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Extensions of Hardy Type Integral Inequality*

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Abstract

In this paper, we prove sharp Hardy type integral inequality by using the generalized Riesz potential generated by the generalized Shift operator. Our results improve and extend the well-known results of Hardy [2].

Keywords and Phrases: *Hardy integral inequality, Hölder's inequality, Maximal function, Riesz potential and Shift operator.*

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1. Introduction

The initial Hardy type integral inequality is the following, (see [2]):

If $p > 1$, $f(x) \geq 0$, and $F(x) = \int\limits_0^x f(t)dt$, then

$$\int\limits_0^\infty \left(\frac{F}{x}\right)^p dx \quad < \quad q^p \int\limits_0^\infty f^p(x)dx, \tag{1.1}$$

unless $f \equiv 0$. The constant q^p is the best possible.

This inequality plays an important role in analysis and its applications. It is obvious that, for parameters a and b such that $0 < a < b < \infty$, the following inequality is also valid

$$\int\limits_a^b \left(\frac{F}{x}\right)^p dx \quad < \quad q^p \int\limits_a^b f^p(x)dx, \tag{1.2}$$

where $0 < \int\limits_0^\infty f^p(x)dx < \infty$. The classical Hardy inequality asserts that if $p > 1$ and f is a non-negative measurable function on (a, b), then (1.2) is true unless $f \equiv 0$ a.e. in (a, b), where the constant q^p is the best possible. This inequality remains true provided that $0 < a < b < \infty$.

In [3], the classical Hardy inequality for fractional integrals states that

$$\left\| x^{\beta-\alpha} \int\limits_0^x \frac{f(y)dy}{y^\beta (x-y)^{1-\alpha}} \right\|_{L_p(0,b)} \quad \leq \quad c \left\| f \right\|_{L_p(0,b)}, \quad 0 < \alpha < 1 \tag{1.3}$$

where $\alpha - \frac{1}{p} < \beta < \frac{1}{p'}$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $0 < b \leq \infty$. Later, Sarikaya and Yıldırım [8] studied the following generalization Hardy inequality

$$\int\limits_{\mathbb{R}^n} |x|_\lambda^\mu \left| I_{\alpha,\lambda} f(x) \right|^p dx \quad \leq \quad c \int\limits_{\mathbb{R}^n} |x|_\lambda^\gamma \left| f(x) \right|^p dx \tag{1.4}$$

for the generalized Riesz potential with the non-isotropic kernel depending on $\lambda-$distance.

Because of their fundamental importance in the discipline over the years much effort and time have been devoted to the improvement and generalizations of Hardy's inequalities (1.1)-(1.4). These include, among others, works in [2-8, 16, 17].

The aim of this paper is to obtain sharp Hardy type integral inequality generated by the generalized Riesz potential by using Maximal function theory. Our results improve and extend the well-known results of Hardy (see [2]).

2. Preliminaries

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$ are vectors in $\mathbb{R}^n$, $x.y = x_1y_1 + ... + x_ny_n$, $|x| = (x.x)^{\frac{1}{2}}$,

$$\mathbb{R}_n^+ = \{x : \ x = (x_1, ..., x_n), \ x_1 > 0, ..., x_n > 0\},$$

and $E_+(x, r) = \{y \in \mathbb{R}_n^+ : \ |x - y| \le r\}$.

The Bessel differential operator is defined by

$$B_i = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}, \quad i = 1, 2, ..., n,$$

$v = (v_1, ..., v_n)$, $v_1 > 0, ..., v_n > 0$, $|v| = v_1 + ... + v_n$.

For $1 \le p < \infty$, let $L_{p,v} = L_{p,v}\left(\mathbb{R}_n^+, \left(\prod\limits_{i=1}^n x_i^{2v_i}\right) dx\right)$ be the space of functions measurable on $\mathbb{R}_n^+$ with the following norms

$$\|f\|_{p,v} = \left(\int\limits_{\mathbb{R}_n^+} |f(x)|^p \left(\prod_{i=1}^n x_i^{2v_i}\right) dx\right)^{\frac{1}{p}},$$

$$\|f\|_{\infty,v} = ess\sup_{x\in\mathbb{R}_n^+} |f(x)|.$$

$$|E_+(0,r)|_v = \int\limits_{E_+(0,r)} \left(\prod_{i=1}^n x_i^{2v_i}\right) dx = Cr^{n+2|v|}.$$

