.

Theorem 1. *Let $u, a, b, c, d, f, g \in D(N_{\alpha,\beta+1}, R_+)$. Suppose that*

$$u(n) \leq a(n) + b(n) \sum_{s=\alpha}^{n-1} f(s) \left[u(s) + d(s) \sum_{\tau=\alpha}^{\beta} g(\tau) u(\tau) \right]$$
$$+ c(n) \sum_{\tau=\alpha}^{\beta} g(\tau) u(\tau), \tag{2.1}$$

for $n \in N_{\alpha,\beta+1}$. If

$$k = \sum_{\xi=\alpha}^{\beta} g(\xi) K_2(\xi) < 1, \tag{2.2}$$

then

$$u(n) \leq K_1(n) + M K_2(n), \tag{2.3}$$

for $n \in N_{\alpha,\beta+1}$, where

$$K_1(n) = a(n) + b(n) \sum_{s=\alpha}^{n-1} f(s) a(s) \prod_{\sigma=s+1}^{n-1} [1 + f(\sigma) b(\sigma)], \tag{2.4}$$

$$K_2(n) = c(n) + b(n) \sum_{s=\alpha}^{n-1} f(s) [c(s) + d(s)] \prod_{\sigma=s+1}^{n-1} [1 + f(\sigma) b(\sigma)], \tag{2.5}$$

and

$$M = \frac{1}{1-k} \sum_{\xi=\alpha}^{\beta} g(\xi) K_1(\xi). \tag{2.6}$$

Proof. Let

$$\lambda = \sum_{\tau=\alpha}^{\beta} g(\tau)\, u(\tau), \tag{2.7}$$

$$z(n) = \sum_{s=\alpha}^{n-1} f(s) \left[u(s) + d(s) \sum_{\tau=\alpha}^{\beta} g(\tau)\, u(\tau) \right]$$
$$= \sum_{s=\alpha}^{n-1} f(s) \left[u(s) + d(s)\, \lambda \right], \tag{2.8}$$

then $z(\alpha) = 0$ and (2.1) can be restated as

$$u(n) \le a(n) + b(n)\, z(n) + c(n)\, \lambda. \tag{2.9}$$

From (2.8) and (2.9), we have

$$\Delta z(n) = f(n) \left[u(n) + d(n)\, \lambda \right]$$

$$\le f(n) \left[a(n) + b(n)\, z(n) + c(n)\, \lambda + d(n)\, \lambda \right]$$
$$= f(n)\, b(n)\, z(n) + f(n) \left[a(n) + \left[c(n) + d(n) \right] \lambda \right]. \tag{2.10}$$

Now a suitable application of Theorem 1.2.1 given in [7] to (2.10) yields

$$z(n) \le \sum_{s=\alpha}^{n-1} f(s) \left[a(s) + \lambda \left[c(s) + d(s) \right] \right] \prod_{\sigma=s+1}^{n-1} \left[1 + f(\sigma)\, b(\sigma) \right]. \tag{2.11}$$

Using (2.11) in (2.9), we get

$$u(n) \le a(n) + b(n) \sum_{s=\alpha}^{n-1} f(s) \left[a(s) + \lambda \left[c(s) + d(s) \right] \right] \prod_{\sigma=s+1}^{n-1} \left[1 + f(\sigma)\, b(\sigma) \right]$$

$$+ c(n)\, \lambda$$
$$= K_1(n) + \lambda\, K_2(n). \tag{2.12}$$

From (2.7) and (2.12), it is easy to observe that

$$\lambda \le M. \tag{2.13}$$

Using (2.13) in (2.12), we get (2.3).

3. Existence and Uniqueness

For a function $z(n)$ and $\Delta z(n)$ in $D(N_{\alpha,\beta+1}, R^m)$ we denote by $|z(n)|_1 = |z(n)| + |\Delta z(n)|$. Let S be the space of functions $z(n), \Delta z(n)$ in $D(N_{\alpha,\beta+1}, R^m)$ which fulfil the condition

$$|z(n)|_1 = O(\exp(Ln)), \tag{3.1}$$

where $L > 0$ is a constant. In the space S we define the norm

$$|z|_S = \max_{n \in N_{\alpha,\beta+1}} \left[|z(n)|_1 \exp(-Ln) \right]. \tag{3.2}$$

It is easily seen that S with norm defined in (3.2) is a Banach space. We note that the condition (3.1) implies that there exists a constant $M \geq 0$ such that $|z(n)|_1 \leq M \exp(Ln)$, $n \in N_{\alpha,\beta+1}$. Using this fact in (3.2) we observe that

$$|z|_S \leq M. \tag{3.3}$$

We are now in a position to formulate the existence and uniqueness result for the solution of equation (1.1).

