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# Notes on Some Identities Related to the Partial Bell Polynomials *

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**Abstract**

The purpose of this paper is to establish several identities involving the partial Bell polynomials by using the generating function. These results generalize some identities by Yang in "Discrete Math., 308(2008)" and Abbas and Bouroubi in "Discrete Math., 293(2005)". Some applications are given.

**Keywords and Phrases:** *Exponential generating function, Partial Bell polynomial, Derivative.*

## 1. Introduction

The partial Bell polynomials are the polynomials $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1)}$ in an infinite number of variables $x_1$, $x_2$, …, defined by the formal double series expansion:

$$exp\left(u\sum_{m\geq 1} x_m \frac{t^m}{m!}\right) = 1 + \sum_{n\geq 1}\frac{t^n}{n!}\left\{\sum_{k=1}^{n} u^k B_{n,k}(x_1, x_2, \ldots, x_{n-k+1)}\right\} \tag{1}$$

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or, by the series expansion:

$$\frac{1}{k!}\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right)^k = \sum_{n\geq k} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})\frac{t^n}{n!}, \qquad k = 0, 1, 2, \ldots. \tag{2}$$

The partial Bell polynomials have the following expression:

$$B_{n,k}(x_1, x_2, \ldots, x_{n-k+1)} = \sum \frac{n!}{c_1!c_2!\ldots(1!)^{c_1}(2!)^{c_2}\ldots} x_1^{c_1} x_2^{c_2}\ldots \tag{3}$$

where the summation takes place over all integers $c_1$, $c_2$, $c_3$, $\ldots \geq 0$, such that $c_1+c_2+c_3+\ldots = k$, and $c_1+2c_2+3c_3+\ldots = n$. The partial Bell polynomials have the following known identity:

$$B_{n,k}(1, 2, 3, \ldots, ) = \binom{n}{k} k^{n-k}. \tag{4}$$

For the detail, see [2].

Let $\varphi_n(x)$ be binomial sequences if it satisfies the following condition: (1) $\varphi_0(1) = 1$, $\varphi_1(x) = x$, (2) for any positive integer $n$, $\varphi_n(x)$ is a polynomial of degree $n$ with $\varphi_n(0) = 0$, and (3) for all nonnegative integer $n$, $\varphi_n(x+y) = \sum_{k=0}^{n} \binom{n}{k}\varphi_k(x)\varphi_{n-k}(y)$.

In [1], Abbas and Bouroubi generalized (4) as the following identity involving the partial Bell polynomials and the binomial sequencss:

$$B_{n,k}(\varphi_0(1), 2\varphi_1(1), 3\varphi_2(1), \ldots, ) = \binom{n}{k}\varphi_{n-k}(k)$$

Recently, Yang [4] generalized further the following identity:

$$B_{n,k}(\varphi_0(x), 2\varphi_1(x), 3\varphi_2(x), \ldots, ) = \binom{n}{k}\varphi_{n-k}(kx). \tag{5}$$

At same time, he supplied the following two identities:

$$\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\varphi_n(jx) = B_{n,k}(\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, ), \tag{6}$$

$$\frac{1}{k!}\sum_{j=0}^{k}\sum_{i=0}^{n}(-1)^{k-j}\binom{k}{j}\binom{n}{j}\binom{k-j}{i}i!\varphi_{n-i}(jx)$$
$$= (n)_k B_{n-k,k}\left(\frac{1}{2}\varphi_2(x), \frac{1}{3}\varphi_3(x), \frac{1}{4}\varphi_4(x), \ldots,\right). \tag{7}$$

These two identities generalized the results of [5].

The purpose of this paper is to establish several identities involving the partial Bell polynomials by using the generating function. These results generalize some identities by Yang in [4] and Abbas and Bouroubi in [1].

## 2. A Class of Function

Let $f(x)$ be a function such that the any order derivatives exist. To avoid any unnecessary confusion, we apply $f^{(k)}(x)$ to denote the $k$-th order derivative of $f(x)$ and use $f^k(x)$ to stand for the $k$-th power of $f(x)$. Obviously, $f^{(0)}(x) = f(x)$ and $f^0(x) = 1$.

Suppose that

$$\Omega_f(x,t) = \sum_{n=0}^{\infty} f^{(n)}(x)\frac{t^n}{n!} \tag{8}$$

**Theorem 2.1.** *We have*

$$a\Omega_f(x,t) + b\Omega_g(x,t) = \Omega_{af+bg}(x,t) \tag{9}$$
$$\Omega_f(x,t)\Omega_g(x,t) = \Omega_{fg}(x,t) \tag{10}$$

**Proof.** The first formula is obviously. Now we prove the second formula. Since

$$\begin{aligned}
\Omega_f(x,t)\Omega_g(x,t) &= \left(\sum_{n\geq 0} f^{(n)}(x)\frac{t^n}{n!}\right)\left(\sum_{n\geq 0} g^{(n)}(x)\frac{t^n}{n!}\right) \\
&= \sum_{n\geq 0}\left(\sum_{i=0}^{n}\binom{n}{i} f^{(i)}(x)g^{(n-i)}(x)\right)\frac{t^n}{n!} \\
&= \sum_{n\geq 0}(f(x)g(x))^{(n)}\frac{t^n}{n!} \\
&= \Omega_{fg}(x,t),
\end{aligned}$$

the proof is completed.

