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On Some Results for Subclass of $\beta-$spirallike Functions of Order α *

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Abstract

This paper introduces two new subclass of the classes of $\beta-$spirallike function of order α and $\beta-$convexlike function of order α. We prove a subordination relation for these classes of functions.

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1. Introduction

Let A denote the class of functions $f(z)$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. Let S denote the subclass of A consisting of analytic and univalent functions in E. For $0 \leqslant \alpha < 1$, a function $f \in A$ is said to be starlike of order α if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in E). \tag{1.2}$$

We denote the class of such functions by $S^*(\alpha)$. A function $f \in A$ is said to be convex in E, if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in E). \tag{1.3}$$

We denote by K the class of all convex functions in E.

For $|\lambda| < \frac{\pi}{2}$ and $0 \leqslant \alpha < 1$, a function $f(z) \in A$ is said to be $\lambda-$spirallike of order α in E if

$$Re\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > \alpha\cos\lambda \quad (z \in E). \tag{1.4}$$

The class of such functions is denoted by $S_p^{\alpha}(\lambda)$ [2].

Definition 1. The convolution or Hadamard product of two power series $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n$ are analytic in E, is defined as then their Hadamard product, $f * g$ is defined by the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

The function $f * g$ is also analytic in E.

Definition 2. Let f be analytic in $E$and $g$be analytic and univalent in E with$f(0) = g(0)$. Then the subordination $f(z) \prec g(z)$ is equivalent to $f(E) \subset g(E)$.

Definition 3. A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if whenever $f(z) = \sum_{k=1}^{\infty} a_k z^k, \quad a_1 = 1$ is regular, univalent and convex in E, we have

$$\sum_{k=1}^{\infty} a_k b_k z^k \prec f(z)$$

in E.

The following lemma is due to Wilf [1].

Lemma 1. *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequences if and only if*

$$Re\left\{1 + 2\sum_{n=1}^{\infty} b_n z^n\right\} > 0$$

for $z \in E$.

2. Some Definitions and Results for $\beta-$spirallike Function of Order α

Definition 4. Let $F(z) = \frac{zf'(z)}{f(z)}$ for $f(z) \in S$. A function $f(z) \in S$ is said to be in the class $S_\beta(\alpha)$ if it satisfies the inequality

$$\left|\frac{1}{e^{i\beta}F(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \quad (z \in E) \tag{2.1}$$

for some real β and $0 < \alpha < 1$.

Theorem 1. $f(z) \in S_\beta(\alpha)$ *iff* $Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > \alpha$.

Proof. Let $F(z) = \frac{zf'(z)}{f(z)}$ for $f(z) \in S$. If $f(z) \in S_\beta(\alpha)$, we can write

$$\left|\frac{2\alpha - e^{i\beta}F(z)}{2\alpha e^{i\beta}F(z)}\right| < \frac{1}{2\alpha}.$$

Then, we can obtain

$$\begin{aligned}
\left|\frac{2\alpha - e^{i\beta}F(z)}{2\alpha e^{i\beta}F(z)}\right| &< \frac{1}{2\alpha} \Leftrightarrow \left|\frac{2\alpha - e^{i\beta}F(z)}{2\alpha e^{i\beta}F(z)}\right|^2 < \left(\frac{1}{2\alpha}\right)^2 \\
&\Leftrightarrow \left(2\alpha - e^{i\beta}F(z)\right)\left[\overline{2\alpha - e^{i\beta}F(z)}\right] < \left[e^{-i\beta}\overline{F(z)}\right]e^{i\beta}F(z) \\
&\Leftrightarrow \left(2\alpha - e^{i\beta}F(z)\right)\left[\overline{2\alpha - e^{i\beta}F(z)}\right] < \left[e^{-i\beta}\overline{F(z)}\right]e^{i\beta}F(z) \\
&\Leftrightarrow 4\alpha^2 - 2\alpha e^{-i\beta}\overline{F(z)} - 2\alpha e^{i\beta}F(z) + F(z)\overline{F(z)} < F(z)\overline{F(z)} \\
&\Leftrightarrow 4\alpha^2 - 2\alpha(e^{-i\beta}\overline{F(z)} + e^{i\beta}F(z)) < 0 \\
&\Leftrightarrow 2\alpha - 2Re(e^{i\beta}F(z)) < 0 \\
&\Leftrightarrow Re(e^{i\beta}F(z)) > \alpha \\
&\Leftrightarrow Re(e^{i\beta}\frac{zf'(z)}{f(z)}) > \alpha.
\end{aligned}$$