Theorem 2. *Suppose that*
(i) the functions f, h in (1.1) satisfy the conditions

$$|f(n,u,v,w) - f(n,\bar{u},\bar{v},\bar{w})| \leq p(n)\left[|u-\bar{u}| + |v-\bar{v}|\right] + |w-\bar{w}|, \tag{3.4}$$

$$|h(n,\tau,u,v) - h(n,\tau,\bar{u},\bar{v})| \leq q(n,\tau)\left[|u-\bar{u}| + |v-\bar{v}|\right], \tag{3.5}$$

where $p \in D(N_{\alpha,\beta+1}, R_+)$, $q \in D\left(N^2_{\alpha,\beta+1}, R_+\right)$,
(ii) for L as in (3.1).
(a_1) there exists a nonnegative constant δ such that $0 \leq \delta < 1$ and

$$E(n) + \sum_{s=\alpha}^{n-1} E(s) \leq \delta \exp(Ln), \tag{3.6}$$

for $n \in N_{\alpha,\beta+1}$, where

$$E(n) = p(n)\exp(Ln) + \sum_{\tau=\alpha}^{\beta} q(n,\tau)\exp(L\tau),$$

(a_2) *there exists a nonnegative constant* d *such that*

$$|u_0| + |f(n,0,0,H0)| + \sum_{s=\alpha}^{n-1} |f(s,0,0,H0)| \leq d \exp(Ln), \tag{3.7}$$

where f, H, u_0 *are as in equation (1.1).*
Under the assumptions (i) and (ii) the equation (1.1) has a unique solution $u(n)$ *on* $N_{\alpha,\beta+1}$ *in* S.

Proof. Let $u \in S$ and define the operator T by

$$(Tu)(n) = u_0 + \sum_{s=\alpha}^{n-1} f(s, u(s), \Delta u(s), Hu(s)). \tag{3.8}$$

From (3.8), we get

$$\Delta(Tu)(n) = f(n, u(n), \Delta u(n), Hu(n)). \tag{3.9}$$

We first show that T maps S into itself. We verify that (3.1) is fulfilled. From (3.8), (3.9), using the hypotheses and (3.3), we have

$$\begin{aligned}
|(Tu)(n)|_1 &\leq |u_0| + \sum_{s=\alpha}^{n-1} |f(s, u(s), \Delta u(s), Hu(s)) - f(s,0,0,H0)| \\
&\quad + \sum_{s=\alpha}^{n-1} |f(s,0,0,H0)| \\
&\quad + |f(n, u(n), \Delta u(n), Hu(n)) - f(n,0,0,H0)| + |f(n,0,0,H0)| \\
&\leq d\exp(Ln) + \sum_{s=\alpha}^{n-1} \left\{ p(s)|u(s)|_1 + \sum_{\tau=\alpha}^{\beta} q(s,\tau)|u(\tau)|_1 \right\} \\
&\quad + p(n)|u(n)|_1 + \sum_{\tau=\alpha}^{\beta} q(n,\tau)|u(\tau)|_1 \\
&\leq d\exp(Ln) + |u|_S \left[\sum_{s=\alpha}^{n-1} \left\{ p(s)\exp(Ls) + \sum_{\tau=\alpha}^{\beta} q(s,\tau)\exp(L\tau) \right\} \right.
\end{aligned}$$

$$+\left\{p(n)\exp(Ln)+\sum_{\tau=\alpha}^{\beta}q(n,\tau)\exp(L\tau)\right\}\Bigg]$$

$$\leq d\exp(Ln)+M\left[\sum_{s=\alpha}^{n-1}E(s)+E(n)\right]$$

$$\leq [d+M\delta]\exp(Ln). \tag{3.10}$$

From (3.10), it follows that $Tx \in S$.