**Corollary 2.2.** *We have*

$$\Omega_f^k(x,t) = \Omega_{f^k}(x,t) \tag{11}$$

**Proof.** Apply inductive on $k$.

# 3. Main Results

**Theorem 3.1.** *Let $f(x)$ be a function such that the any order derivatives exist. Then for all integers, $n$, $k \geq 0$,*

$$\binom{n}{k}\left(f^k(x)\right)^{(n-k)} = B_{n,k}(f(x), 2f^{(1)}(x), 3f^{(2)}(x), \ldots) \tag{12}$$

**Proof.** From the definition of the partial Bell polynomial, we have

$$\begin{aligned}
\frac{1}{k!}\left(t\Omega_f(x,t)\right)^k &= \frac{1}{k!}\left(t\sum_{n=0}^{\infty} f^{(n)}(x)\frac{t^n}{n!}\right)^k \\
&= \frac{1}{k!}\left(\sum_{n=1}^{\infty} nf^{(n-1)}(x)\frac{t^n}{n!}\right)^k \\
&= \sum_{n\geq k} B_{n,k}(f(x),\ 2f^{(1)}(x),\ 3f^{(2)}(x),\ \ldots\ )\frac{t^n}{n!}. \qquad (13)
\end{aligned}$$

By Corollary 2.2, we have

$$\begin{aligned}
\frac{1}{k!}\left(t\Omega_f(x,t)\right)^k &= \frac{1}{k!}t^k\left(\Omega_f(x,t)\right)^k \\
&= \frac{1}{k!}t^k\Omega_{f^k}(x,t) \\
&= \frac{1}{k!}t^k\sum_{n\geq 0}\left(f^k(x)\right)^{(n)}\frac{t^n}{n!} \\
&= \sum_{n\geq k}\binom{n}{k}\left(f^k(x)\right)^{(n-k)}\frac{t^n}{n!} \qquad (14)
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in (13) and (14), the assertion follows.

Similar to the proof of Theorem 3.1, we can obtain the following two results.

**Theorem 3.2.** *Let $f(x)$ be a function such that the any order derivatives exist. Then for all integers, $n,\ k \geq 0$,*

$$\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f^{k-j}(x)(f^j(x))^{(n)} = B_{n,k}(f^{(1)}(x), f^{(2)}(x), f^{(3)}(x), \ldots) \quad (15)$$

**Proof.** Since

$$\begin{aligned}\frac{1}{k!}\left(\Omega_f(x,t) - f(x)\right)^k &= \frac{1}{k!}\left(\sum_{n\geq 1} f^{(n)}(x)\frac{t^n}{n!}\right)^k \\ &= \sum_{n\geq k} B_{n,k}(f^{(1)}(x), f^{(2)}(x), f^{(3)}(x), \ldots)\frac{t^n}{n!}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{k!}\left(\Omega_f(x,t) - f(x)\right)^k &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}f^{k-j}(x)\Omega_f^j(x,t) \\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}f^{k-j}(x)\Omega_{f^j}(x,t) \\ &= \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}f^{k-j}(x)\sum_{n\geq 0}\left(f^j(x)\right)^{(n)}\frac{t^n}{n!} \\ &= \sum_{n\geq 0}\left(\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f^{k-j}(x)(f^j(x))^{(n)}\right)\frac{t^n}{n!},\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, the proof is completed. $\square$

**Theorem 3.3.** *Let $f(x)$ be a function such that the any order derivatives exist. Then for all integers, $n,\ k \geq 0$,*

$$\begin{aligned}&\frac{1}{k!}\sum_{j=0}^{k}\sum_{i=0}^{n}(-1)^{k-j}\binom{k}{j}\binom{n}{i}\binom{k-j}{i}i!f^{k-j-i}(x)\left(f^{(1)}(x)\right)^i\left(f^j(x)\right)^{(n-i)} \\ =\ & (n)_k B_{n-k,k}\left(\frac{1}{2}f^{(2)}(x), \frac{1}{3}f^{(3)}(x), \frac{1}{4}f^{(4)}(x), \ldots\right). \quad (16)\end{aligned}$$

