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Some Properties of Certain Classes of Meromorphic Univalent Functions *

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Abstract

In this paper, we obtain distortion theorem and the Hadamard products of functions belonging the class $\Omega^+(p;\alpha,\beta,\gamma,A,B,\lambda)$ of meromorphic functions with positive and missing coefficients. Also some properties of neighborhoods of functions in the class $\Omega(\alpha,\beta,\gamma,A,B,\lambda)$ are investigated.

Keywords and Phrases: *Meromorphic, Neighborhoods, Hadamard product.*

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1. Introduction

Let $\sum$ denote the class of functions of the form :

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_kz^k \qquad (1.1)$$

which are analytic and univalent in the punctured disc $U^*=\{z:z\in C \text{ and } 0<|z|<1\}=U\backslash\{0\}$ and which have a simple pole at the origin with residue one there. Let $\sum_p$ denote the class of functions $f(z)$ defined by (1.1) with $a_j=0$ $(j=1,2,..,p-1\,; p\in N=\{1,2,...\})$ that is , by

$$f(z)\;=\frac{1}{z}+\sum_{k=p}^{\infty}a_kz^k \qquad (p\in N), \qquad (1.2)$$

which are analytic and univalent in U^* . Setting

$$F(z)=(1-\lambda)f(z)\;+\lambda zf^{'}(z)\;\;(f\in\sum\nolimits_p\;;\;\;0\le\lambda<\frac{1}{2}), \qquad (1.3)$$

so that , obviously ,

$$F(z)=\frac{1-2\lambda}{z}+\sum_{k=p}^{\infty}[1+\lambda(k-1)]a_kz^k\;\;(p\in N\,;\;0\le\lambda<\frac{1}{2}),$$

since $f(z)\in\sum_p$ is given by (1.2) .

For a function $f(z)\in\sum_p$, we say that $f(z)$ is a member of the class $\Omega(p;\alpha,\beta,\gamma,A,B,\lambda)$ if the function $F(z)$ defined by (1.3) satisfies the inequality :

$$\left|\frac{z^2F^{'}(z)+(1-2\lambda)}{[(B-A)\gamma-B]z^2F^{'}(z)+(1-2\lambda)[(B-A)\gamma\alpha-B]}\right|<\beta\quad(z\in U^*), \quad (1.4)$$

where (and throughout this paper) the parameters α,β,γ,A,B and λ are constrained as follows:

$$0\le\alpha<1;\;0<\beta\le1\,;-1\le A<B\le1\,;\;\;0<B\le1\,;$$
$$\frac{B}{(B-A)}<\gamma\le\left\{\begin{array}{ll}\dfrac{B}{(B-A)\alpha} & (\alpha\ne0)\\ 1 & (\alpha=0)\end{array}\right\}.$$

We note that $\Omega(1;\alpha,\beta,\gamma,A,B,\lambda)=\Omega(\alpha,\beta,\gamma,A,B,\lambda)$.

Furthermore, we say that a function $f(z)\in\Omega^{+}(p;\alpha,\beta,\gamma,A,B,\lambda)$ whenever $f(z)$ is of the form [cf. Equation (1.2)] :

$$f(z)=\frac{1}{z}+\sum_{k=p}^{\infty}|a_k|\,z^k \quad (k\geq p\,;\,p\in N). \tag{1.5}$$

The class $\Omega^{+}(p;\alpha,\beta,\gamma,A,B,\lambda)$ was introduced and studied by Joshi et al. [8]. We note that :

(i)$\Omega^{+}(p;\alpha,\beta,\gamma,-1,1,0)=\sum_p(\alpha,\beta,\gamma)(0\leq\alpha<1;0<\beta\leq1;\frac{1}{2}\leq\gamma\leq1)$ (Cho et al. [6]) ;

(ii) $\Omega^{+}(1;\alpha,\beta,\gamma,-1,1,0)=\sum_1(\alpha,\beta,\gamma)(0\leq\alpha<1;0<\beta\leq1;\frac{1}{2}\leq\gamma\leq1)$ (Cho et al. [5]) ;

(iii) $\Omega^{+}(1;0,1,1,-A,-B,0)=\sum_d(A,B)(-1\leq B<A\leq1;-1\leq B<0)$ (Cho [4]).

In order to derive our results , we need the following lemma given by Joshi et al. [8].