Theorem 2. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty}\left\{n + \left|n - 2\alpha e^{-i\beta}\right|\right\}|a_n| \leqslant 1 - \left|1 - 2\alpha e^{-i\beta}\right| \tag{2.2}$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, thenf(z) \in S_\beta(\alpha)$.

Proof. It suffices to show that

$$\left|\frac{2\alpha - e^{i\beta}F(z)}{e^{i\beta}F(z)}\right| < 1$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$,where $F(z) = \frac{zf'(z)}{f(z)}$. Note that

$$\begin{aligned}
\left|\frac{2\alpha - e^{i\beta}F(z)}{e^{i\beta}F(z)}\right| &= \left|\frac{2\alpha f(z) - e^{i\beta}zf'(z)}{e^{i\beta}zf'(z)}\right| \\
&= \left|\frac{2\alpha e^{-i\beta}f(z) - zf'(z)}{zf'(z)}\right| \\
&= \left|\frac{\left(1-2\alpha e^{-i\beta}\right) + \sum_{n=2}^{\infty}\left(n-2\alpha e^{-i\beta}\right)|a_n|\, z^{n-1}}{1+\sum_{n=2}^{\infty} n a_n z^{n-1}}\right| \\
&\leqslant \frac{\left|1-2\alpha e^{-i\beta}\right| + \sum_{n=2}^{\infty}\left|n-2\alpha e^{-i\beta}\right| |a_n|\, |z|^{n-1}}{1-\sum_{n=2}^{\infty} n\, |a_n|\, |z|^{n-1}} \qquad (2.3) \\
&< \frac{\left|1-2\alpha e^{-i\beta}\right| + \sum_{n=2}^{\infty}\left|n-2\alpha e^{-i\beta}\right| |a_n|}{1-\sum_{n=2}^{\infty} n\, |a_n|}.
\end{aligned}$$

Therefore, if

$$\sum_{n=2}^{\infty}\left\{n+\left|n-2\alpha e^{-i\beta}\right|\right\}|a_n| \leqslant 1-\left|1-2\alpha e^{-i\beta}\right|$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then

$$\sum_{n=2}^{\infty}\left|n-2\alpha e^{-i\beta}\right| |a_n| \leqslant 1-\left|1-2\alpha e^{-i\beta}\right| - \sum_{n=2}^{\infty} n\, |a_n|.$$

Using this inequality in (2.3), we obtain

$$\begin{aligned}
\left|\frac{2\alpha - e^{i\beta}F(z)}{e^{i\beta}F(z)}\right| &< \frac{\left|1-2\alpha e^{-i\beta}\right| + \sum_{n=2}^{\infty}\left|n-2\alpha e^{-i\beta}\right| |a_n|}{1-\sum_{n=2}^{\infty} n\, |a_n|} \\
&\leqslant \frac{\left|1-2\alpha e^{-i\beta}\right| + 1 - \left|1-2\alpha e^{-i\beta}\right| - \sum_{n=2}^{\infty} n\, |a_n|}{1-\sum_{n=2}^{\infty} n\, |a_n|} \\
&= 1.
\end{aligned}$$

Therefore, $f(z) \in S_\beta(\alpha)$ for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$.

Taking $\beta = \frac{\pi}{4}$ in Theorem 2, we have the following Corollary 1

Corollary 1. *$f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} \left\{ n + \sqrt{n^2 - 2\sqrt{2}\alpha n + 4\alpha^2} \right\} |a_n| \leqslant 1 - \sqrt{1 - 2\sqrt{2}\alpha + 4\alpha^2}$$

for some $0 < \alpha < \frac{\sqrt{2}}{2}$, then $f(z) \in S_{\frac{\pi}{4}}(\alpha)$.

