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On Some Sequence Spaces *

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Abstract

In this article we introduce some sequence spaces with base space X, a real linear n-normed space. We also use an Orlicz function to construct the spaces. We investigate these spaces for some algebraic and topological structures.

Keywords and Phrases: *n-norm, Orlicz function, Sequence space, Completeness.*

1. Introduction

Let w denote the space of all real or complex sequences. By c, c_0 and ℓ_∞, we denote the Banach spaces of all convergent, null and bounded sequences $x = (x_k)$, respectively normed by $\|x\| = \sup\limits_k |x_k|$.

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An Orlicz function is a function $M : [0, \infty) \longrightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \longrightarrow \infty$, as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [9] used the Orlicz function and introduced the sequence space ℓ_M as follows:

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \quad \text{for some } \rho > 0 \right\}.$$

They proved that ℓ_M is a Banach space normed by

$$\|(x_k)\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

The concept of 2-normed spaces was initially developed by Gähler [2] in the mid of 1960's while that of n-normed spaces can be found in Misiak [10]. Since then, many others have studied this concept and obtained various results, see for instance Gunawan [4, 5], and Gunawan and Mashadi [7].

Let $n \in N$ and let X be a real linear space of dimension d, where $d \geq n$. A real valued function $\|\bullet, \bullet, \ldots, \bullet\|$ on X^n satisfying the following conditions:

nN_1: $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

nN_2: $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,

nN_3: $\|x_1, x_2, \ldots, x_{n-1}, \alpha x_n\| = |\alpha| \; \|x_1, x_2, \ldots, x_n\|$ for all $\alpha \in R$,

nN_4: $\|x_1, x_2, \ldots, x_{n-1}, y+z\| \leq \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$ for all $y, z, x_1, x_2, \ldots, x_{n-1} \in X$,

then the function $\|\bullet, \bullet, \ldots, \bullet\|$ is called an n-norm on X and the pair $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is called an n-normed space.

A trivial example of an n-normed space is $X = R^n$ equipped with the following Euclidean n-norm:

$$\|x_1, x_2, \ldots, x_n\|_E = \text{abs}\left(\begin{vmatrix} x_{11} & L & x_{1n} \\ M & O & M \\ x_{n1} & L & x_{nn} \end{vmatrix} \right),$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in R^n$ for each $i = 1, 2, \ldots, n$.

Gunawan and Mashadi [7] showed that if $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be a linearly independent set in X. Then the following function $\|\bullet,\bullet,\ldots,\bullet\|_\infty$ on X^{n-1} defined by

$$\|x_1,x_2,\ldots,x_{n-1}\|_\infty=\max\{\|x_1,x_2,\ldots,x_{n-1},a_i\|:i=1,2,\ldots,n\} \tag{1.1}$$

defines an $(n-1)$ norm on X with respect to $\{a_1,a_2,\ldots,a_n\}$.

Gunawan and Mashadi [7] also defined the standard n-norm on X, a real inner product space of dimension $d\geq n$ as follows:

$$\|x_1,x_2,\ldots,x_n\|_S=\begin{vmatrix}<x_1,x_1> & L & <x_1,x_n>\\ M & O & M\\ <x_n,x_1> & L & <x_n,x_n>\end{vmatrix}^{\frac{1}{2}},$$

where $<\bullet,\bullet>$ denotes the inner product on X. If $X=R^n$, then this n-norm is exactly the same as the Euclidean n-norm $\|x_1,x_2,\ldots,x_n\|_E$ mentioned earlier. For $n=1$, this n-norm is the usual norm $\|x_1\|=<x_1,x_1>^{\frac{1}{2}}$.

A sequence (x_k) in an n-normed space $(X,\|\bullet,\bullet,\ldots,\bullet\|)$ is said to *converge* to some $L\in X$ in the n-norm if $\lim\limits_{k\to\infty}\|x_k-L,w_2,w_3\ldots,w_n\|=0$, for every $w_2,w_3\ldots,w_n\in X$.

A sequence (x_k) in an n-normed space $(X,\|\bullet,\bullet,\ldots,\bullet\|)$ is said to be *Cauchy* sequence with respect to the n-norm if $\lim\limits_{k,l\to\infty}\|x_k-x_l,w_2,w_3\ldots,w_n\|=0$, for every $w_2,w_3\ldots,w_n\in X$.

If every Cauchy sequence in X converges to some $L\in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Now we state the following three usefull results as Lemmas which can be found in [7].