Denote by T_x^y the generalized shift operator acting according to the law

$$\mathrm{T}_x^y f(x) = \mathrm{C}_v \int\limits_0^\pi ... \int\limits_0^\pi f\left(\sqrt{x_1^2 + y_1^2 - 2x_1y_1\cos\varphi_1}, ..., \sqrt{x_n^2 + y_n^2 - 2x_ny_n\cos\varphi_n}\right) \times \prod_{i=1}^n (\sin^{2v_i-1}\varphi_i d\varphi_i),$$

where $x, y \in \mathbb{R}_n^+$, $C_v = \prod\limits_{i=1}^n \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$ (see [9-15]). Let f be in $L_{p,v}(R_n^+), 1 \le p < \infty$. Then $T_x^y f$ belongs to $L_{p,v}(R_n^+)$, and

$$\|T_x^y f\|_{p,v} \le \|f\|_{p,v}. \tag{2.1}$$

We remark that T_x^y is closely connected with the Bessel differential operator $B = (B_1, ..., B_n)$(see [9-15]).

The convolution operator determined by the T_x^y is defined by

$$(f * \varphi)_B(x) = \int_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x) (\prod_{i=1}^{n} y_i^{2v_i}) dy.$$

This convolution is known as a B-convolution. We note that the following properties hold (see [9-15]):

a.$(f * \varphi)_B = (\varphi * f)_B$

b. $\|f * \varphi\|_{r,v} \leq \|f\|_{p,v} \|\varphi\|_{q,v}$ $1 \leq p, r \leq \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$

c. $T_x^y.1 = 1$

d. If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function all $x_i > 0 (i = 1, ..., n)$ and

$$\int_{\mathbb{R}_n^+} |f(x)| (\prod_{i=1}^{n} x_i^{2v_i}) dx < \infty,$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) (\prod_{i=1}^{n} y_i^{2v_i}) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x) (\prod_{i=1}^{n} y_i^{2v_i}) dy.$$

e. $|T_x^y f(x)| \leq \sup_{x \geq 0} |f(x)|$.

The maximal function $M_B f(x)$ is defined by

$$M_B f(x) = \sup_{r>0} \frac{1}{|E_+(0,r)|_v} \int_{E_+(0,r)} T_x^y |f(x)| (\prod_{i=1}^{n} y_i^{2v_i}) dy.$$

Theorem 2.1.[1] *(i) If* $f \in L_{1,v}\left(\mathbb{R}_n^+, \left(\prod_{i=1}^{n} x_i^{2v_i}\right) dx\right)$, *then for every* $\alpha > 0$

$$|\{x : M_B f(x) > \lambda\}| \leq \frac{C}{\lambda} \int_{\mathbb{R}_n^+} |f(x)| \prod_{i=1}^{n} x_i^{2v_i} dx,$$

where $C > 0$ *is independent on* f.

(ii) If $f \in L_{p,v}\left(\mathbb{R}_n^+, \left(\prod_{i=1}^n x_i^{2v_i}\right) dx\right),\ 1 < p < \infty,$
then $M_B f \in L_{p,v}\left(\mathbb{R}_n^+, \left(\prod_{i=1}^n x_i^{2v_i}\right) dx\right)$ *and*

$$\|M_B f\|_{p,v} \leq C_p \|f\|_{p,v},$$

where $C_p > 0$ *is independent on* f.

Corollary 2.2. *For* $\forall f \in L_{p,v}\left(\mathbb{R}_n^+, \left(\prod_{i=1}^n x_i^{2v_i}\right) dx\right),\ \forall p' > q,$ *we have*

$$\left\|(M_B(|f|^q))^{\frac{1}{q}}\right\|_{p',v} \leq C \|f\|_{p',v}.$$

In fact, as $p' > q$, that is $\frac{p'}{q} > 1$, by Theorem 2.1, we have

$$\begin{aligned} \left\|(M_B(|f|^q))^{\frac{1}{q}}\right\|_{p',v} &= \|M_B(|f|^q)\|_{\frac{p'}{q},v}^{\frac{1}{q}} \\ &\leq C \||f|^q\|_{\frac{p'}{q},v}^{\frac{1}{q}} \\ &= C \|f\|_{p',v}. \end{aligned}$$