Remark 1. *Owa, Ochiai and Srivastava [4] have considered the subclass $M(\alpha)$ of A consisting of functions $f(z)$ such that*

$$\left| \frac{f(z)}{zf'(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \quad (z \in E, 0 < \alpha < 1).$$

Owa, Ochiai and Srivastava [4] have proved the following theorem.

Theorem 3. *Let $0 < \alpha < 1$. If $f(z) \in A$ satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leqslant \frac{1}{2} (1 - |1 - 2\alpha|) = \begin{cases} \alpha; & \left(0 < \alpha \leq \frac{1}{2}\right) \\ 1 - \alpha; & \left(\frac{1}{2} \leq \alpha < 1\right) \end{cases},$$

then $f(z) \in M(\alpha)$.

Putting $\beta = 0$ in Theorem 2, we get the Theorem 3 given by Owa, Ochiai and Srivastava [4].

Let us denote by $G(\beta, \alpha)$, the class of functions $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ whose coefficients satisfy the condition (2.2).

Theorem 4. *Let $f \in G(\beta, \alpha)$. Then*

$$\frac{1 + \left|1 - 2\alpha e^{-i\beta}\right|}{2\left(2 + \left|1 - 2\alpha e^{-i\beta}\right|\right)} (f * g)(z) \prec g(z) \tag{2.4}$$

for $z \in E$ and for every function $g(z)$ in the class K.

In particular

$$Ref(z) > -\frac{2 + \left|1 - 2\alpha e^{-i\beta}\right|}{1 + \left|1 - 2\alpha e^{-i\beta}\right|}$$

for $z \in E$. The constant $\frac{1 + \left|1 - 2\alpha e^{-i\beta}\right|}{2\left(2 + \left|1 - 2\alpha e^{-i\beta}\right|\right)}$ cannot be replace by any larger one.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in $G(\beta, \alpha)$ and let $g(z) = z + \sum_{n=2}^{\infty} c_n z^n$ be in K. Then

$$\frac{1+\left|1-2\alpha e^{-i\beta}\right|}{2\left(2+\left|1-2\alpha e^{-i\beta}\right|\right)}(f*g)(z) = \frac{1+\left|1-2\alpha e^{-i\beta}\right|}{2\left(2+\left|1-2\alpha e^{-i\beta}\right|\right)}\left(z+\sum_{n=2}^{\infty} a_n c_n z^n\right)$$

Thus, by definition 3,the assertion of our theorem will hold if the sequence

$$\left(\frac{\left\{1+\left|1-2\alpha e^{-i\beta}\right|\right\}a_n}{2\left(2+\left|1-2\alpha e^{-i\beta}\right|\right)}\right)_{n=1}^{\infty}$$

is subordinating factor sequence, with $a_1 = 1$.In view of Lemma 1, this will be the case if and only if

$$Re\left[1+2\sum_{n=2}^{\infty}\frac{1+\left|1-2\alpha e^{-i\beta}\right|}{2\left(2+\left|1-2\alpha e^{-i\beta}\right|\right)}a_n z^n\right] > 0 \tag{2.5}$$

for $z \in E$.

Now, we can write

$$\begin{aligned} & Re\left[1+\sum_{n=1}^{\infty}\frac{1+\left|1-2\alpha e^{-i\beta}\right|}{2+\left|1-2\alpha e^{-i\beta}\right|}a_n z^n\right] \\ = \; & Re\left[1+\frac{1+\left|1-2\alpha e^{-i\beta}\right|}{2+\left|1-2\alpha e^{-i\beta}\right|}z+\frac{1}{2+\left|1-2\alpha e^{-i\beta}\right|}\sum_{n=2}^{\infty}\left(1+\left|1-2\alpha e^{-i\beta}\right|\right)a_n z^n\right]. \end{aligned}$$

Because of $1+\left|1-2\alpha e^{-i\beta}\right| \leqslant n+\left|n-2\alpha e^{-i\beta}\right|$ for all $n \geqslant 2$, $|\beta| < \frac{\pi}{2}$ and

