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On the Solutions of Integral Equations of Fredholm type with Special Functions *

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Abstract

In the present paper, we study the various useful methods of solving the one-dimensional integral equation of Fredholm type. We first solve an integral equation involving the product of multivariable H-function by the application of fractional calculus theory. Further the Fredholm integral equation involving the product of I-functions in the kernel is also considered by the Mellin transform techniques. The results obtained here are general in nature and capable of yielding a large number of results (new or known) hitherto scattered in the literature.

Keywords and Phrases: *Fredholm type integral equations, Riemann-Liouville fractional integral, Weyl fractional integral, Mellin inversion theorem, Multivariable H-function, I-function.*

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1. Introduction and Definitions

In the last several years a large number of Fredholm type integral equations involving various polynomials or special functions as their kernels have been studied by many authors notably Buchman [1], Higgins [5], Love ([7] and [8]), Prabhakar and Kashyap [10], Srivastava and Buchman [14], Srivastava and Raina [17], Chaurasia and Patni [2] and others, In the present paper, we obtain the solutions of the following Fredholm integral equations

$$\int_0^\infty y^{-\alpha} H^{0,\lambda' : (\alpha',\beta') ;...; (\alpha^{(r)},\beta^{(r)})}_{A',C' : (M',N') ;...; (M^{(r)},N^{(r)})} \left[\begin{matrix} [(g_j):\gamma',...,\gamma^{(r)}]_{1,A'} : (q',\eta')_{1,M'} ;...; \\ [(f_j):\xi',...,\xi^{(r)}]_{1,C'} : (p',\varepsilon')_{1,N'} ;...; \end{matrix} \right.$$

$$\left. \begin{matrix} (q^{(r)},\eta^{(r)})_{1,M^{(r)}} ; \\ (p^{(r)},\varepsilon^{(r)})_{1,N^{(r)}} ; \end{matrix} u_1(x/y)^p, ..., u_r(x/y)^p \right]$$

$$\times H^{0,\lambda : (u',v') ;...; (u^{(r)},v^{(r)})}_{A,C : (B',D') ;...; (B^{(r)},D^{(r)})} \left[\begin{matrix} [(a_j):\theta',...,\theta^{(r)}] : (b',\phi')_{1,B'} ;...; \\ [(c_j):\psi',...,\psi^{(r)}] : (d',\delta')_{1,D'} ;...; \end{matrix} \right.$$

$$\left. \begin{matrix} (b^{(r)},\phi^{(r)})_{1,B^{(r)}} ; \\ (d^{(r)},\delta^{(r)})_{1,D^{(r)}} ; \end{matrix} z_1(x/y)^q, ..., z_r(x/y)^q \right] f(y)\, dy = g(x) \; (0 < x < \infty) \quad (1)$$

and

$$\int_0^\infty y^{-\alpha} I^{m',n'}_{p'_i,q'_i:r'} \left[u\,(x/y)^p \left| \begin{matrix} (a'_j,\alpha'_j)_{1,n'} ; (a'_{ji},\alpha'_{ji})_{n'+1,p'_i} \\ (b'_j,\beta'_j)_{1,m'} ; (b'_{ji},\beta'_{ji})_{m'+1,q'_i} \end{matrix} \right. \right]$$

$$\times I^{m,n}_{p_i,q_i:r} \left[z\,(x/y)^q \left| \begin{matrix} (a_j,\alpha_j)_{1,n} ; (a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m} ; (b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] f(y)\, dy = g(x) \; (0 < x < \infty). \quad (2)$$

The series representation of the multivariable H-function [16] is given by Olkha and Chaurasia [2] as

$$H\,[u_1, ..., u_r] = H^{0,\lambda' : (\alpha',\beta') ;...; (\alpha^{(r)},\beta^{(r)})}_{A',C' : (M',N') ;...; (M^{(r)},N^{(r)})}$$

$$\times \left[\begin{matrix} [(g_j):\gamma',...,\gamma^{(r)}]_{1,A'} : (q',\eta')_{1,M'} ;...; (q^{(r)},\eta^{(r)})_{1,M^{(r)}} ; \\ [(f_j):\xi',...,\xi^{(r)}]_{1,C'} : (p'\varepsilon')_{1,N'} ;...; (p^{(r)},\varepsilon^{(r)})_{1,N^{(r)}} ; \end{matrix} u_1, ..., u_r \right]$$

$$= \sum_{m_i=1}^{\alpha^{(i)}} \sum_{n_i=0}^{\infty} \phi_1\, \phi_2\, \frac{\prod_{i=1}^{r} (u_i)^{U_i} (-1)^{\sum_{i=1}^{r} n_i}}{\prod_{i=1}^{r} (\varepsilon^{i}_{m_i} n_i\,!)}\,, \quad (3)$$