Lemma 1.1. [8]. *Let $f(z)\in\sum_p^+$ be given by (1.5). Then $f(z)\in$ $\Omega^{+}(p;\alpha,\beta,\gamma,A,B,\lambda)$ if and only if*

$$\sum_{k=p}^{\infty}k\left\{1+\beta[(B-A)\gamma-B]\right\}\left[1+\lambda(k-1)\right]|a_k|\leq(B-A)\beta\gamma(1-\alpha)(1-2\lambda).$$

2. Distortion Theorem

Theorem 2.1. *If a function $f(z)$ defined by (1.5) is in the class $\Omega^{+}(p;\alpha,\beta,\gamma,A,B,\lambda)$, then*

$$\left\{m!-\frac{(p-1)!(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}{(p-m)!\{1+\beta[(B-A)\gamma-B]\}[1+\lambda(p-1)]}r^{p+1}\right\}r^{-(m+1)}$$

$$\leq\left|f^{(m)}(z)\right|\leq$$

$$\left\{m!+\frac{(p-1)!(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}{(p-m)!\{1+\beta[(B-A)\gamma-B]\}[1+\lambda(p-1)]}r^{p+1}\right\}r^{-(m+1)} \tag{2.1}$$

$$(0 < |z| = r < 1; p \in N; 0 \leq m < p).$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \frac{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}{p\left\{1+\beta[(B-A)\gamma - A]\right\}[1+\lambda(p-1)]} z^p \quad (p \in N). \tag{2.2}$$

Proof. In view of Lemma 1.1, we have

$$\frac{p\left\{1+\beta[(B-A)\gamma - B]\right\}[1+\lambda(p-1)]}{p!} \sum_{k=p}^{\infty} k! \, |a_k|$$

$$\leq \sum_{k=p}^{\infty} k\left\{1+\beta[(B-A)\gamma - B]\right\}[1+\lambda(k-1)] \, |a_k| \leq (B-A)\beta\gamma(1-\alpha)(1-2\lambda),$$

which yields

$$\sum_{k=p}^{\infty} k! \, |a_k| \leq \frac{(p-1)!(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}{\left\{1+\beta[(B-A)\gamma - B]\right\}[1+\lambda(p-1)]} \quad (p \in N). \tag{2.3}$$

Now , by differentiating both sides of (1.5) m times with respect to z , we have

$$f^{(m)}(z) = (-1)^m m! \, z^{-(m+1)} + \sum_{k=p}^{\infty} \frac{k!}{(k-m)!} |a_k| \, z^{k-m},$$

$$(p \in N; 0 \leq m < p), \tag{2.4}$$

and Theorem 2.1 follows easily from (2.3) and (2.4) .

Finally , it is easy to see that the bounds in (2.1) are attained for the function $f(z)$ given by (2.2) at the points $z = r, \pm ir$ $(0 < r < 1)$.

3. Neighborhoods and Partial Sums

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [7] and Ruscheweyh [12], and (more recently) by Altintas et al. ([1], [2] and [3]), Liu [9] and Liu and Srivastava (

[10] and [11]) , we begin by introducing here the $\delta-$neighborhood of a function $f(z) \in \sum$ of the form (1.1) by means of the definition given below :

$$N_\delta(f) = \left\{ g \in \sum : g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \quad \text{and} \right.$$

$$\sum_{k=1}^{\infty} \frac{k\left\{1+\beta[(B-A)\gamma - B]\right\}[1+\lambda(k-1)]}{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)} \quad |a_k - b_k| \le \delta,$$

$$\left. (-1 \le A < B \le 1\ , 0 \le \lambda < \frac{1}{2},\ p \in N,\ \delta > 0) \right\}. \tag{3.1}$$

Making use of the definition (3.1) , we now prove Theorem 3.1 below :

Theorem 3.1. *Let the function $f(z)$ defined by (1.1) be in the class $\Omega(\alpha,\beta,\gamma,A,B,\lambda)$.*

