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Higher-Order Duality for Multiobjective Programming Involving (Φ, ρ)-Invexity*

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Abstract

The concepts of (Φ, ρ)-invexity have been given by Caristi, Ferrara and Stefanescu[32]. We consider a higher-order dual model associated to a multiobjective programming problem involving support functions and a weak duality result is established under appropriate higher-order (Φ, ρ)-invexity conditions.

Keywords and Phrases: *Higher-order(Φ,ρ)-(pseudo/quasi)-convexity, Multiobjective programming, Higher-order duality, Duality theorem.*

1. Introduction

For nonlinear programming problems, a number of duals have been suggested among which the Wolfe dual [35,8] is well known. While studying duality under generalized convexity, Mond and Weir [36] proposed a number of different duals for

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nonlinear programming problems with nonnegative variables and proved various duality theorems under appropriate pseudo-convexity/quasi-convexity assumptions.

The study of second order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective function when approximations are used [10,16,24]. Mangasarian[12] considered a nonlinear programming problem and discussed second order duality under inclusion condition. Mond [14] was the first who present second order convexity. He also gave in [14] simpler conditions than Mangasarian using a generalized form of convexity. which was later called bonvexity by Bector and Chandra [2]. Further, Jeyakumar [37,30] and Yang [24] discussed also second order Mangasarian type dual formulation under ρ-convexity and generalized representation conditions respectively. In [20] Zhang and Mond established some duality theorems for second-order duality in nonlinear programming under generalized second-order B-invexity, defined in their paper. In [14] it was shown that second order duality can be useful from computational point of view, since one may obtain better lower bounds for the primal problem than otherwise. The case of some optimization problems that involve n-set functions was studied by Preda [38]. Recently, Yang et al. [24] proposed four second-order dual models for nonlinear programming problems and established some duality results under generalized second-order F -convexity assumptions. In [15] Mishra and Rueda generalized Zhang's Mangasarian type and Mond-Weir type higher-order duality [28] to higher-order type I functions. Yang et al. [26] extended this results to a class of nondifferentiable multiobjective programming problems. They also presented an unified higher-order dual model for nondifferentiable multiobjective programs, where every component of the objective function contains a support function of a compact convex set, also Batatorescu et al. [33].

For $\Phi(x,a,(y,r)) = F(x,a;y) + rd^2(x,a)$, where $F(x,a;.)$ is sublinear on R^n, the definition of (Φ,ρ) - invexity reduces to the definition of (F,ρ) -convexity introduced by Preda[29], which in turn Jeyakumar[30] generalizes the concepts of F-convexity and ρ -invexity[31].

The more recent literature, Xu[21], Ojha [27], Ojha and Mukherjee [22] for duality under generalized (F,ρ) -convexity, Mishra [23] and Yang et al.[24] for duality under second order F -convexity. Liang et al. [25] and Hachimi[26] for optimality criteria and duality involving (F,α,ρ,d) -convexity or generalized $\{F,\alpha,\rho,d)$ -type functions.The (F,ρ)-convexity was recently generalized to (Φ,ρ) - invexity by Caristi , Ferrara and Stefanescu [32],and here we will use this concept to extend some theoretical results of multiobjective programming.

Whenever the objective function and all active restriction functions satisfy simultaneously the same generalized invexity at a Kuhn-Tucker point which is an optimum condition, then all these functions should satisfy the usual invexity, too. This is not the case in multiobjective programming ; Ferrara and Stefanescu[28] showed that sufficiency Kuhn-Tucker condition can be proved under (Φ, ρ) -invexity, even if Hanson's invexity is not satisfied, Puglisi[34].The interested reader may consult[1,3,4,5,6,7,9,11,13,15,17,18,19,33,39,40,41,4243,44,45,46,47.48].

Therefore, the results of this paper are real extensions of the similar results known in the literature.

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Therefore, the results of this paper are real extensions of the similar results known in the literature.

In Section 2 we define the higher-order (Φ, ρ) -invexity . In Section 3 we consider a class of multiobjective programming problems and for the dual model we prove a weak duality result.