where

$$\phi_1 = \frac{\prod_{j=1}^{\lambda'} \Gamma\left(1 - g_j + \sum_{i=1}^{r} \gamma_j^{(i)} U_i\right)}{\prod_{j=\lambda'+1}^{A} \Gamma\left(g_j - \sum_{i=1}^{r} \gamma_j^{(i)} U_i\right) \prod_{j=1}^{C'} \Gamma\left(1 - f_j + \sum_{i=1}^{r} \xi_j^{(i)} U_i\right)}, \tag{4}$$

$$\phi_2 = \frac{\prod_{j=1, j\neq m_i}^{\alpha^{(i)}} \Gamma(p_j^{(i)} - \varepsilon_j^{(i)} U_i) \prod_{j=1}^{\beta^{(i)}} \Gamma(1 - q_j^{(i)} + \eta_j^{(i)} U_i)}{\prod_{j=\alpha^{(i)}+1}^{N^{(i)}} \Gamma(1 - p_j^{(i)} + \varepsilon_j^{(i)} U_i) \prod_{j=\beta^{(i)}+1}^{M^{(i)}} (q_j^{(i)} - \eta_j^{(i)} U_i)} \tag{5}$$

and

$$U_i = \frac{p_{m_i}^{(i)} + n_i}{\varepsilon_{m_i}^{(i)}}, \; i = 1, ..., r, \tag{6}$$

which is valid under the following conditions

$$\varepsilon_{m_i}^{(i)}[p_j^{(i)} + p_i] \neq \varepsilon_j^{(i)} [p_{m_i}^{(i)} + n_i] \tag{7}$$

for $j = m_i,\ m_i = 1, ..., \alpha^{(i)};\ p_i,\ n_i = 0, 1, 2, ... ;\ u_i \neq 0,$

$$\nabla_i = \sum_{j=1}^{A'} \gamma_j^{(i)} - \sum_{j=1}^{C'} \xi_j^{(i)} + \sum_{j=1}^{B^{(i)}} \eta_j^{(i)} - \sum_{j=1}^{D^{(i)}} \varepsilon_j^{(i)} < 0, \; \forall\, i \in \{1, ..., r\}. \tag{8}$$

The series representation of I-function ([11] and [12]) is given by

$$I_{p_i',q_i':r'}^{m_i',n_i'}[u] = I_{p_i',q_i':r'}^{m_i',n_i'}\left[u \left| \begin{matrix} (a_j',\alpha_j')_{1,n'} ; (a_{ji}',\alpha_{ji}')_{n'+1,p_i'} \\ (b_j',\beta_j')_{1,m'} ; (b_{ji}',\beta_{ji}')_{m'+1,q_i'} \end{matrix} \right. \right]$$

$$= \sum_{h=1}^{m'} \sum_{k=0}^{\infty} \frac{(-1)^{-k} \phi'(\eta_{h,k})\, u^{\eta_{h,k}}}{\beta_h \; k\,!}, \tag{9}$$

where

$$\phi'(\eta_{h,k}) = \frac{\prod_{j=1}^{m'} \Gamma(b_j' - \beta_j' \eta_{h,k}) \prod_{j=1}^{n'} \Gamma(1 - a_j' + \alpha_j' \eta_{h,k})}{\sum_{i=1}^{r} \left\{ \prod_{j=m'+1}^{q_i'} [(1 - b_{ji}' + \beta_{ji}' \eta_{h,k}) \prod_{j=n'+1}^{p_i'} \Gamma(a_{ji}' - \alpha_{ji}' \eta_{h,k}) \right\}} \tag{10}$$

and

$$\eta_{h,k} = \frac{b_h' + k}{\beta_h'}.$$

Let $\int$ denote the space of all functions f which are defined on R^+ $[0,\infty)$ and satisfy
(i) $f \in \wp\,(R^+)$,
(ii) $\lim\limits_{x\to\infty} [x^\gamma\, f^r(x)] = 0$ for all non-negative integers γ and r.
(iii) $f(x) = 0(1)$ as $x \to 0$.
For correspondence to the space of good functions defined on the whole real line $(-\infty,\infty)$ see Lighthill [6].
The Riemann-Liouville fractional integral (of order μ) is defined by

$$D^{-\mu}\{f(x)\} = {}_0D_x^{-\mu}\{f(x)\} = \frac{1}{\Gamma(\mu)}\int_0^x (x-\omega)^{\mu-1} f(\omega)\, d\omega$$

$$(\mathrm{Re}(\mu) > 0 : f \in \int), \tag{11}$$

where $D^\mu\{f(x)\} = \phi(x)$is understood to mean that ϕ is a locally integrable solution of $f(x) = D^{-\mu}\{\phi(x)\}$,implying that D^μ is the inverse of the fractional operator $D^{-\mu}$ (whenever necessary, we shall simply write $D_x^{-\mu}$ for ${}_0D_x^{-\mu}$for the Riemann-Liouville fractional integral operator defined by (11) above).
The Weyl fractional integral (of order h) is defined by

$$W^{-h}\{f(x)\} = {}_xD_\infty^{-h}\{f(x)\}$$

$$= \frac{1}{\Gamma(h)}\int_x^\infty (\xi - x)^{h-1} f(\xi)\, d\xi\,,\ (\mathrm{Re}(h) > 0\,;\ f \in \int). \tag{12}$$