If $f(z)$ satisfies the following condition :

$$\frac{f(z)+\varepsilon z^{-1}}{1+\varepsilon} \in \Omega(\alpha,\beta,\gamma,A,B,\lambda)\,(\varepsilon \in C, |\varepsilon| < \delta, \delta > 0), \tag{3.2}$$

then

$$N_\delta(f) \subset \Omega(\alpha,\beta,\gamma,A,B,\lambda). \tag{3.3}$$

Proof. It is easily seen from (1.4) that $g(z) \in \Omega(\alpha,\beta,\gamma,A,B,\lambda)$ if and only if for any complex number σ with $|\sigma| = 1$,

$$\frac{z^2F'(z) + 1 - 2\lambda}{[(B-A)\gamma - B]F'(z) + (1-2\lambda)[(B-A)\gamma\alpha - B]} \ne \sigma\beta \quad (z \in U),$$

which, is equivalent to

$$\frac{(g*h)(z)}{z^{-1}} \ne 0 \quad (z \in U), \tag{3.4}$$

where , for convenience,

$$\begin{aligned} h(z) &= \frac{1}{z} + \sum_{k=1}^{\infty} c_k z^k \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{k\left\{1 - \beta\sigma[(B-A)\gamma - A]\right\}[1+\lambda(k-1)]}{(B-A)\sigma\beta\gamma(1-\alpha)(1-2\lambda)} z^k. \end{aligned} \tag{3.5}$$

From (3.5) , we have

$$|c_k| \le \frac{k\{1+\beta[(B-A)\gamma - A]\}[1+\lambda(k-1)]}{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)} \quad (k \in N, 0 \le \lambda < \frac{1}{2}, p \in N).$$

Now, if $f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \in \sum$ satisfies the condition (3.2), then (3.4) yields

$$\left| \frac{(f * h)(z)}{z^{-1}} \right| \ge \delta \quad (z \in U; \delta > 0).$$

By letting

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k \in N_\delta(f),$$

so that

$$\left| \frac{[g(z) - f(z)] * h(z)}{z^{-1}} \right| = \left| \sum_{k=1}^{\infty} (b_k - a_k) c_k z^{k+1} \right|$$

$$\begin{aligned} &\le \ |z| \sum_{k=1}^{\infty} \frac{k\{1+\beta[(B-A)\gamma - A]\}[1+\lambda(k-1)]}{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)} |b_k - a_k| \\ &< \ \delta \ (z \in U;\ \delta > 0). \end{aligned}$$

Thus we have (3.4), and hence also for any $\sigma \in C$ such that $|\sigma| = 1$, which implies that $g(z) \in \Omega(\alpha, \beta, \gamma, A, B, \lambda)$. This evidently proves the assertion (3.3) of Theorem 3.1.

Next we prove the following result.

Theorem 3.2. *Let $f(z) \in \sum$ be given by (1.1) and define the partial sums $s_1(z)$ and $s_m(z)$ as follows :*

$$s_1(z) = \frac{1}{z} \quad \textit{and} \quad s_m(z) = \frac{1}{z} + \sum_{k=1}^{m-1} a_k z^k \quad (m \in N).$$

Suppose also that

$$\sum_{k=1}^{\infty} d_k z^k \ \le 1 \,(\, d_k \ = \ \frac{k\{1+\beta[(B-A)\gamma - A]\}[1+\lambda(k-1)]}{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}). \tag{3.6}$$

Then we have

$$(i) \qquad f(z) \in \Omega(\alpha, \beta, \gamma, A, B, \lambda),$$

$$(ii) \quad Re\left\{\frac{f(z)}{s_m(z)}\right\} > 1 - \frac{1}{d_m} \quad (z \in U; m \in N), \tag{3.7}$$

and

$$(iii)\ Re\left\{\frac{s_m(z)}{f(z)}\right\} > \frac{d_m}{1 + d_m} \quad (z \in U; m \in N). \tag{3.8}$$

The estimates (3.7) and (3.8) are sharp for each $m \in N$.

Proof. It is not difficult to see that

$$z^{-1} \in \Omega(\alpha, \beta, \gamma, A, B, \lambda).$$

Thus, from Theorem 3.1 and the hypothesis (3.6) of Theorem 3.2, we have

$$N_1(z^{-1}) \subset \Omega(\alpha, \beta, \gamma, A, B, \lambda),$$

which shows that $f(z) \in \Omega(\alpha, \beta, \gamma, A, B, \lambda)$ as asserted by Theorem 3.2.