2. Notation and Preliminaries

we denote by R^n the n -dimensional Euclidean space, and by R^n_+ its nonnegative orthant . Further, $R^n_+ = \{x \in R \,|\, x > 0\}$ .For any vector $x \in R^n$, $y \in R^n$, we denote $x^T y = \sum_{i=1}^{n} x_i y_i$.

We consider $f : R^n \to R^p$, $g : R^n \to R^q$,are differential functions and $X \subset R^n$ is an open set. We define the following multiobjective programming problem:

(P) minimize $f(x) = \left(f_1(x) .... f_p(x) \right)$
$\quad subject\ to\ g(x) \geq 0x, x \in X$

Let X_0 be the set of all feasible solutions of (P) that is, $X_0 = \{x \in X \mid g(x) \geq 0\}$.

We quote some definitions and also give some new ones.

Definition 2.1

A vector $a \in X_0$ is said to be an efficient solution of problem (P) if there exit no $x \in X_0$ such that $f(a) - f(x) \in R_+^p \setminus \{0\}$ i.e., $f_i(x) \leq f_i(a)$ for all $i \in \{1,.,.,., p\}$, and for at least one $j \in \{1,.,.,., p\}$ we have $f_i(x) < f_i(a)$.

Definition 2.2

A point $a \in X_0$ is said to be a weak efficient solution of problem (VP) if there is no $x \in X$ such that $f(x) < f(a)$.

Definition 2.3

A point $a \in X_0$ is said to be a properly efficient solution of (VP) if it is efficient and there exist a positive constant K such that for each $x \in X_0$ and for each $i \in \{1, 2......p\}$ $satisfying$ $f_i(x) < f_i(a)$, there exist at least one $i \in \{1, 2......p\}$ $such\ that$ $f_j(a) < f_j(x)$ and $f_i(a) - f_i(x) \leq K\left(f_j(x) - f_j(a) \right)$ Denoting by WE(P), E(P) and PE(P) the sets of all weakly efficient, efficient and properly efficient solutions of (VP), we have $\mathrm{PE(P)} \subseteq \mathrm{E(P)} \subseteq \mathrm{WE(P)}$.

We denote by $\nabla f(a)$ the gradient vector at a of a differentiable function $f : R^p \to R$, and by $\nabla^2 f(a)$ the Hessian matrix of f at a. For a real valued twice differentiable function $\psi(x, y)$ defined on an open set in $R^p \times R^q$, we denote by $\nabla_x \psi(a,b)$ the gradient vector of ψ with respect to x at (a,b), and by $\nabla_{xx}\psi(a,b)$ the Hessian matrix with respect to x at (a,b). Similarly, we may define $\nabla_y \psi(a,b)$, $\nabla_{xy}\psi(a,b)$ and $\nabla_{yy}\psi(a,b)$ .For convenience, let us write the definitions of (Φ, ρ)-invexity from[32], Let $\varphi : X_0 \to R$ be a differentiable function $(X_0 \subseteq R^n)$, $X \subseteq X_0$, and $a \in X_0$. An element of all (n+1)- dimensional Euclidean Space R^{n+1} is represented as the ordered pair (z, r) with $z \in R^n$ and $r \in R, \rho$ is a real number and Φ is real valued function defined on $X_0 \times X_0 \times R^{n+1}$, *such that* $\Phi(x,a,.)$ *is* convex on R^{n+1} *and* $\Phi(x,a,(0,r)) \geq 0$, for every $(x,a) \in X_0 \times X_0$ and $r \in R_+$, $h: X \times R^n \to R$ be differentiable function.

Definition 2.4

A function $f: X \to R$ is said to be higher-order (Φ,ρ) -invex at $u \in X$ with respect to h ,both f and h are differentiable function, if for all $(x,y) \in X \times R^n$, $\Phi: X \times X \times R^{n+1} \to R$, ρ is a real number, we have,

$$\{f(x) - f(u) - h(u,y) + y^T \nabla_y h(u,y)\} \geqq \Phi(x,u;(\nabla f(u) + \nabla_y h(u,y), \rho)) \qquad (2.1)$$