2. Solution of the Integral Equation Associated with Product of Multivariable H-functions

Lemma 1. *Let (i)* λ, $u^{(i)}$, $v^{(i)}$, A, $B^{(i)}$, C, $D^{(i)}$ *be positive integers such that* $0 \le \lambda \le A$, $0 \le u^{(i)} \le D^{(i)}$, $C \ge 0$, and $0 \le v^{(i)} \le B^{(i)}$, $i = 1,...,r$;
(ii) $\mathrm{Re}(\alpha) > \mathrm{Re}(\beta)$; $\mathrm{Re}\left[\beta + q\sum_{i=1}^r\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] > 0\,(j = 1,...,u^{(i)};\ q > 0)$;
(iii) $|\arg\,(z_i)| < \frac{1}{2}\pi\, T_i$, *where*

$$T_i = \sum_{j=\lambda+1}^{A}\theta_j^{(i)} + \sum_{j=1}^{v^{(i)}}\phi_j^{(i)} - \sum_{j=v^{(i)}+1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{u^{(i)}}\delta_j^{(i)} - \sum_{j=u^{(i)}+1}^{D^{(i)}}\delta_j^{(i)} > 0,$$

$$\forall\, i \in (1, ..., r);$$

(iv) $\varepsilon_{m_i}[p_j^{(i)} + p_i] \neq \varepsilon_j^{(i)}[p_{m_i}^{(i)} + n_i]$, for $j \neq m_i$

$$m_i = 1, ..., \alpha^{(i)};\ p_i,\ n_i = 0, 1, 2, \ldots;$$

(v) $u_i \neq 0$, $\nabla_i < 0$, $\forall\, i \in \{1, ..., r\}$, where ∇_i is given by (11). *Then*

$$W^{\beta-\alpha}\left[y^{-\alpha} H^{0,\lambda' :\ (\alpha',\beta')\ ;...;\ (\alpha^{(r)},\beta^{(r)})}_{A',C' :\ (M',N')\ ;...;\ (M^{(r)},N^{(r)})}\left[\begin{matrix}[(g_j):\gamma',...,\gamma^{(r)}]_{1,A'} : (q',\eta')_{1,M'} ;...; (q^{(r)},\eta^{(r)})_{1,M^{(r)}} ;\\ [(f_j):\xi',...,\xi^{(r)}]_{1,C'} : (p',\varepsilon')_{1,N'} ;...; (p^{(r)},\varepsilon^{(r)})_{1,N^{(r)}} ;\end{matrix}\ u_1(x/y)^p, ..., u_r(x/y)^p\right]\right.$$

$$\left.\times H^{0,\lambda :\ (u',v')\ ;...;\ (u^{(r)},v^{(r)})}_{A,C :\ (B',D')\ ;...;\ (B^{(r)},D^{(r)})}\left[\begin{matrix}[(a_j):\theta',...,\theta^{(r)}]_{1,A} : (b',\phi')_{1,B'} ;...; (b^{(r)},\phi^{(r)})_{1,B^{(r)}} ;\\ [(c_j):\psi',...,\psi^{(r)}]_{1,C} : (d',\delta')_{1,D'} ;...; (b^{(r)},\phi^{(r)})_{1,D^{(r)}} ;\end{matrix}\ z_1(x/y)^q, ..., z_r(x/y)^q\right]\right]$$

$$= y^{-\beta} \sum_{m_i=1}^{\alpha^{(i)}} \sum_{n_i=0}^{\infty} \phi_1\, \phi_2\, \frac{\prod_{i=1}^{r}(u_i)^{U_i}(-1)^{\sum_{i=1}^{r} n_i}}{\prod_{i=1}^{r}(\varepsilon_{m_i}^{(i)}\, n_i\,!)} \left(\frac{x}{y}\right)^{p\sum_{i=1}^{r} U_i}$$

$$\times H^{0,\,\lambda+1\ :\ (u',v')\ ;...;\ (u^{(r)},v^{(r)})}_{A+1,C+1 :\ (B',D')\ ;...;\ (B^{(r)},D^{(r)})}\left[\begin{matrix}(1-\beta-p\sum_{i=1}^{r} U_i : q,...,q), [(a_j):\theta_j',...,\theta_j^{(r)}]_{1,A} : (b_j',\phi_j')_{1,B'} ;...; (b_j^{(r)},\phi_j^{(r)})_{1,B^{(r)}} ; z_1(x/y)^q\\ [(c_j):\psi_j',...,\psi_j^{(r)}]_{1,C}, (1-\alpha-p\sum_{i=1}^{r} U_i : q,...,q) : (d_j',\delta_j')_{1,D'} ;...; (d_j^{(r)},\delta_j^{(r)})_{1,D^{(r)}} ; z_r(x/y)^q\end{matrix}\right]. \tag{13}$$