The Fourier-Bessel transformation and its inverse transformation are defined as follows:

$$\begin{aligned} (F_B f)(x) &= C_v^* \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^n j_{v_i-\frac{1}{2}}(x_i y_i)\, y_i^{2v_i}\right) dy, \\ \left(F_B^{-1} f\right)(x) &= (F_B f)(-x),\ C_v^* = \left(\prod_{i=1}^n 2^{v_i-\frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right)\right)^{-1} \end{aligned}$$

where $j_{v_i-\frac{1}{2}}(x_i y_i)$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. There is the following identity for Fourier-Bessel transformation,

$$F_B(f * g)_B(x) = F_B f(x).F_B g(x).$$

The generalized Riesz potential $I_{\alpha,v}f$ of a function f which is sufficiently smooth and small at infinity are defined in terms of the Fourier Bessel transform by

$$F_B(I_{\alpha,v}f)(x) = |x|^{-\alpha} F_B f(x) \quad (x \in \mathbb{R}_n^+,\ \alpha > 0),$$

where the identity is to be understood in the sense of the distribution theory (see [11-15]). This potential is interpreted as the negative powers of the minus Laplace Bessel

$$-\Delta_B = -\sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial}{\partial x_i} \right), \quad \left(x_i \in \mathbb{R}_n^+,\ v_i > 0,\ i = 1, 2, ..., n \right),$$

and have the following $B-$convolution type operator $I_{\alpha,v}f$ is defined by

$$(I_{\alpha,v}f)(x) = (-\Delta_B)^{-\frac{\alpha}{2}} f(x) = C_{n,\alpha,v} \int\limits_{\mathbb{R}_n^+} f(y) T_x^y \, |x|^{\alpha-n-2|v|} \, (\prod_{i=1}^{n} y_i^{2v_i}) dy,$$

where

$$0 < \alpha < n + 2\,|v|\,,\ C_{n,\alpha,v} = \left\{ 2^{\alpha-n} \Gamma(\frac{\alpha}{2}) \frac{1}{\Gamma\left(\frac{n+2|v|-\alpha}{2}\right)} \prod_{i=1}^{n} \Gamma(v_i + \frac{1}{2}) \right\}^{-1}.$$

which is obtained by the generalized shift operator. The operator $I_{\alpha,v}$ is called the generalized Riesz potential generated by the generalized Shift operator.

In recent years, this potential is known as important technical tools in Fourier and Fourier Bessel harmonic analysis. The important properties of this potential were studied by many authors. The readers are advised to find more detailed information about this potential from [9-15].

Lemma 2.3. *Let $x, y \in \mathbb{R}^+$. In this case, there is the following inequality for the generalized shift operator*

$$|x - y|^2 \leq x^2 + y^2 - 2xy\cos\theta \leq (x + y)^2\,,$$

where $\theta \in [0, \pi]$.

3. Main Results

In this section, we will state our main results and give their proofs as follows.

Lemma 3.1. *Let* $p' = \frac{p}{p-1},\ q = \frac{q'}{q'-1},\ p' > q,\ \alpha q' < n + 2\,|v|,$

$$g \in L_{p',v}\left(\mathbb{R}_n^+, \left(\prod_{i=1}^{n} x_i^{2v_i}\right) dx\right),$$

and

$$Ig(y) = \frac{C}{|y|^{n+2|v|-\alpha}} \int\limits_{|x|<\frac{|y|}{2}} \frac{g(x)}{|x|^{\alpha}} (\prod_{i=1}^{n} x_i^{2v_i}) dx.$$

Then

$$\|Ig(y)\|_{p',v} \leq C\,\|g\|_{p',v}\,.$$

Proof. For $\alpha > 0$ and $\alpha q' < n+2\,|v|$,we take $q > \frac{n+2|v|}{n+2|v|-\alpha}$. Thus, by Hölder's inequality, **(c)** and **(d)** we have

$$\begin{aligned}
|Ig(y)| &\leq \frac{C}{|y|^{n+2|v|-\alpha}} \left(\int\limits_{|x|<\frac{|y|}{2}} |g(x)|^q \, (\prod_{i=1}^{n} x_i^{2v_i}) dx \right)^{\frac{1}{q}} \left(\int\limits_{|x|<\frac{|y|}{2}} \frac{1}{|x|^{\alpha q'}} (\prod_{i=1}^{n} x_i^{2v_i}) dx \right)^{\frac{1}{q'}} \\
&\leq \frac{C}{|y|^{n+2|v|-\alpha}} \left(\int\limits_{|x|<\frac{|y|}{2}} |g(x)|^q \, (\prod_{i=1}^{n} x_i^{2v_i}) dx \right)^{\frac{1}{q}} |y|^{(n+2|v|-\alpha q')\frac{1}{q'}} \\
&= \frac{C}{|y|^{\frac{n+2|v|}{q}}} \left(\int\limits_{|x|<\frac{|y|}{2}} |g(x)|^q \, (\prod_{i=1}^{n} x_i^{2v_i}) dx \right)^{\frac{1}{q}} \\
&= C \left(\frac{1}{|y|^{n+2|v|}} \int\limits_{|x|<\frac{|y|}{2}} T_y^x \, |g(y)|^q \, (\prod_{i=1}^{n} x_i^{2v_i}) dx \right)^{\frac{1}{q}} \\
&\leq C \left(M(|g|^q) \right)^{\frac{1}{q}} (y)
\end{aligned}$$