$0<\alpha<\cos\beta$, we have

$$\begin{aligned}
& Re\left[1+\sum_{n=1}^{\infty}\frac{1+\left|1-2\alpha e^{-i\beta}\right|}{2+\left|1-2\alpha e^{-i\beta}\right|}a_n z^n\right] \\
> & \left[1-\frac{1+\left|1-2\alpha e^{-i\beta}\right|}{2+\left|1-2\alpha e^{-i\beta}\right|}r-\frac{1-\left|1-2\alpha e^{-i\beta}\right|}{2+\left|1-2\alpha e^{-i\beta}\right|}\sum_{n=2}^{\infty}\frac{n+\left|n-2\alpha e^{-i\beta}\right|}{1-\left|1-2\alpha e^{-i\beta}\right|}\left|a_n\right|r^n\right] \\
\geqslant & \left[1-\frac{1+\left|1-2\alpha e^{-i\beta}\right|}{2+\left|1-2\alpha e^{-i\beta}\right|}r-\frac{1-\left|1-2\alpha e^{-i\beta}\right|}{2+\left|1-2\alpha e^{-i\beta}\right|}r\right] \\
= & \left[1-\frac{r}{2+\left|1-2\alpha e^{-i\beta}\right|}\left(1+\left|1-2\alpha e^{-i\beta}\right|+1-\left|1-2\alpha e^{-i\beta}\right|\right)\right] \\
= & \ 1-\frac{2r}{2+\left|1-2\alpha e^{-i\beta}\right|} \\
> & \ 0.
\end{aligned}$$

This prove the first assertion. That

$$Ref\left(z\right)>-\frac{2+\left|1-2\alpha e^{-i\beta}\right|}{1+\left|1-2\alpha e^{-i\beta}\right|}$$

for $f\left(z\right)\in G\left(\beta,\alpha\right)$ follows by taking $g\left(z\right)=\frac{z}{1-z}$ in (2.4) .

To prove the sharpness of the constant $\left\{1+\left|1-2\alpha e^{-i\beta}\right|\right\}/2\left\{2+\left|1-2\alpha e^{-i\beta}\right|\right\}$ we consider the function

$$f_0\left(z\right)=z-\frac{1}{1+\left|1-2\alpha e^{-i\beta}\right|}z^2 \quad \left(\left|\beta\right|<\frac{\pi}{2},0<\alpha<\cos\beta\right), \tag{2.6}$$

$$\left(\left|\beta\right|<\frac{\pi}{2},0<\alpha<\cos\beta\right),$$

which is a member of the class $G\left(\beta,\alpha\right)$ .Thus from the relation (2.4), we obtain

$$\frac{1+\left|1-2\alpha e^{-i\beta}\right|}{2\left(2+\left|1-2\alpha e^{-i\beta}\right|\right)}f_0\left(z\right)\prec\frac{z}{1-z}. \tag{2.7}$$

It can be verified that

$$\min_{|z|\leqslant 1} Re\left[\frac{1+\left|1-2\alpha e^{-i\beta}\right|}{2\left(2+\left|1-2\alpha e^{-i\beta}\right|\right)}f_0\left(z\right)\right]=-\frac{1}{2}. \tag{2.8}$$

This shows that the constant $\frac{1+\left|1-2\alpha e^{-i\beta}\right|}{2\left(2+\left|1-2\alpha e^{-i\beta}\right|\right)}$ is the best possible.

To prove Theorem 5, first we give the following result due to Miller and Mocanu [5].

Lemma 2. *Let $\phi(u,v)$ be a complex -valued function such that*

$$\phi : D \to \mathbb{C}, \quad D \subset \mathbb{C} \times \mathbb{C}$$

$\mathbb{C}$ being (as usual) the complex plane, and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$.