Next, we verify that the operator T is a contraction map. Let $u, v \in S$. From (3.8), (3.9), and using the hypotheses, we have

$$|(Tu)(n)-(Tv)(n)|_1$$

$$\leq \sum_{s=\alpha}^{n-1}|f(s,u(s),\Delta u(s),Hu(s))-f(s,v(s),\Delta v(s),Hv(s))|$$

$$+|f(n,u(n),\Delta u(n),Hu(n))-f(n,v(n),\Delta v(n),Hv(n))|$$

$$\leq \sum_{s=\alpha}^{n-1}\left\{p(s)|u(s)-v(s)|_1+\sum_{\tau=\alpha}^{\beta}q(s,\tau)|u(s)-v(s)|_1\right\}$$

$$+p(n)|u(n)-v(n)|_1+\sum_{\tau=\alpha}^{\beta}q(n,\tau)|u(\tau)-v(\tau)|_1$$

$$\leq |u-v|_S\left[\sum_{s=\alpha}^{n-1}E(s)+E(n)\right]$$

$$\leq |u-v|_S\,\delta\exp(Ln). \tag{3.11}$$

From (3.11), we get

$$|Tu-Tv|_S \leq \delta|u-v|_S.$$

Since $\delta < 1$, it follows from Banach fixed point theorem (see [3, Theorem 9.1, p. 372]) that T has a unique fixed point in S. The fixed point of T is however a solution of equation (1.1). The proof is complete.

4. Estimates On the Solutions

In this section we apply the inequality in Theorem 1 to obtain estimates on the solutions of IVP (1.1) under some suitable conditions on the functions involved therein.

The following theorem concerning the estimate on the solutions of IVP (1.1) holds.

Theorem 3. *Suppose that the functions f, h in (1.1) satisfy the conditions*

$$|f(n,u,v,w)| \leq \gamma\left[|u|+|v|+|w|\right], \tag{4.1}$$

$$|h(n,\tau,u,v)| \leq q(n)\, r(\tau)\left[|u|+|v|\right], \tag{4.2}$$

for all $u, v, w \in R^m$, where $0 \leq \gamma < 1$ is a constant and $q, r \in D(N_{\alpha,\beta+1}, R_+)$. Let

$$L_1(n) = \frac{|u_0|}{1-\gamma} + \frac{1}{1-\gamma}\sum_{s=\alpha}^{n-1}\gamma\frac{|u_0|}{1-\gamma}\prod_{\sigma=s+1}^{n-1}\left[1+\frac{1}{1-\gamma}\right], \tag{4.3}$$

$$L_2(n) = \frac{\gamma}{1-\gamma}q(n) + \frac{1}{1-\gamma}\sum_{s=\alpha}^{n-1}\gamma\left[\frac{\gamma}{1-\gamma}+1\right]q(s)\prod_{\sigma=s+1}^{n-1}\left[1+\frac{1}{1-\gamma}\right] \tag{4.4}$$

for $n \in N_{\alpha,\beta+1}$ and

$$\lambda = \sum_{\xi=\alpha}^{\beta} r(\xi) L_2(\xi) < 1, \tag{4.5}$$

$$Q = \frac{1}{1-\lambda}\sum_{\xi=\alpha}^{\beta} r(\xi) L_1(\xi). \tag{4.6}$$

If $u(n)$ is any solution of IVP (1.1) on $N_{\alpha,\beta+1}$, then

$$|u(n)| + |\Delta u(n)| \leq L_1(n) + Q\, L_2(n), \tag{4.7}$$

for $n \in N_{\alpha,\beta+1}$.

Proof. Let $m(n) = |u(n)| + |\Delta u(n)|, n \in N_{\alpha,\beta+1}$. Using the hypotheses, we observe that

$$m(n) = \left| u_0 + \sum_{s=\alpha}^{n-1} f(s, u(s), \Delta u(s), Hu(s)) \right| + |f(n, u(n), \Delta u(n), Hu(n))|$$

$$\leq |u_0| + \sum_{s=\alpha}^{n-1} \gamma \left[m(s) + \sum_{\tau=\alpha}^{\beta} q(s) r(\tau) m(\tau) \right] + \gamma \left[m(n) + \sum_{\tau=\alpha}^{\beta} q(n) r(\tau) m(\tau) \right]. \tag{4.8}$$

From (4.8) it is easy to observe that

$$m(n) \leq \frac{|u_0|}{1-\gamma} + \frac{1}{1-\gamma} \sum_{s=\alpha}^{n-1} \gamma \left[m(s) + q(s) \sum_{\tau=\alpha}^{\beta} r(\tau) m(\tau) \right]$$
$$+ \frac{\gamma}{1-\gamma} q(n) \sum_{\tau=\alpha}^{\beta} r(\tau) m(\tau). \tag{4.9}$$

Now a suitable application of Theorem 1 to (4.9) yields (4.7).