(ii) For the coefficients d_k given by (3.6), it is not difficult to verify that

$$d_{k+1} > d_k > 1 \ \ (k \in N).$$

Therefore , we have

$$\sum_{k=1}^{m-1} |a_k| + d_m \sum_{k=m}^{\infty} |a_k| \le \sum_{k=1}^{\infty} d_k |a_k| \le 1, \tag{3.9}$$

where we have used the hypothesis (3.6) again .

By setting

$$h_1(z) = d_m \left\{\frac{f(z)}{s_m(z)} - (1 - \frac{1}{d_m})\right\}$$

$$= 1 + \frac{d_m \sum\limits_{k=m}^{\infty} a_k z^{k+1}}{1 + \sum\limits_{k=1}^{m-1} a_k z^{k+1}} \ ,$$

and applying (3.9), we find that

$$\left|\frac{h_1(z)-1}{h_1(z)+1}\right| \leq \frac{d_m \sum\limits_{k=m}^{\infty} |a_k|}{2 - z\sum\limits_{k=1}^{m-1} |a_k| - d_m \sum\limits_{k=m}^{\infty} |a_k|} \leq 1 \quad (z \in U),$$

which readily yields the assertion (3.7) of Theorem 3.2. If we take

$$f(z) = \frac{1}{z} - \frac{z^m}{d_m} \ , \tag{3.10}$$

then

$$\frac{f(z)}{s_m(z)} = 1 - \frac{z^{m+1}}{d_m} \to 1 - \frac{1}{d_m}, \ \text{ as } \quad z \to 1^- \ ,$$

which shows that the bound in (3.7) is the best possible for each $m \in N$.

(iii) Just as in Part (ii) above , if we put

$$h_2(z) = (1+d_m)\Big(\frac{s_m(z)}{f(z)} - \frac{d_m}{1+d_m}\Big)$$

$$= 1 - \frac{(1+d_m)\sum\limits_{k=m}^{\infty} a_k z^{k+1}}{1+\sum\limits_{k=1}^{\infty} a_k z^{k+1}} \ ,$$

and make use of (3.9), we can deduce that

$$\left|\frac{h_2(z)-1}{h_2(z)+1}\right| \leq \frac{(1+d_m)\sum\limits_{k=m}^{\infty} |a_k|}{2 - 2\sum\limits_{k=1}^{m-1} |a_k| - (1-d_m)\sum\limits_{k=m}^{\infty} |a_k|} \leq 1 \quad (z \in U),$$

which leads us immediately to the assertion (3.8) of Theorem 3.2.

The bound in (3.8) is sharp for each $m \in N$, with the extremal function $f(z)$ given by (3.10). The proof of Theorem 3.2 is thus completed.

4. Convolution Properties

For the functions

$$f_j(z) = \frac{1}{z} + \sum_{k=p}^{\infty} |a_{k,j}|\, z^k \quad (j = 1, 2; p \in N), \tag{4.1}$$

we denote by $(f_1 * f_2)(z)$ the Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=p}^{\infty} |a_{k,1}|\, |a_{k,2}|\, z^k.$$

Theorem 4.1. *Let the functions $f_j(z)(j = 1, 2)$ defined by (4.1) be in the class $\Omega^+(p; \alpha, \beta, \gamma, A, B, \lambda)$.*

Then $(f_1 * f_2)(z) \in \Omega^+(p; \delta, \beta, \gamma, A, B, \lambda)$, where

$$\delta = 1 - \frac{(B - A)\beta\gamma(1 - \alpha)^2(1 - 2\lambda)}{p\,\{1 + \beta[(B - A)\gamma - B]\}\,[1 + \lambda(p - 1)]}\,.$$

The result is sharp for the functions

$$f_j(z) = \frac{1}{z} + \frac{(B - A)\beta\gamma(1 - \alpha)^2(1 - 2\lambda)}{p\,\{1 + \beta[(B - A)\gamma - B]\}\,[1 + \lambda(p - 1)]} z^p \quad (j = 1, 2;\ p \in N). \tag{4.2}$$