Definition 2.5

A function $f: X \to R$ is said to be higher-order (Φ,ρ) -incave at $u \in X$ with respect to h ,both f and h are differentiable function, if for all $(x,y) \in X \times R^n$, $\Phi: X \times X \times R^{n+1} \to R$, ρ is a real number, we have,

$$\{f(x) - f(u) - h(u,y) + y^T \nabla_y h(u,y)\} \geqq \Phi(x,u;(-\nabla f(u) - \nabla_y h(u,y), \rho))$$

Definition 2.6

A function $f: X \to R$ is said to be higher-order (Φ,ρ) -pseudoinvex at $u \in X$ with respect to h ,both f and h are differentiable function, if for all $(x,y) \in X \times R^n$, $\Phi: X \times X \times R^{n+1} \to R$, ρ is a real number, we have

$$\Phi(x,u;(\nabla f(u) + \nabla_y h(u,y), \rho)) \geqq 0 \Rightarrow \{f(x) - f(u) - h(u,y) + y^T \nabla_y h(u,y)\} \geqq 0 \qquad (2.2)$$

Definition 2.7

A function $f: X \to R$ is said to be higher-order (Φ,ρ) -quasiinvex at $u \in X$ with respect to h ,both f and h are differentiable function, if for all $(x,y) \in X \times R^n$, $\Phi: X \times X \times R^{n+1} \to R$, ρ is a real number, we have,

$$\{f(x) - f(u) - h(u,y) + y^T \nabla_y h(u,y)\} \leqq 0 \Rightarrow \Phi(x,u;(\nabla f(u) + \nabla_y h(u,y), \rho)) \leqq 0 \qquad (2.3)$$

Remark 2.1

(i) If we consider the case, $\Phi(x,u;(\nabla f(u),\rho)) = F(x,u;\nabla f(u))$ (with F is sublinear in third argument, then the above definition reduce to Definition 4 of Chandra et al.[4] .

Example 2.1

We present here a function which is higher-order (Φ,ρ)-invex. Let us consider $X=(0,\infty)$ and

$f:X\to R$, $f(x)=x\log x$, $h:X\times R\to R, h(u,y)=-y\log u$. We have

$\nabla_u f(u)=1+\log u, \nabla_{uu} f(u)=\dfrac{1}{u}, \nabla_y h(u,y)=-\log u$, $\Phi: X\times X\times R^{n+1}\to R$, taking $\rho=0$ $\Phi(x,y;b)=|b|+|b|^2$.

It is obvious our mapping is more generalized rather than previous ones.

Hence $f(x)=x\log x$ is higher-order (Φ,ρ)-invex at $u\in X$, with respect to $h(u,y)=-y\log u$.

3. Higher-order Mond-Weir type Symmetric Duality

We consider in this section twice differentiable functions $f_i=R^n\times R^m\to R, g_i=R^n\times R^m\times R^n\to R, h_i=R^n\times R^m\times R^m\to R$, and compact convex sets $C_i\subset R^n$ and $D_i\subset R^m$, for $i=1,2,.,.,p$.

We define the following pair of higher-order symmetric multiobjective dual problems.

(MP)

minimize

$$\begin{pmatrix} f_1(x,y)+h_1(x,y,\pi_1)-\pi_1^T(\nabla_{\pi_1} h_1(x,y,\pi_1)) \\ \vdots \\ \vdots \\ f_p(x,y)+h_p(x,y,\pi_p)-\pi_p^T(\nabla_{\pi_p} h_p(x,y,\pi_p)) \end{pmatrix}$$

subject to

$$\sum_{i=1}^{p}\lambda_i(\nabla_y f_i(x,y)+\nabla_{\pi_i} h_i(x,y,\pi_i))\le 0 \qquad (3.1)$$

$$y^T \sum_{i=1}^{p} \lambda_i (\nabla_y f_i(x,y) + \nabla_{\pi_i} h_i(x,y,\pi_i)) \geq 0 \tag{3.2}$$

$$i = 1,.,., p, \lambda > 0, \sum_{i=1}^{p} \lambda_i = 1 \tag{3.3}$$

(MD)