Proof. To prove Lemma 1, we first use the definition of Weyl fractional integral given in (12) express the one multivariable H-function in series form and other in Mellin-Barnes type, then we change the order of summations and integrations (which is justified under the stated conditions), evaluate the t-integral and reinterpreting the resulting Mellin-Barnes contour integral in terms of the multivariable H-function, we easily arrive at the desired result.

Theorem 1. *With the set of sufficient conditions (i), (ii), (iii), (iv) and (v) of Lemma 1,*

$$\int_0^\infty y^{-\beta} \sum_{m_i=1}^{\alpha^{(i)}} \sum_{n_i=0}^{\infty} \phi_1\,\phi_2 \frac{\prod_{i=1}^r (u_i)^{U_i} (-1)^{\sum_{i=1}^r n_i}}{\prod_{i=1}^r (\varepsilon_{m_i}^{(i)}\, n_i\,!)} \left(\frac{x}{y}\right)^{p\,\sum_{i=1}^r U_i}$$

$$\times\, H^{0,\lambda+1\,:\,(u',v')\,;\ldots;\,(u^{(r)},v^{(r)})}_{A+1,C+1:\,(B',D')\,;\ldots;\,(B^{(r)},D^{(r)})} \left[\begin{matrix}(1-\beta-p\,\sum_{i=1}^r U_i : q,\ldots,q), [(a_j):\theta_j',\ldots,\theta_j^{(r)}]_{1,A} : \\ [(c_j):\psi_j',\ldots,\psi_j^{(r)}]_{1,C},\end{matrix}\right.$$

$$\left.\begin{matrix}(b_j',\phi_j')_{1,B'}\,;\ldots;(b_j^{(r)},\phi_j^{(r)})_{1,B^{(r)}}\,; z_1(x/y)^q \\ (1-\alpha-p\,\sum_{i=1}^r U_i : q,\ldots,q) : (d_j',\delta_j')_{1,D'}\,;\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,D^{(r)}}\,; z_r\,(x/y)^q\end{matrix}\right] f(y)\,dy$$

$$= \int_0^\infty y^{-\alpha}\, H^{0,\lambda'\,:\;(\alpha',\beta')\;;\ldots;\,(\alpha^{(r)},\beta^{(r)})}_{A',C'\,:\,(M',N')\,;\ldots;\,(M^{(r)},N^{(r)})} \left[\begin{matrix}[(g_j):\gamma',\ldots,\gamma^{(r)}]_{1,A'} : (q',\eta')_{1,M'}\,;\ldots; \\ [(f_j):\xi',\ldots,\xi^{(r)}]_{1,C'} : (p',\varepsilon')_{1,N'}\;;\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}(q^{(r)},\eta^{(r)})_{1,M^{(r)}}\,; u_1(x/y)^p \\ (p^{(r)},\varepsilon^{(r)})_{1,N^{(r)}}\;; u_r(x/y)^p\end{matrix}\right]$$

$$\times\, H^{0,\lambda\,:\,(u',v')\,;\ldots;\;(u^{(r)},v^{(r)})}_{A,C\,:\,(B',D')\,;\ldots;\,(B^{(r)},D^{(r)})} \left[\begin{matrix}[(a_j):\theta',\ldots,\theta^{(r)}]_{1A} : (b',\phi')_{1,B'}\;;\ldots; \\ [(c_j):\psi',\ldots,\psi^{(r)}]_{1C} : (d',\delta')_{1,D'}\,;\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}(b^{(r)},\phi^{(r)})_{1,B^{(r)}};\; z_1(x/y)^q \\ (d^{(r)},\delta^{(r)})_{1,D^{(r)}}\,;\; z_r(x/y)^q\end{matrix}\right] D^{\beta-\alpha}\,[f(y)]\,dy, \tag{14}$$

provided further $f \in \int$ and $x > 0$.