Remark 1. We note that the estimate obtained in (4.7) yields not only the bound on the solution $u(n)$ of IVP (1.1) but also the bound on $\Delta u(n)$. If the estimate on the right hand side in (4.7) is bounded, then the solution $u(n)$ of IVP (1.1) and $\Delta u(n)$ are also bounded on $N_{\alpha,\beta+1}$.

Consider the IVP (1.1) together with the following IVP:

$$\Delta z(n) = g(n, z(n), \Delta z(n), Hz(n)),\ z(\alpha) = z_0, \tag{4.10}$$

for $n \in N_{\alpha,\beta+1}$, where H is given by (1.2) and $g \in D(N_{\alpha,\beta+1} \times R^m \times R^m \times R^m, R^m)$.

In the following theorem we provide conditions concerning the closeness of solutions of IVP (1.1) and IVP (4.10).

Theorem 4. *Suppose that the functions f, h in (1.1) satisfy the conditions*

$$|f(n, u, v, w) - f(n, \bar{u}, \bar{v}, \bar{w})| \leq \gamma [|u - \bar{u}| + |v - \bar{v}| + |w - \bar{w}|], \tag{4.11}$$

$$|h(n,\tau,u,v) - h(n,\tau,\bar{u},\bar{v})| \le q(n)\, r(\tau)\left[|u-\bar{u}| + |v-\bar{v}|\right], \qquad (4.12)$$

where $0 \le \gamma < 1$ *is a constant,* $q, r \in D(N_{\alpha,\beta+1}, R_+)$ *and there exist constants* $\varepsilon \ge 0, \delta \ge 0$ *such that*

$$|f(n,u,v,w) - g(n,u,v,w)| \le \varepsilon, \qquad (4.13)$$

$$|u_0 - z_0| \le \delta, \qquad (4.14)$$

where f, u_0 *and* g, z_0 *are as in (1.1) and (4.10). Let* $\lambda, L_2(n)$ *be as in (4.5), (4.4) and*

$$e(n) = \delta + \varepsilon\left[1 + n - \alpha\right], \qquad (4.15)$$

$$Q_0 = \frac{1}{1-\lambda}\sum_{\xi=\alpha}^{\beta} r(\xi)\, A_0(\xi), \qquad (4.16)$$

in which

$$A_0(n) = \frac{e(n)}{1-\gamma} + \frac{1}{1-\gamma}\sum_{s=\alpha}^{n-1}\gamma\frac{e(s)}{1-\gamma}\prod_{\sigma=s+1}^{n-1}\left[1 + \frac{\gamma}{1-\gamma}\right]. \qquad (4.17)$$

Let $u(n)$ *and* $z(n)$ *be respectively, solutions of IVP (1.1) and IVP (4.10) on* $N_{\alpha,\beta+1}$*, then*

$$|u(n) - z(n)| + |\Delta u(n) - \Delta z(n)| \le A_0(n) + Q_0\, L_2(n), \qquad (4.18)$$

for $n \in N_{\alpha,\beta+1}$.

Proof. Let $w(n) = |u(n) - z(n)| + |\Delta u(n) - \Delta z(n)|$, $n \in N_{\alpha,\beta+1}$. Using the hypotheses, we observe that

$$w(n) \le |u_0 - z_0| + \sum_{s=\alpha}^{n-1}|f(s,u(s),\Delta u(s),Hu(s)) - f(s,z(s),\Delta z(s),Hz(s))|$$

$$+\sum_{s=\alpha}^{n-1}|f(s,z(s),\Delta z(s),Hz(s)) - g(s,z(s),\Delta z(s),Hz(s))|$$

$$+|f(n,u(n),\Delta u(n),Hu(n)) - f(n,z(n),\Delta z(n),Hz(n))|$$

$$+|f(n,z(n),\Delta z(n),Hz(n)) - g(n,z(n),\Delta z(n),Hz(n))|$$

$$\leq \delta + \sum_{s=\alpha}^{n-1} \gamma \left[w(s) + q(s) \sum_{\tau=\alpha}^{\beta} r(\tau) w(\tau) \right] + \sum_{s=\alpha}^{n-1} \varepsilon$$
$$+\gamma \left[w(n) + q(n) \sum_{\tau=\alpha}^{\beta} r(\tau) w(\tau) \right] + \varepsilon$$
$$= e(n) + \sum_{s=\alpha}^{n-1} \gamma \left[w(s) + q(s) \sum_{\tau=\alpha}^{\beta} r(\tau) w(\tau) \right] + \gamma \left[w(n) + q(n) \sum_{\tau=\alpha}^{\beta} r(\tau) w(\tau) \right]. \tag{4.19}$$

From (4.19) it is easy to observe that

$$w(n) \leq \frac{e(n)}{1-\gamma} + \frac{1}{1-\gamma} \sum_{s=\alpha}^{n-1} \gamma \left[w(s) + \sum_{\tau=\alpha}^{\beta} q(s) r(\tau) w(\tau) \right] + \frac{\gamma}{1-\gamma} q(n) \sum_{\tau=\alpha}^{\beta} r(\tau) w(\tau). \tag{4.20}$$

Now a suitable application of Theorem 1 to (4.20) yields (4.18).