Suppose that the function $\phi(u,v)$ satisfies each of the following conditions:

(i) $\phi(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $Re\{\phi(1,0)\} > 0$;

(iii) $Re\{\phi(iu_2, v_1)\} \leqslant 0$ for all $(iu_2, v_1) \in D$ such that

$$v_1 \leqslant -\left(1+u_2^2\right)/2.$$

Let

$$p(z) = 1 + p_1 z + p_2 z^2 + ...$$

be analytic (reguler) in the unit disk E such that

$$\left(p(z), zp'(z)\right) \in D$$

for all $z \in E$. If

$$Re\left\{\phi\left(p(z), zp'(z)\right)\right\} > 0 \quad (z \in E),$$

then

$$Re\{p(z)\} > 0 \quad (z \in E).$$

Theorem 5. *Let the function $f(z)$ defined by (1.1) be in the class $S_\beta(\alpha)$ and let*

$$0 < \xi \leqslant \frac{1}{2(\cos\beta - \alpha)} \text{ and } 0 < \alpha < \cos\beta. \tag{2.9}$$

Then we have

$$Re\left\{\left(\frac{f(z)}{z}\right)^{\xi e^{i\beta}}\right\} > \frac{1}{2\xi(\cos\beta - \alpha) + 1} \quad (z \in E). \tag{2.10}$$

Proof. If we put

$$A = \frac{1}{2\xi\left(\cos\beta - \alpha\right) + 1}$$

and

$$\left(\frac{f(z)}{z}\right)^{\xi e^{i\beta}} = (1 - A)\, p(z) + A \tag{2.11}$$

where ξ satisfies (2.9) then $p(z)$ is regular in the unit disk E and $p(z) = 1 + p_1 z + ...$

From (2.11) after taking the logarithmical differentiation we have that

$$\xi e^{i\beta}\left[\frac{f'(z)}{f(z)} - \frac{1}{z}\right] = (1 - A)\,\frac{p'(z)}{(1 - A)\, p(z) + A}$$

$$\Rightarrow e^{i\beta}\frac{zf'(z)}{f(z)} - e^{i\beta} = (1 - A)\,\frac{zp'(z)}{\xi\left\{(1 - A)\, p(z) + A\right\}},$$

and from there

$$e^{i\beta}\frac{zf'(z)}{f(z)} - \alpha = e^{i\beta} - \alpha + (1 - A)\,\frac{zp'(z)}{\xi\left\{(1 - A)\, p(z) + A\right\}}. \tag{2.12}$$

Since $f(z) \in S_\beta(\alpha)$ then from (2.12) we get

$$Re\left\{e^{i\beta} - \alpha + (1 - A)\,\frac{zp'(z)}{\xi\left\{(1 - A)\, p(z) + A\right\}}\right\} > 0, \quad (z \in E, 0 < \alpha < \cos\beta)\,.$$

Let consider the function $\theta\,(u, v)$ defined by

$$\theta(u, v) = e^{i\beta} - \alpha + (1 - A)\,\frac{v}{\xi\left\{(1 - A)\, u + A\right\}},$$

where $u = p(z)$ and $v = zp'(z)$. Then $\theta\,(u, v)$ is continuous in $D = \left(\mathbb{C} - \left\{\frac{A}{A-1}\right\}\right) \times \mathbb{C}$.

Also, $(1, 0) \in D$ and $Re\left\{\theta\,(1, 0)\right\} = \cos\beta - \alpha > 0$. Furthermore, for all $(iu_2, v_1) \in D$ such that

$$v_1 \leqslant -\frac{1 + u_2^2}{2},$$

we have

$$\begin{aligned} Re\left\{\theta\left(iu_2, v_1\right)\right\} &= \cos\beta - \alpha + Re\left\{(1-A)\frac{v_1}{\xi\left[(1-A)\,iu_2 + A\right]}\right\} \\ &= \cos\beta - \alpha + \frac{A\,(1-A)\,v_1}{\xi\left[(1-A)^2 u_2^2 + A^2\right]} \\ &\leqslant \cos\beta - \alpha - \frac{A\,(1-A)\,(1+u_2^2)}{2\xi\left[(1-A)^2 u_2^2 + A^2\right]} \\ &= (\cos\beta - \alpha)\frac{A^2\left[4\xi^2(\cos\beta-\alpha)^2 - 1\right]u_2^2}{\left[(1-A)^2 u_2^2 + A^2\right]} \\ &\leqslant 0 \end{aligned}$$

because $0 < \alpha < \cos\beta$ and $4\xi^2\left(\cos\beta - \alpha\right)^2 - 1 \leqslant 0 \quad \Rightarrow \quad \xi \leqslant \frac{1}{2(\cos\beta-\alpha)}$. Therefore, the function $\theta\left(u, v\right)$ satisfies the conditions in Lemma 2. This proves that $Re\left\{p(z)\right\} > 0$ for $z \in E$, that is, that from (2.11)