Proof. Let Δ denote the first member of the assertion (14). Then using Lemma 1 and applying (12), we have

$$\Delta = \int_0^\infty \frac{f(y)}{\Gamma(\alpha-\beta)} \left\{\int_0^\infty (\xi-y)^{\alpha-\beta-1}\,\xi^{-\alpha}\,H\left[u_1(x/\xi)^p,\ldots,u_r(x/\xi)^p\right]\right.$$

$$\left.\times\; H\left[z_1(x/\xi)^q,\ldots,z_r(x/\xi)^q\right]\,d\xi\right\}\,dy$$

$$= \int_0^\infty \xi^{-\alpha}\,H\left[u_1(x/\xi)^p,\ldots,u_r(x/\xi)^p\right] H\left[z_1(x/\xi)^q,\ldots,z_r(x/\xi)^q\right]$$

$$\times \left\{\int_0^\xi \frac{(\xi-y)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\,f(y)\,dy\right\} d\xi. \tag{15}$$

The change in the order of integration is assumed to be permissible just as in the proof of Lemma 1.
Now, by appealing to definition (11), (15) gives

$$\Delta = \int_0^\infty \xi^{-\alpha}\, H\,[u_1(x/\xi)^p, ..., u_r(x/\xi)^p]\, H\,[z_1(x/\xi)^q, ..., z_r(x/\xi)^q] \times D^{\beta-\alpha}\,\{f(\xi)\}\, d\xi, \tag{16}$$

which is precisely the right-hand member of (14). This completes the proof of Theorem 1.

3. Solution of the Integral Equation Involving the Product of I-functions

Lemma 2. *Let*
(i) $p_i\,(i = 1, ..., r)$, $q_i(i = 1, ..., r)$, *m,n be positive integers such that* $0 \le n \le p_i$, $0 \le m \le q_i$ $(i = 1,\ldots,r)$;
(ii) $\mathrm{Re}\,(\alpha) > \mathrm{Re}\,(\beta)$; $\mathrm{Re}\left[\beta + q\left(\frac{b_j}{\beta_j}\right)\right] > 0$, $j = (1, ..., m)$; $q > 0$;
(iii) $|\arg\,(z)| < \frac{1}{2}\pi\, T$, *where*

$$T = \sum_{j=1}^{m}\alpha_j + \sum_{j=1}^{n}\beta_j - \max_{1\le j\le r}\left[\sum_{j=n+1}^{p_i}\alpha_{ji} + \sum_{j=m+1}^{q_i}\beta_{ji}\right];$$

(iv) $p_i' < q_i'$ and $|u| < 1$.
Then

$$W^{\beta-\alpha}\left[y^{-\alpha}\, I^{m',n'}_{p_i',q_i':r'}\left[u\,(x/y)^p \left|\begin{matrix}(a_j',\alpha_j')_{1,n'};(a_{ji}',\alpha_{ji}')_{n'+1,p_i'}\\(b_j,\beta_j)_{1,m'};(b_{ji}',\beta_{ji}')_{m'+1,q_i'}\end{matrix}\right.\right]\right.$$
$$\left.\times I^{m,n}_{p_i,q_i:\,r}\left[z\,(x/y)^q \left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{m+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]\right]$$

$$= y^{-\beta}\sum_{h=1}^{m'}\sum_{k=0}^{\infty}\frac{(-1)^k\,\phi'(\eta_{h,k})}{\beta_h'\;k\,!}\,u^{\eta_{h,k}}\,(x/y)^{p\eta_{h,k}}$$
$$\times\, I^{m,\,n+1}_{p_i+1,q_i+1:\,r}\left[\begin{matrix}(1-\beta-p\eta_{h,k},q),\,(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(1-\alpha-p\,\eta_{h,k},q)\end{matrix}\right]. \tag{17}$$

Proof. The Lemma 2 can be easily established by using the same technique as used in Lemma 1.

Theorem 2. *Under the sufficient conditions (i), (ii), (iii), and (iv) of Lemma 2,*

$$\int_0^\infty y^{-\beta} \sum_{h=1}^{m'} \sum_{k=0}^{\infty} \frac{(-1)^k \phi'(\eta_{h,k})}{\beta_h' \; k\,!} u^{\eta_{h,k}} (x/y)^{p\, \eta_{h,k}}$$

$$\times I^{m,\, n+1}_{p_i+1, q_i+1: r} \left[z\,(x/y)^q \left| \begin{matrix} (1-\beta-p\, \eta_{h,k}, q), (a_j,\alpha_j)_{1,n}; (a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m}; (b_{ji},\beta_{ji})_{m+1,q_i}, (1-\alpha-p\, \eta_{h,k}, q) \end{matrix} \right. \right] f(y)\, dy$$

$$= \int_0^\infty y^{-\alpha} I^{m',n'}_{p_i',q_i':r'} \left[u(x/y)^p \left| \begin{matrix} (a_j',\alpha_j')_{1,n'}; (a_{ji}',\alpha_{ji}')_{n'+1,p_i'} \\ (b_j',\beta_j')_{1,m'}; (b_{ji}',\beta_{ji}')_{m'+1,q_i'} \end{matrix} \right. \right]$$

$$\times I^{m,\, n}_{p_i,q_i: r} \left[z\,(x/y)^q \left| \begin{matrix} (a_j,\alpha_j)_{1,n}; (a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m}; (b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] D^{\beta-\alpha} \{f(y)\}\, dy, \qquad (18)$$

provided further $f \in \int$ and $x > 0$.