Lemma 1.1. *Every n-normed space is an $(n-r)$-normed space for all $r=1,2,\ldots,n-1$. In particular, every n-normed space is a normed space.*

Lemma 1.2. *A standard n-normed space is complete if and only if it is complete with respect to the usual norm $\|\bullet\|_S=<\bullet,\bullet>^{\frac{1}{2}}$.*

Lemma 1.3. *On a standard n-normed space X, the derived (n-1)-norm $\|.,\ldots,.\|_\infty$, defined with respect to orthonormal set $\{e_1,e_2,\ldots,e_n\}$, is equivalent to the standard (n-1)-norm $\|\bullet,\bullet,\ldots,\bullet\|_S$. Precisely, we have*

$$\|x_1,x_2,\ldots,x_{n-1}\|_\infty\leq\|x_1,x_2,\ldots,x_{n-1}\|_S\leq\sqrt{n}\|x_1,x_2,\ldots,x_{n-1}\|_\infty$$

for all $x_1,x_2,\ldots,x_{n-1}$, where

$$\|x_1,x_2,\ldots,x_{n-1}\|_\infty=\max\{\|x_1,x_2,\ldots,x_{n-1},e_i\|_S:i=1,2,\ldots,n\}.$$

Let $(\|\bullet, \bullet, \ldots, \bullet\|_X)$ be a real linear n-normed space and $w(X)$ denotes the X-valued sequence space. Then for an Orlicz function M we define the following sequence spaces:

$c_0(X, M) =$

$$\left\{(x_k) \in w(X) : \lim_{k\to\infty} M\left(\left\|\tfrac{x_k}{\rho}, z_1, \ldots, z_{n-1}\right\|_X\right) = 0,\ z_1, \ldots, z_{n-1} \in X,\ \text{for some } \rho > 0\right\},$$

$$c(X, M) = \{(x_k) \in w(X) : \lim_{k\to\infty} M\left(\left\|\frac{x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right\|_X\right) = 0,$$
$$z_1, \ldots, z_{n-1} \in X \text{ and for some } L \in X, \rho > 0\},$$

$\ell_\infty(X, M) =$

$$\left\{(x_k) \in w(X) : \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\tfrac{x_k}{\rho}, z_1, \ldots, z_{n-1}\right\|_X\right) < \infty,\ \text{for some } \rho > 0\right\}.$$

In the above definition of spaces, n-norm $\|\bullet, \bullet, \ldots, \bullet\|_X$ on X is either a standard n-norm or a non-standard n-norm. In general we write $\|\bullet, \bullet, \ldots, \bullet\|_X$ and for standard case we write $\|\bullet, \bullet, \ldots, \bullet\|_S$. Again for derived norm we use $\|\bullet, \bullet, \ldots, \bullet\|_\infty$.

It is obvious that $c_0(X, M) \subset c(X, M)$. Again $c(X, M) \subset \ell_\infty(X, M)$ follows from the following inequality:

$$\begin{aligned} M\left(\left\|\frac{x_k}{2\rho}, z_1, \ldots, z_{n-1}\right\|_X\right) &\leq \frac{1}{2} M\left(\left\|\frac{x_k - L}{\rho}, z_1, \ldots, z_{n-1}\right\|_X\right) \\ &+ \frac{1}{2} M\left(\left\|\frac{L}{\rho}, z_1, \ldots, z_{n-1}\right\|_X\right) \end{aligned}$$

2. Main Results

In this section we investigate the main results of this article involving the sequence spaces $c_0(X, M)$, $c(X, M)$ and $\ell_\infty(X, M)$.

Theorem 2.1. *The spaces $c_0(X, M)$, $c(X, M)$ and $\ell_\infty(X, M)$ are linear spaces.*

Proof. The proof of this theorem can be proved very easily.

Theorem 2.2. *The spaces $c_0(X, M)$, $c(X, M)$ and $\ell_\infty(X, M)$ are normed linear spaces, normed by $\| \bullet \|_0$ defined by*

$$\|x\|_0 = \inf\left\{\rho > 0 : \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{x_k}{\rho}, z_1, \ldots, z_{n-1}\right\|_X\right) \leq 1\right\} \qquad (2.1)$$

Proof. If $x=\theta$, then clearly $\|x\|_0=0$. Conversely assume $\|x\|_0=0$, Then using equation (2.1), we have

$$\inf\left\{\rho>0:\sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{x_k}{\rho},z_1,\ldots,z_{n-1}\right\|_X\right)\leq 1\right\}=0.$$