maximize

$$\begin{pmatrix} f_1(u,v) + g_1(u,v,\mu_1) - \mu_1^T (\nabla_{\mu_1} g_1(u,v,\mu_1)) \\ : \\ : \\ f_p(u,v) + g_p(u,v,\mu_p) - \mu_p^T (\nabla_{\mu_p} g_p(u,v,\mu_p)) \end{pmatrix}$$

subject to

$$\sum_{i=1}^{p} \lambda_i (\nabla_u f_i(u,v) + \nabla_{\mu_i} g_i(u,v,\mu_i)) \geq 0 \tag{3.4}$$

$$u^T \sum_{i=1}^{p} \lambda_i (\nabla_u f_i(u,v) + \nabla_{\mu_i} g_i(u,v,\mu_i)) \leq 0 \tag{3.5}$$

$$i = 1,.,., p, \lambda > 0, \lambda^T e = 1 \tag{3.6}$$

In the sequel we shall establish weak, strong and converse duality theorems under (Φ, ρ)-univex type assumptions. For this, the number $. \rho_i \in R, i = 1, 2,.,., p$ .Further, we suppose that the functions $\Phi_0 : R^n \times R^n \times R^{n+1} \to R$ and $\Phi_1 : R^n \times R^n \times R^{n+1} \to R$ also satisfy the condition

$\Phi_0(x,u;(\xi,\rho)) + u^T \xi \geq 0$, for all $\xi \in R_+^n$

$\Phi_1(v,y;(\zeta,\rho')) + y^T \zeta \geq 0$, for all $\zeta \in R_+^m$ (3.7)

We suppose also that following conditions are satisfied:

(1) the functions $f_i(.,v)$ are higher-order (Φ_0, ρ)-invex at u .

(2) $f_i(x,.)$ are higher-order (Φ_1, ρ)-incave at y .

Theorem 3.1 (Weak duality)

Let $(x, y, \lambda, \pi_1, \pi_2, ., ., \pi_p)$ *be a feasible solution of (MP) and* $(u, v, \lambda, \mu_1, \mu_2, ., ., \mu_p)$ *a feasible solution of (MD). Then the inequalities can not hold simultaneously:*

(i) for all $i \in \{1, 2, ., ., p\}$,

$$f_i(x, y) + h_i(x, y, \pi_i) - \pi_i^T(\nabla_\pi h_i(x, y, \pi_i)) \le f_i(u, v) + g_i(u, v, \mu_i) - \mu_i^T(\nabla_\mu g_i(u, v, \mu_i)) \tag{3.8}$$

(ii) for at least one $j \in \{1, 2, ., ., p\}$,

$$f_i(x, y) + h_i(x, y, \pi_i) - \pi_i^T(\nabla_\pi h_i(x, y, \pi_i)) < f_i(u, v) + g_i(u, v, \mu_i) - \mu_i^T(\nabla_\mu g_i(u, v, \mu_i)) \tag{3.9}$$

Proof.

Since, $(x, y, \lambda, \pi_1, \pi_2, ., ., \pi_p)$ be a feasible solution of (MP) and $(u, v, \lambda, \mu_1, \mu_2, ., ., \mu_p)$ a feasible solution of (MD), by (3.7) and (3.4), we get

$$\Phi_0(x, u; (\sum_{i=1}^{p} \lambda_i \{\nabla_u f_i(u, v) + \nabla_\mu g_i(u, v, \mu_i)\}, \rho_i)) + u^T \sum_{i=1}^{p} \lambda_i \{\nabla_u f_i(u, v) + \nabla_\mu g_i(u, v, \mu_i)\} \ge 0$$

By (3.5) we have

$$\Phi_0(x, u; (\sum_{i=1}^{p} \lambda_i \{\nabla_u f_i(u, v) + \nabla_\mu g_i(u, v, \mu_i)\}, \rho_i)) \ge 0 \tag{3.10}$$

It follows from the higher-order (Φ_0, ρ)-invexity of $f_i(., v)$ at u with respect to $g_i(u, v, \mu_i)$ that

$$\{f_i(x, v) - f_i(u, v)\} \ge$$

$$\Phi_0(x, u; (\sum_{i=1}^{p} \lambda_i \{\nabla_u f_i(u, v) + \nabla_\mu g_i(u, v, \mu_i)\}, \rho_i)) + \{g_i(u, v, \mu_i) - \mu_i^T \nabla_\mu g_i(u, v, \mu_i)\} \tag{3.11}$$