for $\forall p' > q > \frac{n+2|v|}{n+2|v|-\alpha}$, that is $1 < p < \frac{n+2|v|}{\alpha}$, follow by Corollary 2.2, we get I strong (p', p') type.

Theorem 3.2. *Let $p > 1,\ 0 \leq \alpha < \frac{n+2|v|}{p}$, then there exists a constant C such that for $\forall u \in L_{p,v}\left(\mathbb{R}_n^+, \left(\prod\limits_{i=1}^{n} x_i^{2v_i}\right) dx\right)$,*

$$\int\limits_{\mathbb{R}_n^+} \left(\frac{|u(x)|}{|x|^{\alpha}}\right)^p \left(\prod_{i=1}^{n} x_i^{2v_i}\right) dx \quad \leq \quad C \left\|u\right\|_{p,v}^p . \tag{3.1}$$

Remark 3.3. ***i.****If $\alpha = 0$, it is obvious that the result holds true for every $1 < p < \infty$.*

ii. *Without considering the constant C, the inequality is sharp. In details, for $p = \frac{n+2|v|}{\alpha}$, , if we take $u(x) \in C_c^{\infty}(\mathbb{R}_n^+)$ satisfying $u(x) = 1$ for $|x| \leq 1$, $u(x) = 0$ for $|x| \geq 1$, then the above theorem can not hold. In fact*

$$\int\limits_{\mathbb{R}_n^+} \frac{|u(x)|^p}{|x|^{n+2|v|}} \left(\prod_{i=1}^{n} x_i^{2v_i}\right) dx \quad \geq \quad \int\limits_{|x|\leq 1} \frac{1}{|x|^{n+2|v|}} \left(\prod_{i=1}^{n} x_i^{2v_i}\right) dx = \infty$$

while

$$\left\|u\right\|_{p,v} \leq \infty.$$

Proof of the Theorem 3.2. Let $I_{-\alpha,v}u = f$, then $u = I_{\alpha,v}f$. To prove the inequality (3.1), it is sufficient to show that

$$\left\|\frac{I_{\alpha,v}f}{|x|^{\alpha}}\right\|_{p,v} \leq C \left\|f\right\|_{p,v} . \tag{3.2}$$

Let $f \in L_{p,v}\left(\mathbb{R}_n^+, \left(\prod\limits_{i=1}^{n} x_i^{2v_i}\right) dx\right),\ 0 \leq \alpha < \frac{n+2|v|}{p}$ and

$$Af = \frac{I_{\alpha,v}f}{|x|^{\alpha}} = \int\limits_{\mathbb{R}_n^+} \frac{f(y)T_x^y \left|x\right|^{\alpha-n-2|v|}}{|x|^{\alpha}} (\prod_{i=1}^{n} y_i^{2v_i}) dy.$$

Then we have

$$\begin{aligned} Af = \frac{I_{\alpha,v}f}{|x|^{\alpha}} &= \int\limits_{\mathbb{R}_n^+} \frac{f(y)T_x^y\,|x|^{\alpha-n-2|v|}}{|x|^{\alpha}}(\prod_{i=1}^{n} y_i^{2v_i})dy \\ &= \int\limits_{E_+(0,2|x|)} \frac{f(y)T_x^y\,|x|^{\alpha-n-2|v|}}{|x|^{\alpha}}(\prod_{i=1}^{n} y_i^{2v_i})dy \\ &\quad + \int\limits_{\mathbb{R}_n^+\setminus E_+(0,2|x|)} \frac{f(y)T_x^y\,|x|^{\alpha-n-2|v|}}{|x|^{\alpha}}(\prod_{i=1}^{n} y_i^{2v_i})dy \\ &= A_1f + A_2f. \end{aligned}$$