This implies that for a given $\varepsilon>0$, there exists some ρ_ε $(0<\rho_\varepsilon<\varepsilon)$ such that

$$\sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{x_k}{\rho_\varepsilon},z_1,\ldots,z_{n-1}\right\|_X\right)\leq 1. \text{ So, } M\left(\left\|\frac{x_k}{\rho_\varepsilon},z_1,\ldots,z_{n-1}\right\|_X\right)\leq 1,$$

for every $k\geq 1$ and $z_1,\ldots,z_{n-1}\in X$. Hence

$$M\left(\left\|\frac{x_k}{\varepsilon},z_1,\ldots,z_{n-1}\right\|_X\right)\leq M\left(\left\|\frac{x_k}{\rho_\varepsilon},z_1,\ldots,z_{n-1}\right\|_X\right)\leq 1$$

for every $k\geq 1$ and $z_1,\ldots,z_{n-1}\in X$. Suppose $x_{n_i}\neq 0$, for some i. Let $\varepsilon\longrightarrow 0$ then $\left\|\frac{x_{n_i}}{\varepsilon},z_1,\ldots,z_{n-1}\right\|_X\longrightarrow\infty$. It follows that $M\left(\left\|\frac{x_{n_i}}{\varepsilon},z_1,\ldots,z_{n-1}\right\|_X\right)\longrightarrow\infty$ as $\varepsilon\longrightarrow 0$ for some $n_i\in N$. This is a contradiction. Therefore $x_k=0$ for all $k\geq 1$. Thus $x=\theta$.

Let $x=(x_k)$ and $y=(y_k)$ be any two elements. Then there exist $\rho_1,\rho_2>0$ such that

$$\sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{x_k}{\rho_1},z_1,\ldots,z_{n-1}\right\|_X\right)\leq 1$$
$$\text{and}\quad \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{y_k}{\rho_2},z_1,\ldots,z_{n-1}\right\|_X\right)\leq 1$$

Let $\rho=\rho_1+\rho_2$. Then by the convexity of M, we have

$$\begin{aligned}
\sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} & M\left(\left\|\frac{x_k+y_k}{\rho}, z_1,\ldots,z_{n-1}\right\|_X\right) \\
&\leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right) \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{x_k}{\rho_1}, z_1,\ldots,z_{n-1}\right\|_X\right) \\
&+ \left(\frac{\rho_2}{\rho_1+\rho_2}\right) \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{y_k}{\rho_2}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1.
\end{aligned}$$

$$\begin{aligned}
\text{Now } \|x+y\|_0 &= \inf\left\{\rho : \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{(x_k+y_k)}{\rho}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1\right\} \\
&\leq \inf\left\{\rho_1 : \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{x_k}{\rho_1}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1\right\} \\
&+ \inf\left\{\rho_2 : \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{y_k}{\rho_2}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1\right\}.
\end{aligned}$$

Thus $\|x+y\|_0 \leq \|x\|_0 + \|y\|_0$. Finally let α be any scalar. Then

$$\begin{aligned}
\|\alpha x\|_0 &= \inf\left\{\rho : \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{\alpha x_k}{\rho}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1\right\} \\
&= \inf\left\{|\alpha|\lambda > 0 : \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{x_k}{\lambda}, z_1,\ldots,z_{n-1}\right\|_X\right) \leq 1\right\}, \\
&\qquad \text{where } \lambda = \frac{\rho}{|\alpha|} \\
&= |\alpha|\ \|x\|_0
\end{aligned}$$

Remark 2.3. *Let $\{a_1, a_2, \ldots, a_n\}$ be a linearly independent set in X. Then by equation (1.1), $\|x, z_1, z_2, \ldots, z_{n-r-1}\|_\infty = \max\{\|x, z_1, z_2, \ldots, z_{n-r-1}, a_{i_1}, \ldots, a_{i_r}\|_X\}$, $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, n\}$ is an derived $(n-r)$-norm on X, for each $r = 1, 2, \ldots, n-1$. Hence we have the following theorem.*

Theorem 2.4. *Let $\{a_1, a_2, \ldots, a_n\}$ be a linearly independent set in X. Then $c_0(X, M)$, $c(X, M)$ and $\ell_\infty(X, M)$ are normed linear spaces, normed by $\|\bullet\|_r$, defined by*

$$\|x\|_r = \inf\left\{\rho > 0 : \sup_{k\geq 1,\ z_1,\ldots,z_{n-r-1}\in X} M\left(\left\|\frac{x_k}{\rho}, z_1,\ldots,z_{n-r-1}\right\|_\infty\right) \leq 1\right\}, \qquad (2.2)$$

for each $i = 1, 2, \ldots, n-1$. We call these norms as derived norms.