**Proof.** Since

$$\begin{aligned}
& \frac{1}{k!}\left(\Omega_f(x,t)-f(x)-f^{(1)}(x)t\right)^k \\
= & \frac{1}{k!}\left(\sum_{n\geq 2} f^{(n)}(x)\frac{t^n}{n!}\right)^k \\
= & \frac{t^k}{k!}\left(\sum_{n\geq 1} f^{(n+1)}(x)\frac{t^n}{(n+1)!}\right)^k \\
= & t^k\sum_{n\geq k} B_{n,k}\left(\frac{1}{2}f^{(2)}(x),\frac{1}{3}f^{(3)}(x),\frac{1}{4}f^{(4)}(x),\ldots\right)\frac{t^n}{n!} \\
= & \sum_{n\geq 2k}(n)_k B_{n-k,k}\left(\frac{1}{2}f^{(2)}(x),\frac{1}{3}f^{(3)}(x),\frac{1}{4}f^{(4)}(x),\ldots\right)\frac{t^n}{n!}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{k!}\left(\Omega_f(x,t)-f(x)-f^{(1)}(x)t\right)^k \\
= & \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(f(x)+f^{(1)}(x)t\right)^{k-j}\Omega_f^j(x,t) \\
= & \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(f(x)+f^{(1)}(x)t\right)^{k-j}\Omega_{f^j}(x,t) \\
= & \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\sum_{n\geq 0}\left(f^j(x)\right)^{(n)}\frac{t^n}{n!}\sum_{i=0}^{k-j}\binom{k-j}{i}f^{k-j-i}(x)\left(f^{(1)}(x)\right)^i t^i \\
= & \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\sum_{n\geq 0}\sum_{i=0}^{n}\binom{n}{i}\binom{k-j}{i}i!f^{k-j-i}(x)\left(f^{(1)}(x)\right)^i\left(f^j(x)\right)^{(n-i)}\frac{t^n}{n!},
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, the proof is completed.

# 4. Applications

**Example 4.1.** Let $g(x,z)=\sum_{n=0}^{\infty}\varphi_n(x)\frac{z^n}{n!}$, where $\varphi_n(x)$ is a binomial sequence. Then $g^l(x,z)=g(lx,z)$, (see [3]). If taking $f^{(k)}(x)=\frac{d^k}{dz^k}g(x,z)|_{z\to 0}=$

$\varphi_k(x)$, we have $(f^l(x))^{(k)} = \frac{d^k}{dz^k}g^l(x,z)|_{z\to 0} = \frac{d^k}{dz^k}g(lx,z)|_{z\to 0} = \varphi_k(lx)$. From Theorems 3.1-3.3, the results of Yang [4] are obtained. From the binomial sequences: $x^n$, $(x)_n = x(x-1)\dots(x-n+1)$, $x^{n|\lambda} = x(x+\lambda)(x+2\lambda)\dots(x+(n-1)\lambda)$, $x(x-na)^{n-1}$, $\sum_{k=0}^n s(n,k)$, $\sum_{k=0}^n S(n,k)$, where $s(n,k)$ and $S(n,k)$ are the Stirling number of the first and second kinds, respectively, the corresponding identities can be obtained.

**Example 4.2.** If taking $f(x) = e^x - 1$ and $x \to 0$, from Theorems 3.1-3.3, the following identities are obtained:

$$\binom{n}{k} k! S(n-k,k) = B_{n,k}(0,\ 2,\ 3,\ 4,\ \dots,\ ),$$

$$S(n,k) = B_{n,k}(1,\ 1,\ 1,\ \dots,\ ),$$

$$\sum_{i=0}^{k} (-1)^i \binom{n}{i} S(n-i,k-i) = (n)_k B_{n-k,k}\left(\frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \dots,\ \right).$$

**Example 4.3.** If taking $f(x) = \ln(1+x)$ and $x \to 0$, from Theorems 3.1-3.3, the following identities are obtained:

$$\binom{n}{k} k! s(n-k,k) = B_{n,k}(0,\ 2,\ -3\cdot 1!,\ 4\cdot 2!,\ -5\cdot 3!,\ 6\cdot 4!,\ \dots,\ ),$$

$$s(n,k) = B_{n,k}(0!,\ -1!,\ 2!,\ -3!,\ 4!,\ \dots,\ ),$$

$$\sum_{i=0}^{k} (-1)^i \binom{n}{i} s(n-i,k-i) = (n)_k B_{n-k,k}\left(-\frac{1!}{2},\ \frac{2!}{3},\ -\frac{3!}{4},\ -\frac{4!}{5},\ \dots,\ \right).$$

**Example 4.4.** If taking $f(x) = \frac{1}{1-x^2} = \sum_{n\geq 0}(2n)!\frac{x^{2n}}{(2n)!}$ and $x \to 0$, noting that

$$\left(f^k(x)\right)^{(n)}\Big|_{x\to 0} = \left(\frac{1}{(1-x^2)^k}\right)^{(n)}\Bigg|_{x\to 0} = \begin{cases} (2n)!\binom{k+n-1}{n}, & if \quad n\ even \\ 0, & if \quad n\ odd, \end{cases}$$

from Theorem 3.1, we have

$$B_{n,k}(2!-1!,\ 0,\ 3!-2!,\ 0,\ 5!-4!,\ \dots,) = \binom{n}{k}\binom{n-1}{k-1}(2n-2k)!,$$

if $n-k$ is even and

$$B_{n,k}(2!-1!,\ 0,\ 3!-2!,\ 0,\ 5!-4!,\ \ldots,)=0.$$

if $n-k$ is odd.