$$\begin{aligned} &Re\left\{\left(\frac{f\left(z\right)}{z}\right)^{\xi e^{i\beta}}\right\} > A \\ \Rightarrow\ &Re\left\{\left(\frac{f(z)}{z}\right)^{\xi e^{i\beta}}\right\} > \frac{1}{2\xi\left(\cos\beta - \alpha\right) + 1} \end{aligned}$$

which is equivalent to the statement Theorem 5.

Taking $\beta = 0$ in Theorem 5, we obtain the following Corollary 2.

Corollary 2. *Let the function $f(z)$ defined by (1.1) be in the class $S_0(\alpha)$ and let $0 < \xi \leqslant \frac{1}{2(1-\alpha)}$ and $0 < \alpha < 0$.*

Then we have

$$Re\left\{\left(\frac{f(z)}{z}\right)^{\xi}\right\} > \frac{1}{2\xi\left(1-\alpha\right)+1} \quad (z \in E).$$

Remark 2. *If we take $\lambda = 0$ in Theorem 1 given by Obradovic and Owa [3], we can obtain Corollary 2.*

3. Some definitions and results for $\beta-$convexlike function of order α

Definition 5. *Let* $G(z)=1+\frac{zf''(z)}{f'(z)}$ *for* $f(z)\in S$. *A function* $f(z)\in S$ *is said to be in the class* $K_\beta(\alpha)$ *if it satisfies the inequality*

$$\left|\frac{1}{e^{i\beta}G(z)}-\frac{1}{2\alpha}\right|<\frac{1}{2\alpha}\quad (z\in E) \tag{3.1}$$

for some real β *and* $0<\alpha<1$.

Theorem 6. $f(z)\in K_\beta(\alpha)$ *iff* $Re\left\{e^{i\beta}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha$.

Proof. Let $G(z)=1+\frac{zf''(z)}{f'(z)}$ for $f(z)\in S$.If $f(z)\in K_\beta(\alpha)$,we can write

$$\left|\frac{1}{e^{i\beta}G(z)}-\frac{1}{2\alpha}\right|<\frac{1}{2\alpha}.$$

Then, we can obtain

$$\begin{aligned}
\left|\frac{1}{e^{i\beta}G(z)}-\frac{1}{2\alpha}\right| &< \frac{1}{2\alpha}\Leftrightarrow\left|\frac{2\alpha-e^{i\beta}G(z)}{2\alpha e^{i\beta}G(z)}\right|<\frac{1}{2\alpha}\\
&\Leftrightarrow \left|\frac{2\alpha-e^{i\beta}G(z)}{2\alpha e^{i\beta}G(z)}\right|<\frac{1}{2\alpha}\\
&\Leftrightarrow \left|\frac{2\alpha-e^{i\beta}G(z)}{2\alpha e^{i\beta}G(z)}\right|^2<\left(\frac{1}{2\alpha}\right)^2\\
&\Leftrightarrow \left(2\alpha-e^{i\beta}G(z)\right).\left(\overline{2\alpha-e^{i\beta}G(z)}\right)<\left(e^{i\beta}G(z)\right).\left(e^{-i\beta}\overline{G(z)}\right)\\
&\Leftrightarrow 4\alpha^2-2\alpha\left(e^{-i\beta}\overline{G(z)}+e^{i\beta}G(z)\right)<0\\
&\Leftrightarrow Re\left\{e^{i\beta}G(z)\right\}>\alpha\\
&\Leftrightarrow Re\left\{e^{i\beta}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha.
\end{aligned}$$

Theorem 7. *If* $f(z)\in A$ *satisfies*

$$\sum_{n=2}^{\infty}n\left(n+\left|n-2\alpha e^{-i\beta}\right|\right)|a_n|\leqslant 1-\left|1-2\alpha e^{-i\beta}\right| \tag{3.2}$$

for some $|\beta|<\frac{\pi}{2}$ *and* $0<\alpha<Cos\beta$, *then* $f(z)\in K_\beta(\alpha)$.