Since $\lambda > 0, \lambda^T e = 1$, from (3.4),(3.10) and (3.11), we get

$$\sum_{i=1}^{p} \lambda_i \{f_i(x, v) - f_i(u, v)\} \ge$$

$$\Phi_0(x, u; (\sum_{i=1}^{p} \lambda_i \{\nabla_u f_i(u, v) + \nabla_\mu g_i(u, v, \mu_i)\}, \rho_i)) + \sum_{i=1}^{p} \lambda_i \{g_i(u, v, \mu_i) - \mu_i^T \nabla_\mu g_i(u, v, \mu_i)\}$$

That is

$$\sum_{i=1}^{p}\lambda_i f_i(x,v) \geq \sum_{i=1}^{p}\lambda_i f_i(u,v) + \sum_{i=1}^{p}\lambda_i\{g_i(u,v,\mu_i)-\mu_i^T\nabla_\mu g_i(u,v,\mu_i)\} \tag{3.12}$$

On the other hand , from (3.1) and (3.7) we get

$$\Phi_1(v,y;(-\sum_{i=1}^{p}\lambda_i(\nabla_y f_i(x,y)+\nabla_\pi h_i(x,y,\pi_i),\rho_i)) - y^T\sum_{i=1}^{p}\lambda_i(\nabla_y f_i(x,y)+\nabla_\pi h_i(x,y,\pi_i) \geq 0 ,$$

which, by using (3.2), imply

$$\Phi_1(v,y;(-\sum_{i=1}^{p}\lambda_i(\nabla_y f_i(x,y)+\nabla_\pi h_i(x,y,\pi_i),\rho_i)\geq 0 \tag{3.13}$$

Now, using the fact that $f_i(x,.)$ is higher-order (Φ_1,ρ) -incavity at y , with respect to $-h_i(x,y,\pi_i), i=1,2,.,.,p$, we have,

$$-\{f_i(x,v)-f_i(x,y)\}\geq$$

$$\Phi_1(v,y;(-\sum_{i=1}^{p}\lambda_i(\nabla_y f_i(x,y)+\nabla_\pi h_i(x,y,\pi_i),\rho_i)) + \{-h_i(x,y,\pi_i)+\pi_i^T\nabla_\pi h_i(x,y,\pi_i)\} \tag{3.14}$$

Since $\lambda>0, \lambda^T e=1$, from (3.13)and (3.14) , we get

$$\sum_{i=1}^{p}\lambda_i f_i(x,v) \leq \sum_{i=1}^{p}\lambda_i f_i(x,y) + \sum_{i=1}^{p}\lambda_i\{h_i(x,y,\pi_i)-\pi_i^T\nabla_\pi h_i(x,y,\pi_i)\} \tag{3.15}$$

from (3.12) and (3.15) we obtain

$$\sum_{i=1}^{p}\lambda_i\{f_i(u,v)+g_i(u,v,\mu_i)-\mu_i^T\nabla_\mu g_i(u,v,\mu_i)\} \leq \sum_{i=1}^{p}\lambda_i[f_i(x,y)+h_i(x,y,\pi_i)-\pi_i^T\nabla_\pi h_i(x,y,\pi_i)]$$

which proves the assertion of the theorem.

Remark 3.2.

Following the same lines as in the previous proof, we easily can prove other variants of Theorem 3.1 under the same assumptions, but replacing in the statement the corresponding conditions by those below:

(1) the functions $f_i(.,v)$ are higher-order (Φ_0, ρ) -pseudoinvex at u , with respect to $g_i(u, v, \mu_i), i = 1, 2,.,., p$;

(2) $f_i(x,.)$ are higher-order (Φ_1, ρ) -incave at y , with respect to

$-h_i(x, y, \pi_i), i = 1, 2,.,., p$; respectively

(3) the functions $f_i(.,v)$ are higher-order (Φ_0, ρ) -pseudoinvex at u , with respect to $g_i(u, v, \mu_i), i = 1, 2,.,., p$;

(4) $f_i(x,.)$ are higher-order (Φ_1, ρ) -incave at y , with respect to

$-h_i(x, y, \pi_i), i = 1, 2,.,., p$; respectively

Now, under appropriate conditions, we state a strong duality and a converse duality theorem relative to problems, (MP) and (MD).