Theorem 2 is established with the help of Lemma 2 and the equation (11), on proceeding on similar lines as indicated in the proof of Theorem 1.

4. Use of other Methods

One-dimensional Fredholm integral equation (2) involving the product of I-functions in the kernel can also be solved by resorting to the application of Mellin transforms.

Theorem 3. *If $f \in \int$, $D^{\alpha-\beta}\{f(x)\}$ exists $q > 0$, $x > 0$, $|arg\ (z)\ | < \frac{1}{2}\ \pi\ T$, $T > 0$ (T is given in Lemma 2), Re $(\alpha) >$ Re $(\beta) > 0$, $p_i' < q_i'$ and $|u\ | < 1$, then the solution of the integral equation*

$$\int_0^\infty y^{-\alpha} I^{m',n'}_{p_i',q_i':r'} \left[u(x/y)^p \left| \begin{matrix} (a_j',\alpha_j')_{1,n'}; (a_{ji}',\alpha_{ji}')_{n'+1,p_i'} \\ (b_j',\beta_j')_{1,m'}; (b_{ji}',\beta_{ji}')_{m'+1,q_i'} \end{matrix} \right. \right]$$

$$\times I^{m,\, n}_{p_i,q_i: r} \left[z\,(x/y)^q \left| \begin{matrix} (a_j,\alpha_j)_{1,n}; (a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m}; (b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] f(y)\, dy$$

$$= \mathrm{g(x)} \ \ (0 < x < \infty) \tag{19}$$

is given by

$$\mathrm{f(x)} = \frac{\mathrm{q}}{2\pi \mathrm{i}} \mathrm{x}^{\alpha-1} \lim_{\rho\to\infty} \int_{\sigma-i\rho}^{\sigma+i\rho} \mathrm{x}^{-\mathrm{s}} \left\{ \sum_{\mathrm{h}=1}^{\mathrm{m}'} \sum_{\mathrm{k}=0}^{\infty} \frac{(-1)^{\mathrm{k}}\, \phi' \,(\eta_{\mathrm{h,k}})}{\beta'_{\mathrm{h}}\ \mathrm{k}\,!} \mathrm{u}^{\eta_{\mathrm{h,k}}} \right.$$

$$\left. \times\, \theta \left(\frac{-\mathrm{p}\eta_{\mathrm{h,k}} - \mathrm{s}}{\mathrm{q}} \right) \mathrm{z}^{-\left(\frac{\mathrm{p}\eta_{\mathrm{h,k}}+\mathrm{s}}{\mathrm{q}}\right)} \right\}^{-1} \phi(\mathrm{s})\, \mathrm{ds}\,, \tag{20}$$

provided further that

$$\max\left\{\mathrm{Re}[(\mathrm{a}_\ell - 1)/\alpha_\ell]\right\} < -\mathrm{Re}\left(\frac{\mathrm{p}\eta_{\mathrm{h,k}} + s}{\mathrm{q}}\right) < \min\left\{\mathrm{Re}\left(\frac{\mathrm{b_j}}{\beta_{\mathrm{j}}}\right)\right\},$$

$$(\mathrm{j} = 1, ..., \mathrm{m}\,,\, \ell = 1, ..., \mathrm{n}). \tag{21}$$

Proof. On replacing f by $\mathrm{D}^{\alpha-\beta}\{\mathrm{f}\}$in (18) and applying (19), we have

$$\mathrm{g\,(x)} = \int_0^\infty \mathrm{y}^{-\beta} \sum_{\mathrm{h}=1}^{\mathrm{m}'} \sum_{k=0}^{\infty} \frac{(-1)^{\mathrm{k}}\, \phi'(\eta_{\mathrm{h,k}})}{\beta'_{\mathrm{h}}\ \mathrm{k}\,!} \mathrm{u}^{\eta_{\mathrm{h,k}}} (\mathrm{x/y})^{\mathrm{p}\ \eta_{\mathrm{h,k}}}$$