Proof. It suffices to show that

$$\left|\frac{2\alpha - e^{i\beta}G(z)}{e^{i\beta}G(z)}\right| < 1$$

for some $|\beta| < \frac{\pi}{2}, 0 < \alpha < 1$. By definition $G(z) = 1 + \frac{zf''(z)}{f'(z)}$, we can write

$$\begin{aligned} \left|\frac{2\alpha - e^{i\beta}G(z)}{e^{i\beta}G(z)}\right| &= \left|\frac{2\alpha e^{-i\beta}f'(z) - \left\{f'(z) + zf''(z)\right\}}{f'(z) + zf''(z)}\right| \\ &= \left|\frac{\left(2\alpha e^{-i\beta} - 1\right)f'(z) - zf''(z)}{f'(z) + zf''(z)}\right| \\ &= \left|\frac{\left(2\alpha e^{-i\beta} - 1\right) + \sum_{n=2}^{\infty}\left(2\alpha e^{-i\beta} - n\right)na_n z^{n-1}}{1 + \sum_{n=2}^{\infty} n^2 a_n z^{n-1}}\right| \\ &< \frac{\left|1 - 2\alpha e^{-i\beta}\right| + \sum_{n=2}^{\infty} n\left|n - 2\alpha e^{-i\beta}\right||a_n|}{1 - \sum_{n=2}^{\infty} n^2|a_n|}. \end{aligned} \tag{3.3}$$

Therefore, if

$$\sum_{n=2}^{\infty} n\left(n + \left|n - 2\alpha e^{-i\beta}\right|\right)|a_n| \leqslant 1 - \left|1 - 2\alpha e^{-i\beta}\right|$$

for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < 1$, then

$$\sum_{n=2}^{\infty} n\left|n - 2\alpha e^{-i\beta}\right||a_n| \leqslant 1 - \left|1 - 2\alpha e^{-i\beta}\right| - \sum_{n=2}^{\infty} n^2|a_n|.$$

Using this inequality in (3.3), we obtain

$$\begin{aligned}
\left|\frac{2\alpha - e^{i\beta}G(z)}{e^{i\beta}G(z)}\right| &< \frac{\left|1-2\alpha e^{-i\beta}\right| + \sum\limits_{n=2}^{\infty} n\left|n-2\alpha e^{-i\beta}\right||a_n|}{1-\sum\limits_{n=2}^{\infty} n^2|a_n|} \\
&\leqslant \frac{\left|1-2\alpha e^{-i\beta}\right| + 1 - \left|1-2\alpha e^{-i\beta}\right| \sum\limits_{n=2}^{\infty} n^2|a_n|}{1-\sum\limits_{n=2}^{\infty} n^2|a_n|} \\
&= \frac{1-\sum\limits_{n=2}^{\infty} n^2|a_n|}{1-\sum\limits_{n=2}^{\infty} n^2|a_n|} = 1.
\end{aligned}$$

Therefore $f(z) \in K_\beta(\alpha)$ for some $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < Cos\beta$.

Taking $\beta = 0$ in Theorem 7, we obtain Theorem 8[4].

Theorem 8. *Let $0 < \alpha < 1$. If $f(z) \in A$ satisfies the following coefficient inequality:*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leqslant \frac{1}{2}(1-|1-2\alpha|) = \begin{cases} \alpha; & \left(0 < \alpha \leq \frac{1}{2}\right) \\ 1-\alpha; & \left(\frac{1}{2} \leq \alpha < 1\right) \end{cases},$$

then $f(z) \in N(\alpha)$.

Taking $\beta = \frac{\pi}{4}$ in Theorem 7, we can write the following Corollary 3.

Corollary 3. *If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty} n\left(n + \sqrt{n^2 - 2\sqrt{2}\alpha n + 4\alpha^2}\right)|a_n| \leqslant 1 - \sqrt{1 - 2\sqrt{2}\alpha + 4\alpha^2}$$

for some $0 < \alpha < \frac{\sqrt{2}}{2}$, then $f(z) \in K_{\frac{\pi}{4}}(\alpha)$.

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