Theorem 3.2 (Strong duality)

Let $(x', y', \lambda', \pi_1', \pi_2',.,., \pi_p')$ *be a feasible solution of (MP) and assume that*

(i) for all $i \in \{1, 2,.,., p\}$ *we have*

$$h_i(x', y', 0) = 0, g_i(x', y', 0) = 0, \nabla_\pi h_i(x', y', 0) = 0,$$
$$\nabla_y h_i(x', y', 0) = 0, \nabla_x h_i(x', y', 0) = \nabla_\mu g_i(x', y', 0) \ ;$$

(ii) for all $i \in \{1, 2,.,., p\}$ *the Hessian matrix* $\nabla_{\pi\pi} h_i(x', y', \pi_i')$ *is positive or negative definite ;*

(iii) the vectors $\nabla_y f_i(x', y') + \nabla_\pi h_i(x', y', \pi_i'), i = 1, 2,.\ p$ *, are linearly independent;*

(iv) for any $\beta \in R_+^p, \beta \neq 0$ *, and* $\pi_i \in R^m, \pi_i \neq 0, i = 1, 2,.,., p$ *, we have*

$$\sum_{i=1}^{p} \beta_i \pi_i^T \{\nabla_y f_i(x', y') + \nabla_\pi h_i(x', y', \pi_i)\} \neq 0$$ *, then*

(a) $\pi_i' = 0, i = 1, 2,.,., p$ *;*

(b) there exist $w_i' \in C_i$ *such that* $(x', y', \lambda', 0_1, 0_2,.,.,0_p)$ *is a feasible solution of (MD).*

Furthermore, if the assumptions of Theorem 3.1 are satisfied and the functions $b_i(x', y', u', v') > 0, i = 1, 2, ., ., p$, then $(x', y', \lambda', 0_1, 0_2, ., ., 0_p)$ is a properly efficient solution of (MD) and the values of both problems are equal.

Theorem 3.3 (Converse duality)

Let $(u', v', \lambda', \mu_1', \mu_2', ., \mu_p')$ be a properly efficient solution of (MD) and assume that

(i)for all $i \in \{1, 2, ., ., p\}$ we have $h_i(u', v', 0) = 0, g_i(u', v', 0) = 0, \nabla_\mu g_i(u', v', 0) = 0,$ $\nabla_x g_i(u', v', 0) = 0, \nabla_y g_i(u', v', 0) = \nabla_\pi h_i(u', v', 0)$;

(ii) for all $i \in \{1, 2, ., ., p\}$ the Hessian matrix $\nabla_{\mu\mu} g_i(u', v', \mu_i')$ is positive or negative definite ;

(iii)the vectors $\nabla_x f_i(u', v') + \nabla_\mu g_i(u', v', \mu_i'), i = 1, 2, .\ p$, are linearly independent;

(iv)for any $\beta \in R_+^p, \beta \neq 0$, and $\mu_i \in R^n, \mu_i \neq 0, i = 1, 2, ., ., p$, we have

$$\sum_{i=1}^{p} \beta_i \mu_i^T \{\nabla_x f_i(u', v') + \nabla_\mu g_i(u', v', \mu_i')\} \neq 0,$$ *then*

(a) $\mu_i = 0, i = 1, 2, ., ., p$;

(b) there exist $z_i' \in D_i$ such that $(u', v', \lambda', 0_1, 0_2, ., ., 0_p)$ is a feasible solution of (MP).

Furthermore, if the assumptions of Theorem 3.1 are satisfied and the functions $b_i(x', y', u', v') > 0, i = 1, 2, ., ., p$, then $(u', v', \lambda', 0_1, 0_2, ., ., 0_p)$ is a properly efficient solution of (MP) and the values of both problems are equal.