Noting that

$$\begin{aligned}
&\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{j+n-1}{n}\\
=\ &(-1)^{k-1}\sum_{j=0}^{k}\binom{k}{k-j}\binom{-n-1}{j-1}\\
=\ &(-1)^{k-1}\binom{-(n+1-k)}{k-1}\\
=\ &\binom{n-1}{k-1},
\end{aligned}$$

from Theorems 3.2 and 3.3, we have

$$\begin{aligned}
&B_{2n,k}(0,\ 2!,\ 0,\ 4!,\ 0,\ 6!,\ \ldots,)=\frac{(2n)!}{k!}\binom{n-1}{k-1},\\
&B_{2n-1,k}(0,\ 2!,\ 0,\ 4!,\ 0,\ 6!,\ \ldots,)=0,\\
&B_{2n-k,k}(1!,\ 0,\ 3!,\ 0,\ 5!,\ 0,\ \ldots,)=\binom{n-1}{k-1},\\
&B_{2n-k-1,k}(1!,\ 0,\ 3!,\ 0,\ 5!,\ 0,\ \ldots,)=0.
\end{aligned}$$

**Example 4.5.** If taking $f(x)=\frac{1}{(1-x)^m}$ $(m\geq 1)$ and $x\to 0$, from Theorems 3.1-3.3, the following identities are obtained:

$$\begin{aligned}
&B_{n,k}\left(1,\ 2!\binom{m}{1},\ 3!\binom{m+1}{2},\ 4!\binom{m+2}{3},\ 5!\binom{m+3}{4},\ \ldots,\right)\\
=\ &\binom{n}{k}\binom{mk+n-k-1}{n-k}(n-k)!,
\end{aligned}$$

$$\begin{aligned}
&B_{n,k}\left(1!\binom{m}{1},\ 2!\binom{m+1}{2},\ 3!\binom{m+2}{3},\ 4!\binom{m+3}{4},\ \ldots,\right)\\
=\ &\frac{n!}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{mj+n-1}{n}
\end{aligned}$$

$$
\begin{aligned}
& B_{n-k,k}\left(1!\binom{m+1}{2},\ 2!\binom{m+2}{3},\ 3!\binom{m+3}{4},\ 4!\binom{m+4}{5},\ \ldots,\ \right) \\
= & \sum_{j=0}^{k}\sum_{i=0}^{n}(-1)^{k-j}\binom{k}{j}\binom{k-j}{i}\binom{mj+n-i-1}{n-i}m^{i}.
\end{aligned}
$$

Specially,

$$
B_{n,k}\left(1!,\ 2!,\ 3!,\ 4!,\ \ldots,\ \right)=\frac{n!}{k!}\binom{n-1}{k-1}=L(n,k),
$$

where $L(n,k)$ is the Lah number.

**Example 4.6.** If taking $f(x)=\frac{x}{e^x-1}=1-\frac{1}{2}t+\sum_{n\geq 1}B_{2n}\frac{t^{2n}}{(2n)!}$, where $B_n$ are the Bernoulli numbers, and $x\to 0$, from Theorems 3.1-3.3, the following identities are obtained:

$$
\binom{n}{k}B_{n-k}^{(k)}=B_{n,k}(1,\ -1,\ 3B_2,\ 0,\ 5B_4,\ 0,\ 7B_6,\ 0,\ldots,\ )
$$

$$
\frac{1}{k!}B_{n}^{(k)}=B_{n,k}\left(-\frac{1}{2},\ B_2,\ 0,\ B_4,\ 0,\ B_6,\ 0,\ \ldots,\ \right)
$$

$$
\sum_{i=0}^{k}\binom{n}{i}\frac{B_{n-i}^{(k-i)}}{2^i(k-i)!}=(n)_kB_{n-k,k}\left(\frac{1}{2}B_2,\ 0,\ \frac{1}{4}B_4,\ 0,\ \frac{1}{6}B_6,\ 0,\ \ldots,\ \right),
$$

where $B_n^{(k)}$ are the Bernoulli numbers of $k$-order defined by

$$
\sum_{n\geq 0}B_n^{(k)}\frac{t^n}{n!}=\left(\frac{x}{e^x-1}\right)^k
$$

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