$$\times\, \mathrm{I}^{\mathrm{m,\,n+1}}_{\mathrm{p_i+1,q_i+1:r}} \left[\mathrm{z\,(x/y)^q} \left| \begin{matrix} (1-\beta-\mathrm{p}\ \eta_{\mathrm{h,k}},\mathrm{q}),\, (\mathrm{a_j},\alpha_{\mathrm{j}})_{1,\mathrm{n}}\,;\, (\mathrm{a_{ji}},\alpha_{\mathrm{ji}})_{\mathrm{n+1,p_i}} \\ (\mathrm{b_j},\beta_{\mathrm{j}})_{1,\mathrm{m}};\, (\mathrm{b_{ji}},\beta_{\mathrm{ji}})_{\mathrm{m+1,q_i}},(1-\alpha-\mathrm{p}\ \eta_{\mathrm{h,k}},\mathrm{q}) \end{matrix} \right. \right] \mathrm{D}^{\alpha-\beta}\{\mathrm{f(y)}\}\ \mathrm{dy}. \tag{22}$$

Multiplying both sides of (22) by $\mathrm{x}^{\mathrm{s}-1}$and integrating with respect to x from 0 to ∞, we have

$$\phi(\mathrm{s}) = \int_0^\infty \mathrm{x}^{\mathrm{s}-1}\, \mathrm{g(x)\, dx}$$

$$= \int_0^\infty \mathrm{y}^{-\beta}\, \mathrm{D}^{\alpha-\beta}\, \{\mathrm{f(y)}\} \sum_{\mathrm{h}=1}^{\mathrm{m}'} \sum_{\mathrm{k}=0}^{\infty} \frac{(-1)^{\mathrm{k}}\, \phi'(\eta_{\mathrm{h,k}})}{\beta'_{\mathrm{h}}\ \mathrm{k}\,!} \frac{\mathrm{u}^{\eta_{\mathrm{h,k}}}}{\mathrm{y}^{\mathrm{p}\eta_{\mathrm{h,k}}}}$$

$$\times \left(\int_0^\infty \mathrm{x}^{\mathrm{s}+\mathrm{p}\eta_{\mathrm{h,k}}-1}\, \mathrm{I}^{\mathrm{m,\,n+1}}_{\mathrm{p_i+1,q_i+1:r}} \left[\mathrm{z\,(x/y)^q} \left| \begin{matrix} (1-\beta-\mathrm{p}\ \eta_{\mathrm{h,k}},\mathrm{q}), \\ (\mathrm{b_j},\beta_{\mathrm{j}})_{1,\mathrm{m}}\,; \end{matrix} \right. \right. \right.$$

$$\left. \left. \begin{matrix} (\mathrm{a_j},\alpha_{\mathrm{j}})_{1,\mathrm{n}};\, (\mathrm{a_{ji}},\alpha_{\mathrm{ji}})_{\mathrm{n+1,p_i}} \\ (\mathrm{b_{ji}},\beta_{\mathrm{ji}})_{\mathrm{m+1,q_i}},\, (1-\alpha-\mathrm{p}\ \eta_{\mathrm{h,k}},\mathrm{q}) \end{matrix} \right] \mathrm{dx} \right) \mathrm{dy}\,, \tag{23}$$

where we have assumed the absolute (and uniform) convergence of the integrals involved, with a view to justifying the inversion of the order of integration.

Now, evaluate the inner integral in (23) by a simple change of variables in the familiar results (cf. for example, [3] and [13]), equation (23) reduces to

$$\phi(\mathrm{s}) = \frac{1}{\mathrm{q}} \sum_{\mathrm{h}=1}^{\mathrm{m}'} \sum_{\mathrm{k}=0}^{\infty} \frac{(-1)^{\mathrm{k}}\, \phi'(\eta_{\mathrm{h,k}})}{\beta_{\mathrm{h}}'\ \mathrm{k}\,!}\, \mathrm{u}^{\eta_{\mathrm{h,k}}}\, \mathrm{z}^{-\left(\frac{\mathrm{p}\ \eta_{\mathrm{h,k}}+\mathrm{s}}{\mathrm{q}}\right)}\, \theta\left(\frac{-\mathrm{p}\ \eta_{\mathrm{h,k}} - \mathrm{s}}{\mathrm{q}}\right)$$

$$\times \frac{\Gamma(\beta - \mathrm{s})}{\Gamma(\alpha - \mathrm{s})} \int_0^{\infty} \mathrm{y}^{\mathrm{s}-\beta}\, \mathrm{D}^{\alpha-\beta}\left\{\mathrm{f(y)}\right\} \mathrm{dy}. \tag{24}$$