To prove (3.2), we need only prove that both A_1 and A_2 are strong (p,p) type. Firstly, we estimate A_1f. By taking sum with respect to all integer $k<0$, we get

$$\begin{aligned} |A_1f| &\leq \int\limits_{E_+(0,2|x|)} \frac{|y|^{\alpha-n-2|v|}\,|T_x^y f(x)|}{|x|^{\alpha}}(\prod_{i=1}^{n} y_i^{2v_i})dy \\ &= \sum_{k=-\infty}^{0} \int\limits_{2^k|x|\leq|y|<2^{k+1}|x|} \frac{|y|^{\alpha-n-2|v|}\,|T_x^y f(x)|}{|x|^{\alpha}}(\prod_{i=1}^{n} y_i^{2v_i})dy \\ &\leq C\sum_{k=-\infty}^{0}(2^k\,|x|)^{-n-2|v|} \int\limits_{2^k|x|\leq|y|<2^{k+1}|x|} T_x^y\,|f(x)|\,(\prod_{i=1}^{n} y_i^{2v_i})dy \\ &\leq CM_{\alpha}f. \end{aligned}$$

From Theorem 2.1, we have $\|A_1f\|_{p,v} \leq C\,\|f\|_{p,v}$.

Now, we consider $A_2 f$. From Lemma 2.3 and the properties of T_x^y, we have

$$\begin{aligned} A_2 f(x) &= \int\limits_{\mathbb{R}_n^+ \setminus E_+(0,2|x|)} \frac{f(y) T_x^y \left|x\right|^{\alpha-n-2|v|}}{\left|x\right|^{\alpha}} (\prod_{i=1}^n y_i^{2v_i}) dy \\ &\leq \int\limits_{|y|>2|x|} \frac{\left|y-x\right|^{\alpha-n-2|v|}}{\left|x\right|^{\alpha}} f(y) (\prod_{i=1}^n y_i^{2v_i}) dy \\ &\leq C \int\limits_{|y|>2|x|} \frac{f(y)}{\left|y\right|^{n+2|v|-\alpha} \left|x\right|^{\alpha}} (\prod_{i=1}^n y_i^{2v_i}) dy \triangleq B_2 f(x). \end{aligned}$$

For every $g \in L_{p',v}\left(\mathbb{R}_n^+, \left(\prod_{i=1}^n x_i^{2v_i}\right) dx\right)$, we get

$$\begin{aligned} < B_2 f(x),\ g(x) > &= \int\limits_{\mathbb{R}_n^+} B_2 f(x) g(x) (\prod_{i=1}^n x_i^{2v_i}) dx \\ &= C \int\limits_{\mathbb{R}_n^+} \left[\int\limits_{|y|>2|x|} \frac{f(y)}{\left|y\right|^{n+2|v|-\alpha} \left|x\right|^{\alpha}} (\prod_{i=1}^n y_i^{2v_i}) dy \right] g(x) (\prod_{i=1}^n x_i^{2v_i}) dx \\ &= C \int\limits_{\mathbb{R}_n^+} \left[\frac{1}{\left|y\right|^{n+2|v|-\alpha}} \int\limits_{|x|<\frac{|y|}{2}} \frac{g(x)}{\left|x\right|^{\alpha}} (\prod_{i=1}^n x_i^{2v_i}) dx \right] f(y) (\prod_{i=1}^n y_i^{2v_i}) dy \\ &= < Ig(y),\ f(y) > \end{aligned}$$

where

$$Ig(y) = \frac{C}{\left|y\right|^{n+2|v|-\alpha}} \int\limits_{|x|<\frac{|\mathbf{y}|}{2}} \frac{g(x)}{\left|x\right|^{\alpha}} (\prod_{i=1}^n x_i^{2v_i}) dx.$$

From Hölder's inequality and Lemma 3.1, we have

$$\begin{aligned} < B_2 f(x),\ g(x) > &= < Ig(y),\ f(y) > \\ &= \left\|Ig(y)\right\|_{p',v} \left\|f(y)\right\|_{p,v} \\ &\leq C \left\|g\right\|_{p',v} \left\|f(y)\right\|_{p,v}. \end{aligned}$$

Here, B_2 is strong (p,p) type by definition of norm. Hence, A_2 is also a strong (p,p). This proves the theorem.

Remark 3.4 *Using our method for $p=1$, we can only get A_1 is weak* $(1,1)$ *type.*

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