Proof. Employing the technique used earlier by Schild and Silverman [13], we need to find the largest δ such that

$$\sum_{k=p}^{\infty} \frac{k\,\{1 + \beta[(B - A)\gamma - B]\}\,[1 + \lambda(k - 1)]}{(B - A)\beta\gamma(1 - \delta)(1 - 2\lambda)}\, |a_{k,1}|\, |a_{k,2}| \le 1$$

for $f_j(z) \in \Omega^+(p; \alpha, \beta, \gamma, A, B, \lambda)(j = 1, 2)$. Since $f_j(z) \in \Omega^+(p; \alpha, \beta, \gamma, A, B, \lambda)$ $(j = 1, 2)$, we readily see that

$$\sum_{k=p}^{\infty} \frac{k\,\{1 + \beta[(B - A)\gamma - B]\}\,[1 + \lambda(k - 1)]}{(B - A)\beta\gamma(1 - \alpha)(1 - 2\lambda)}\, |a_{k,j}| \le 1 \quad (j = 1, 2).$$

Therefore, by the Cauchy - Schwarz inequality, we obtain

$$\sum_{k=p}^{\infty} \frac{k\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(k-1)]}{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)} \sqrt{|a_{k,1}|\,|a_{k,2}|} \leq 1. \tag{4.3}$$

This implies that we need only to show that

$$\frac{|a_{k,1}|\,|a_{k,2}|}{(1-\delta)} \leq \frac{\sqrt{|a_{k,1}|\,|a_{k,2}|}}{(1-\alpha)} \quad (k \geq p)$$

or, equivalently, that

$$\sqrt{|a_{k,1}|\,|a_{k,2}|} \leq \frac{(1-\delta)}{(1-\alpha)} \quad (k \geq p).$$

Hence, by the inequality (4.3), it is sufficient to prove that

$$\frac{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}{k\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(k-1)]} \leq \frac{(1-\delta)}{(1-\alpha)} \quad (k \geq p). \tag{4.4}$$

It follows from (4.10) that

$$\delta \leq 1 - \frac{(B-A)\beta\gamma(1-\alpha)^2(1-2\lambda)}{k\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(k-1)]} \quad (k \geq p).$$

Now, defining the function $\varphi(k)$ by

$$\varphi(k) = 1 - \frac{(B-A)\beta\gamma(1-\alpha)^2(1-2\lambda)}{k\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(k-1)]} \quad (k \geq p).$$

We see that $\varphi(k)$ is an increasing function of k. Therefore, we conclude that

$$\delta \leq \varphi(p) = 1 - \frac{(B-A)\beta\gamma(1-\alpha)^2(1-2\lambda)}{p\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(p-1)]} \, ,$$

which evidently completes the proof of Theorem 4.1.

Using arguments similar to those in the proof of Theorem 4.1, we obtain the following result.

Theorem 4.2. *Let the function $f_1(z)$ defined by (4.1) be in the class $\Omega^+(p;\alpha,\beta,\gamma,A,B,\lambda)$. Suppose also that the function $f_2(z)$ defined by (4.1) be in the class $\Omega^+(p;\zeta,\beta,\gamma,A,B,\lambda)$. Then $(f_1 * f_2)(z) \in \Omega^+(p;\zeta,\beta,\gamma,A,B,\lambda)$, where*

$$\xi = 1 - \frac{(B-A)\beta\gamma(1-\alpha)(1-\zeta)(1-2\lambda)}{p\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(p-1)]}.$$

The result is sharp for the functions $f_j(z)(j=1,2)$ given by

$$f_1(z)=\frac{1}{z}+\frac{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}{p\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(p-1)]}z^p \ \ (p\in N),$$

and

$$f_2(z)=\frac{1}{z}+\frac{(B-A)\beta\gamma(1-\zeta)(1-2\lambda)}{p\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(p-1)]}z^p \ \ (p\in N).$$

Theorem 4.3. *Let the functions $f_j(z)(j=1,2)$ defined by (4.1) be in the class $\Omega^+(p;\alpha,\beta,\gamma,A,B,\lambda)$.*

Then the function $h(z)$ defined by

$$h(z)=\frac{1}{z}+\sum_{k=p}^{\infty}(|a_{k,1}|^2+|a_{k,2}|^2)z^k$$

belongs to the class $\Omega^+(p;\tau,\beta,\gamma,A,B,\lambda)$, where

$$\tau=1-\frac{2(B-A)\beta\gamma(1-\alpha)^2(1-2\lambda)}{p\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(p-1)]}.$$

This result is sharp for the functions $f_j(z)(j=1,2)$ given already by (4.2).