References

[1] I. Ahmad and Z. Husain, Nondifferentiable second order symmetric duality in multiobjective programming, *Applied Mathematics Letters*, **18** (2005), 721–728.

[2] C. R. Bector and S. Chandra, Generalized bonvexity and higher order duality for fractional programming, *Opsearch*, **24** (1987), 143–154.

[3] S. Chandra, B. D. Craven, and B. Mond, Generalized concavity and duality with a square root term, *Optimization*, **16** (1985), 653–662.

[4] S. Chandra, A. Goyal, and I. Husain, On symmetric duality in mathematical programming with F-convexity, *Optimization*, **43** (1998), 1–18.

[5] S. Chandra and I. Husain, Nondifferentiable symmetric dual programs, *Bull. Austral. Math. Soc.*, **24** (1981), 295–307.

[6] S. Chandra and D. Prasad, Symmetric duality in multiobjective programming, *J. Austral. Math. Soc.*, **35** Ser. B (1993), 198–206.

[7] B. D. Craven, Lagrangian conditions and quasiduality, *Bull. Austral. Math. Soc.*, **16** (1977), 325–339.

[8] W. S. Dorn, A symmetric dual theorem for quadratic programs, *J. Oper. Res.Soc.*, **2** Japan (1960), 93–97.

[9] T. R. Gulati, I. Ahmad, and I. Husain, Second order symmetric duality with generalized convexity, *Opsearch*, **38** (2001), 210–222.

[10] T. R. Gulati, I. Husain, and A. Ahmed, Multiobjective symmetric duality with invexity, *Bull. Austral. Math. Soc.*, **56** (1997), 25–36.

[11] D. S. Kim, Y. B. Yun, and H. Kuk, Second order symmetric and self duality in multiobjective programming, *Appl. Math. Lett.*, **10** (1997), 17–22.

[12] O. L. Mangasarian, Second and higher order duality in nonlinear programming, *J. Math. Anal. Appl.*, **51** (1975), 607–620.

[13] S. K. Mishra, Second order symmetric duality in mathematical programming with F-convexity, *European J. Oper. Res.*, **127** (2000), 507–518.

[14] B. Mond, Second order duality for nonlinear programs, *Opsearch,* **11** (1974), 90–99.

[15] B. Mond, I. Husain, and M. V. Durga Prasad, Duality for a class of nondifferentiable multiobjective programming, *Util. Math.*, **39** (1991), 3–19.

[16] S. K. Suneja, C. S. Lalitha, and S. Khurana, Second order symmetric duality in multiobjective programming, *European. J. Oper.Res.*, **144** (2003), 492–500.

[17] P. S. Unger and A. P. Hunter Jr. , The dual of the dual as a linear approximation of the primal, *Int. J. Syst. Sci.*, **12** (1974), 1119–1130.

[18] T. Weir and B. Mond, Symmetric and self duality in multiobjective programming, *Asia-Pacific J. Oper. Res.,* **5** (1988), 124–133.

[19] X. M. Yang and S.H. Hou, Second order symmetric duality in multiobjective programming, Appl. Math. Lett. 14 (2001)587–592.

[20] J. Zhang and B. Mond, Second order B-invexity and duality in mathematical programming, *Utilitas Math.*, **50** (1996), 19–31.

[21] Z. Xu, Mixed type duality in multiobjective programming problems, *J. Math.Anal.Appl.*, **198** (1995), 621-635.

[22] D. B. Ojha and R. N. Mukherjee, Some results on symmetric duality of multiobjective programmes with (F, ρ)-invexity, *European Journal of Operational Reaearch*, **168** (2006), 333-339.

[23] S. K. Mishra, Second order symmetric duality in mathematical programming with F-convexity, *European Journal of Operational Reaearch,* **127** (2000), 507-518.

[24] X. M. Yang, X. Q. yang, and K. L. Teo, nodifferentiable second order symmetric duality in mathematical programming with F-convexity, *European Journal of Operational Reaearch,* **144** (2003), 554-559.

[25] Z. A. Liang, H. X. Huang, and P. M. Pardalos, Efficiency conditions and duality for a class of multiobjective fractional programming problems, *Journal of Global Optimization,* **27** (2003), 447-471.