Remark 2. We note that the result given in Theorem 3 relates the solutions of IVP (1.1) and IVP (4.10) in the sense that if f is close to g; u_0 is close to z_0, then it is easy to observe that the solutions of IVP (1.1) and IVP (4.10) are also close to each other.

5. Continuous Dependence

In this section we present results on the dependency of solutions of (1.1) on initial values and also the solutions of equations of the form (1.1) on parameters.

The following theorem deals with the continuous dependence of solutions of (1.1) on given initial values.

Theorem 5. *Let $u_i(n) (i = 1, 2)$ be respectively the solutions of equation*

$$\Delta u(n) = f(n, u(n), \Delta u(n), Hu(n)), \tag{5.1}$$

with the given initial conditions

$$u_i(\alpha) = c_i, \tag{5.2}$$

where f,h are as in (1.1) and c_i are given constants. Suppose that the functions f,h in (5.1) satisfy the conditions (4.11), (4.12). Let $\lambda, L_2(n)$ be as in (4.5), (4.4) and

$$Q_1 = \frac{1}{1-\lambda}\sum_{\xi=\alpha}^{\beta} r(\xi)\, A_1(\xi), \tag{5.3}$$

where $A_1(n)$ is defined by the right hand side of (4.17) by replacing $e(n)$ by $|c_1 - c_2|$. Then

$$|u_1(n) - u_2(n)| + |\Delta u_1(n) - \Delta u_2(n)| \leq A_1(n) + Q_1 L_2(n), \tag{5.4}$$

for $n \in N_{\alpha,\beta+1}$.

Proof. Let $u(n) = |u_1(n) - u_2(n)| + |\Delta u_1(n) - \Delta u_2(n)|,\ n \in N_{\alpha,\beta+1}$. From the hypotheses, we observe that

$$\begin{aligned} u(n) &\leq |c_1 - c_2| + \sum_{s=\alpha}^{n-1} |f(s, u_1(s), \Delta u_1(s), Hu_1(s)) - f(s, u_2(s), \Delta u_2(s), Hu_2(s))| \\ &\quad + |f(n, u_1(n), \Delta u_1(n), Hu_1(n)) - f(n, u_2(n), \Delta u_2(n), Hu_2(n))| \\ &\leq |c_1 - c_2| + \sum_{s=\alpha}^{n-1} \gamma \left[u(s) + \sum_{\tau=\alpha}^{\beta} q(s)\, r(\tau)\, u(\tau)\right] + \gamma \left[u(n) + \sum_{\tau=\alpha}^{\beta} q(n)\, r(\tau)\, u(\tau)\right]. \end{aligned} \tag{5.5}$$

From (5.5) it is easy to observe that

$$\begin{aligned} u(n) \leq \frac{|c_1 - c_2|}{1-\gamma} &+ \frac{1}{1-\gamma}\sum_{s=\alpha}^{n-1} \gamma \left[u(s) + q(s) \sum_{\tau=\alpha}^{\beta} r(\tau)\, u(\tau)\right] \\ &+ \frac{\gamma}{1-\gamma} q(n) \sum_{\tau=\alpha}^{\beta} r(\tau)\, u(\tau). \end{aligned} \tag{5.6}$$

Now a suitable application of Theorem 1 to (5.6) yields (5.4), which shows the dependency of solutions of (5.1) on given initial values.

Remark 3. If we choose $c_1 = c_2$ in Theorem 5, then it is easy to see that $A_1(n) = 0, Q_1 = 0$ and consequently from (5.4) the uniqueness of solutions of (5.1) follows.

Next, we consider the following IVPs for implicit Fredholm type sum-difference equations

$$\Delta z(n) = f(n, z(n), \Delta z(n), Hz(n), \mu), \ z(\alpha) = z_0, \tag{5.7}$$

$$\Delta z(n) = f(n, z(n), \Delta z(n), Hz(n), \mu_0), \ z(\alpha) = z_0, \tag{5.8}$$

where H is given as in (1.2), $f \in D(N_{\alpha,\beta+1} \times R^m \times R^m \times R^m \times R, R^m)$ and μ, μ_0 are parameters.