Proof. Noting that

$$\sum_{k=p}^{\infty}\frac{(k\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(k-1)])^2}{[(B-A)\beta\gamma(1-\alpha)(1-2\lambda)]^2}|a_{k,j}|^2$$

$$\leq(\sum_{k=p}^{\infty}\frac{k\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(k-1)]}{(B-A)\beta\gamma(1-\alpha)(1-2\lambda)}|a_{k,j}|)^2\leq 1 \ \ (j=1,2),$$

for $f_j(z)\in\Omega^+(p;\alpha,\beta,\gamma,A,B,\lambda)(j=1,2)$, we have

$$\sum_{k=p}^{\infty}\frac{(k\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(k-1)])^2}{2[(B-A)\beta\gamma(1-\alpha)(1-2\lambda)]^2}(|a_{k,1}|^2+|a_{k,2}|^2)\leq 1.$$

Therefore, we have to find the largest τ such that

$$\frac{1}{(1-\tau)}\leq\frac{k\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(k-1)]}{2(B-A)\beta\gamma(1-\alpha)(1-2\lambda)} \quad (k\geq p),$$

that is, that

$$\tau \leq 1 - \frac{2(B-A)\beta\gamma(1-\alpha)^2(1-2\lambda)}{k\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(k-1)]} \quad (k \geq p).$$

Now, defining a function $\Psi(k)$ by

$$\Psi(k) = 1 - \frac{2(B-A)\beta\gamma(1-\alpha)^2(1-2\lambda)}{k\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(k-1)]} \quad (k \geq p).$$

We observe that $\Psi(k)$ is an increasing function of k. We thus conclude that

$$\tau \leq \Psi(p) = 1 - \frac{2(B-A)\beta\gamma(1-\alpha)^2(1-2\lambda)}{p\left\{1+\beta[(B-A)\gamma-B]\right\}[1+\lambda(k-1)]},$$

which completes the proof of Theorem 4.3 .

References

[1] O. Altintas and S. Owa, Neighborhoods of certain analytic functions with negative coefficients, *Internat. J. Math. Math. Sci.*, **19**(1996), 797-800.

[2] O. Altintas, O. Ozkan, and H. M. Srivastava, Neighborhoods of a class of analytic functions with negative coefficients, *Appl. Math. Lett.*, **13** no.3 (2000), 63-67.

[3] O. Altintas, O. Ozkan, and H. M. Srivastava, Neighborhoods of a certain family of multivalent functions with negative coefficient, *Comput. Math. Appl.*, **47**(2004), 1667-1672.

[4] N. E. Cho, On certain class of meromorphic functions with positive coefficients, *J. Inst. Math. Comput. Sci. (Math. Ser.)*, **3** no.2 (1990) , 119-125.

[5] N. E. Cho, S. H. Lee, and S. Owa, A class of meromorphic univalent functions with positive coefficients, *Kobe J. Math.*, **4**(1987) , 43-50.

[6] N. E. Cho, S. Owa, S. H. Lee, and O. Altintas, Generalization class of certain meromorphic univalent functions with positive coefficients, *Kyungpook Math. J.*, **29** no.2 (1989), 133-139.

[7] A. W. Goodman, Univalent functions and non analytic curves, *Proc. Amer. math. Soc.*, **8**(1957), 598-601.

[8] S. B. Joshi, S. R. Kulkarni, and H. M. Srivastava, Certain classes of meromorphic functions with positive and missing coefficients, *J. Math. Anal. Appl.*, **193**(1995), 1-14.

[9] J.- L. Liu, Properties of some families of meromorphically p-valent functions, *Math. Japon.*, **52** no.3 (2000), 425-434.

[10] J.- L. Liu and H. M. Srivastava, A linear operator and associated families of meromorphically multivalent functions, *J. Math. Anal. Appl.*, **259**(2001), 566-581.

[11] J.- L. Liu and H. M. Srivastava, Subclasses meromorphically multivalent functions associated with a certain linear operator, *Math. Comput. Modelling*, **39**(2004), 35-44.

[12] St. Ruscheweyh, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, **81**(1981), 521-527.

[13] A. Schild and H. Silverman, Convolution of univalent functions with negative coefficients, *Ann. Univ. Mariae Curie - Sklodswska Sect. A*, **29**(1975), 99-107.