Inverting (24) by applying Mellin inversion theorem [18], we get

$$\mathrm{D}^{\alpha-\beta}\left\{\mathrm{f(y)}\right\} = \frac{\mathrm{q}}{2\pi \mathrm{i}} \lim_{\rho\to\infty} \int_{\sigma-\mathrm{i}\rho}^{\sigma+\mathrm{i}\rho} \left\{\sum_{\mathrm{h}=1}^{\mathrm{m}'} \sum_{\mathrm{k}=0}^{\infty} \frac{(-1)^{\mathrm{k}}\, \phi'(\eta_{\mathrm{h,k}})}{\beta_{\mathrm{h}}'\,\mathrm{k}\,!}\, \mathrm{u}^{\eta_{\mathrm{h,k}}}\right.$$

$$\left.\times \frac{\Gamma(\beta - \mathrm{s})}{\Gamma(\alpha - \mathrm{s})}\, \theta\left(\frac{-\mathrm{p}\eta_{\mathrm{h,k}} - \mathrm{s}}{\mathrm{q}}\right) \mathrm{z}^{-\left(\frac{\mathrm{p}\ \eta_{\mathrm{h,k}}+\mathrm{s}}{\mathrm{q}}\right)}\right\}^{-1} \mathrm{y}^{\beta-\mathrm{s}-1}\, \phi(\mathrm{s})\ \mathrm{ds}. \tag{25}$$

Operating upon both sides by $\mathrm{D}^{\beta-\alpha}$,(25) gives us

$$\mathrm{f(y)} = \frac{\mathrm{q}}{2\pi \mathrm{i}}\, \mathrm{D}^{\beta-\alpha}\left[\lim_{\rho\to\infty} \int_{\sigma-\mathrm{i}\rho}^{\sigma+\mathrm{i}\rho} \mathrm{y}^{\beta-\mathrm{s}-1} \left\{\sum_{\mathrm{h}=1}^{\mathrm{m}'} \sum_{\mathrm{k}=0}^{\infty} \frac{(-1)^{\mathrm{k}}\, \phi'(\eta_{\mathrm{h,k}})}{\beta_{\mathrm{h}}'\,\mathrm{k}\,!}\, \mathrm{u}^{\eta_{\mathrm{h,k}}}\right.\right.$$

$$\left.\left.\times \frac{\Gamma(\beta - \mathrm{s})}{\Gamma(\alpha - \mathrm{s})}\, \theta\left(\frac{-\mathrm{p}\eta_{\mathrm{h,k}} - \mathrm{s}}{\mathrm{q}}\right) \mathrm{z}^{-\left(\frac{\mathrm{p}\ \eta_{\mathrm{h,k}}+\mathrm{s}}{\mathrm{q}}\right)}\right\}^{-1} \phi(\mathrm{s})\ \mathrm{ds}\right], \tag{26}$$

which finally yields

$$\mathrm{f(x)} = \frac{\mathrm{q}}{2\pi \mathrm{i}}\, \mathrm{x}^{\alpha-1} \lim_{\rho\to\infty} \int_{\sigma-\mathrm{i}\rho}^{\sigma+\mathrm{i}\rho} \mathrm{x}^{-\mathrm{s}} \left\{\sum_{\mathrm{h}=1}^{\mathrm{m}'} \sum_{\mathrm{k}=0}^{\infty} \frac{(-1)^{\mathrm{k}}\, \phi'\ (\eta_{\mathrm{h,k}})}{\beta_{\mathrm{h}}'\ \mathrm{k}\,!}\, \mathrm{u}^{\eta_{\mathrm{h,k}}}\right.$$

$$\left.\times\, \theta\left(\frac{-\mathrm{p}\eta_{\mathrm{h,k}} - \mathrm{s}}{\mathrm{q}}\right) \mathrm{z}^{-\left(\frac{\mathrm{p}\eta_{\mathrm{h,k}}+\mathrm{s}}{\mathrm{q}}\right)}\right\}^{-1} \phi(\mathrm{s})\ \mathrm{ds}\,, \tag{27}$$

as the solution of the integral equation (19).

5. Special Cases

(i) If we set p = 0, the results in (13) and (14) reduce to the known results obtained by Chaurasia and Patni [2].
(ii) Taking r = 1 and p = 0, then the results in (17), (18) and (19) reduce to the known results with a slight modification obtained by Chaurasia and Patni [2].
(iii) On specializing the parameters in (13), (14), (17), (18) and (19), we arrive at the results obtained by Srivastava and Raina [17].

The importance of our results lies in its manifold generality. In view of the generality of the multivariable H-function, on specializing the various parameters and variables, we can obtained from our results, several integral equations and solutions involving a remarkably wide variety of useful functions (or product of several functions), which are expressible in terms of E, F, G and H functions of one and several variables. Secondly, by suitably specializing the various parameters in the I-functions, our results can be reduces to a large number of integral equations and solutions involving a product of H-functions, G functions and their various special cases. Thus the results presented in this paper would at once yield a very large number of results involving a large variety of special functions occurring in the problems of mathematical analysis, applied mathematics, mathematical physics and engineering.

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