[26] M. Hachimi, Sufficiency and duality in differentiable multiobjective programming involving generalized type-I functions, *J. Math.Anal. Appl.*, **296** (2004), 382-392.

[27] D. B. Ojha, Some results on symmetric duality on mathematical fractional programming with generalized F-convexity in complex spaces, *Tamkang Journal of Math*, **vol.36** No.2 (2005).

[28] M. Ferrara and M. V. Stefanescu, Optimality condition and duality in multiobjective programming with (Φ, ρ) -invexity, *Yugoslav Journal of Operations Research*, **vol.18** No.2 (2008), 153-165.

[29] V. Preda, On efficiency and duality for multiobjective programs, *J. Math. Anal. Appl.*, **166** (1992), 365-377.

[30] V. Jeyakumar, Strong and weak invexity in mathematical programming, *In: Methods of Operations Research*, **vol.55** (1985), 109-125.

[31] J. P. Vial, Strong and weak convexity of sets and functions, *Math. Operations Research*, **8** (1983), 231-259.

[32] G. Caristi, M. Ferrara, and A. Stefanescu, Mathematical programming with (Φ, ρ) -invexity, *In: V.Igor,Konnov, Dinh The Luc,Alexander, M.Rubinov,(eds.), Generalized Convexity and Related Topics, Lecture Notes in Economics and Mathematical Systems*, **vol.583** Springer (2006), 167-176.

[33] A. Puglisi, Generalized convexity and invexity in optimization theory: Some new results, *Applied Mathematical Sciences*, **vol.3** No.47 (2009), 2311-2325.

[34] W. S. Dorn, Self dual quadratic programs, *SIAM J. Appl. Math.*, **9** (1961), 51-54.

[35] M. Hanson and B. Mond, Further generalization of convexity in mathematical programming, *J. Inform. Optim. Sci.*, **3** (1982), 22-35.

[36] B. Mond and T. Weir, *Generalized convexity and duality*, In: S. Schaible, W. T. Ziemba(Eds.), Generalized convexity in optimization and Economics,263-280,Academic Press, New York, 1981.

[37] V. Jeyakumar, *p*-convexity and second order duality, *Utilitas Math.* **29** (1986), 71-85.

[38] V. Preda, Duality for multiobjective fractional programming problems involving n-set functions, *In: C. A. Cazacu, W. E. Lehto and T. M. Rassias(Eds.)Analysis and Topology*, **Academic Press** (1998), 569-583.

[39] X. Chen, Higher-order symmetric duality in nondifferentiable multiobjective programming problems, *J. Math. Anal. Appl.*, **290** (2004), 423–435.

[40] G. Devi, Symmetric duality for nonlinear programming problem involving η-bonvex functions, *European J. Oper. Res.*, **104** (1998), 615–621.

[41] D. J. Mahajan, *Contributions to optimality conditions and duality theory in nonlinear programming*, Ph. D. Thesis, 1977.

[42] S. K. Mishra and N. G. Rueda, Higher order generalized invexity and duality in mathematical programming, *J. Math. Anal. Appl.*, **247** (2000), 173–182.

[43] B. Mond and M. Schechter, Nondifferentiable symmetric duality, *Bull. Austral. Math.Soc.* **53** (1996), 177–188.

[44] B. Mond and J. Zhang, *Higher order invexity and duality in mathematical programming*, In: J.P. Crouzeix et al. (Eds.), Generalized Convexity, Generalized Monotonicity: Recent Results, pp. 357–372. Kluwer, Dordrecht, 1998.

[45] S. Pandey, Duality for multiobjective fractional programming involving generalized η-bonvex functions, *Opsearch*, **28** (1991), 36–43.

[46] R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, 1970.

[47] X. Q. Yang, Second order global optimality conditions for convex composite optimization, *Math. Programming*, **81** (1998), 327–347.

[48] A. Batatorescu, V. Preda and M. Beldiman , Higher-order symmetric multiobjective duality involving generalized (F,ρ,γ,b) -convexity, *Rev. Rou. Math. Pur. Appl.*, **52** no.6 (2007), 619-630.