The next theorem deals with the dependency of solutions of IVP (5.7) and IVP (5.8) on parameters.

Theorem 6. *Suppose that the functions h and f in (5.7), (5.8) respectively satisfy the condition (4.12) and*

$$|f(n, u, v, w, \mu) - f(n, \bar{u}, \bar{v}, \bar{w}, \mu)| \leq \gamma [|u - \bar{u}| + |v - \bar{v}| + |w - \bar{w}|], \tag{5.9}$$

$$|f(n, u, v, w, \mu) - f(n, u, v, w, \mu_0)| \leq m(n) |\mu - \mu_0|, \tag{5.10}$$

where $0 \leq \gamma < 1$ is a constant and $m \in D(N_{\alpha,\beta+1}, R_+)$. Let

$$\bar{m}(n) = m(n) + \sum_{\tau=\alpha}^{\beta} m(\tau), \tag{5.11}$$

$\lambda, L_2(n)$ be as in (4.5), (4.4) and

$$Q_2 = \frac{1}{1-\lambda} \sum_{\xi=\alpha}^{\beta} r(\xi) A_2(\xi), \tag{5.12}$$

where $A_2(n)$ is defined by the right hand side of (4.17) by replacing $e(n)$ by $|\mu - \mu_0| \bar{m}(n)$. Let $z_1(n)$ and $z_2(n)$ be respectively the solutions of IVP (5.7) and IVP (5.8) on $N_{\alpha,\beta+1}$. Then

$$|z_1(n) - z_2(n)| + |\Delta z_1(n) - \Delta z_2(n)| \leq A_2(n) + Q_2 L_2(n), \tag{5.13}$$

for $n \in N_{\alpha,\beta+1}$.

Proof. Let $w(n) = |z_1(n) - z_2(n)| + |\Delta z_1(n) - \Delta z_2(n)|,\ n \in N_{\alpha,\beta+1}$. From the hypotheses, we observe that

$$w(n) \le \sum_{s=\alpha}^{n-1} |f(s, z_1(s), \Delta z_1(s), Hz_1(s), \mu) - f(s, z_2(s), \Delta z_2(s), Hz_2(s), \mu)|$$

$$+ \sum_{s=\alpha}^{n-1} |f(s, z_2(s), \Delta z_2(s), Hz_2(s), \mu) - f(s, z_2(s), \Delta z_2(s), Hz_2(s), \mu_0)|$$

$$+ |f(n, z_1(n), \Delta z_1(n), Hz_1(n), \mu) - f(n, z_2(n), \Delta z_2(n), Hz_2(n), \mu)|$$

$$+ |f(n, z_2(n), \Delta z_2(n), Hz_2(n), \mu) - f(n, z_2(n), \Delta z_2(n), Hz_2(n), \mu_0)|$$

$$\le \sum_{s=\alpha}^{n-1} \gamma \left[w(s) + \sum_{\tau=\alpha}^{\beta} q(s) r(\tau) w(\tau) \right] + \sum_{s=\alpha}^{n-1} m(s) |\mu - \mu_0|$$

$$+ \gamma \left[w(n) + \sum_{\tau=\alpha}^{\beta} q(n) r(\tau) w(\tau) \right] + m(n) |\mu - \mu_0|. \tag{5.14}$$

From (5.14), we observe that

$$w(n) \le \frac{|\mu - \mu_0| \bar{m}(n)}{1 - \gamma} + \frac{1}{1 - \gamma} \sum_{s=\alpha}^{n-1} \gamma \left[w(s) + q(s) \sum_{\tau=\alpha}^{\beta} r(\tau) w(\tau) \right]$$

$$+ \frac{\gamma}{1 - \gamma} q(n) \sum_{\tau=\alpha}^{\beta} r(\tau) w(\tau). \tag{5.15}$$

Now an application of Theorem 1 to (5.15) yields (5.13), which shows the dependency of solutions of IVP (5.7) and IVP (5.8) on parameters.

Remark 4. We note that the results given in this paper can be extended very easily to study the sum-difference equation of the form (1.1) involving functions of two independent variables (see [10]) by making use of the two independent variable generalization of Theorem 1 given above (see also [8, Theorem 5.5.1]). For further results on the qualitative properties of solutions of various sum-difference equations, see [7-10] and the references cited therein.

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