Proof. Proof is same with Theorem 2.2.

Theorem 2.5. *Let X be an n-Banach space. Then $c_0(X, M)$, $c(X, M)$ and $\ell_\infty(X, M)$ are Banach spaces under the norm defined in equation* (2.1)

Proof. Let Y be any one of the spaces $c_0(X, M)$, $c(X, M)$ and $\ell_\infty(X, M)$. Let (x^i) be any Cauchy sequence in Y. Let $x_0 > 0$ be fixed and $t > 0$ be such that for $0 < \varepsilon < 1$, $\frac{\varepsilon}{x_0 t} \geq 1$. Then there exists a positive integer n_0 such that $\|x^i - x^j\|_0 < \frac{\varepsilon}{x_0 t}$, for all $i, j \geq n_0$. Using equation (2.1), we get

$$\inf\left\{\rho > 0 : \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{x_k^i - x_k^j}{\rho}, z_1, \ldots, z_{n-1}\right\|_X\right) \leq 1\right\} < \frac{\varepsilon}{x_0 t}\ ,$$

for all $i, j \geq n_0$. Hence we have,

$$\sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{x_k^i - x_k^j}{\|x^i - x^j\|_0}, z_1, \ldots, z_{n-1}\right\|_X\right) \leq 1, \quad \text{for all } i, j \geq n_0$$

It follows that for all $i, j \geq n_0$,

$$M\left(\left\|\frac{x_k^i - x_k^j}{\|x^i - x^j\|_0}, z_1, \ldots, z_{n-1}\right\|_X\right) \leq 1, \text{ for each } k \geq 1 \text{ and } z_1, \ldots, z_{n-1} \in X.$$

For $t > 0$ with $M(\frac{tx_0}{2}) \geq 1$, we have

$$M\left(\left\|\frac{x_k^i - x_k^j}{\|x^i - x^j\|_0}, z_1, \ldots, z_{n-1}\right\|_X\right) \leq M\left(\frac{tx_0}{2}\right)$$

This implies that $\|x_k^i - x_k^j, z_1, \ldots, z_{n-1}\|_X \leq \frac{tx_0}{2}.\frac{\varepsilon}{tx_0} = \frac{\varepsilon}{2}$, for each $k \geq 1$ and $z_1, \ldots, z_{n-1} \in X$. Hence (x_k^i) is a Cauchy sequence in X for all $k \in N$. Since X is an n-Banach space, (x_k^i) is convergent in X for all $k \in N$. For simplicity, let $\lim\limits_{i\longrightarrow\infty} x_k^i = x_k$, for each $k \in N$. Again we can find that

$$\inf\left\{\rho : \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{x_k^i - \lim\limits_{j\to\infty} x_k^j}{\rho}, z_1, \ldots, z_{n-1}\right\|_X\right) \leq 1\right\} < \varepsilon,$$

for all $i \geq n_0$. Thus,

$$\inf\left\{\rho : \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{x_k^i - x_k}{\rho}, z_1, \ldots, z_{n-1}\right\|_X\right) \leq 1\right\} < \varepsilon, \text{ for all } i \geq n_0$$

It follows that $(x^i - x) \in Y$. Since $(x^i) \in Y$ and Y is a linear space, so we have $x = x^i - (x^i - x) \in Y$. This completes the proof of the theorem.

The following Corollary is due to Lemma 1.2.

Corollary 2.6. *If X is a Banach space under the standard n-norm then the spaces $c_0(X, M)$, $c(X, M)$ and $\ell_\infty(X, M)$ are Banach spaces under the norm defined by equation* (2.1)

Theorem 2.7. *Let Y be any one of the spaces $c_0(X, M)$, $c(X, M)$ and $\ell_\infty(X, M)$. If (x^i) converges to x in Y in the the norm $\|\bullet\|_0$ defined by equation (2.1), then (x^i) also converges to x in the derived norm $\|\bullet\|_r$ defined by equation (2.2), for $r = 1$.*

Proof. Let (x^i) converges to x in Y in the norm $\|\bullet\|_0$. Then $\|x^i - x\|_0 \longrightarrow 0$, as $i \longrightarrow \infty$. Using definition of norm equation (2.1), we get

$$\inf\left\{\rho > 0 : \sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{x_k^i - x_k}{\rho}, z_1, \ldots, z_{n-1}\right\|_X\right) \leq 1\right\} \longrightarrow 0, \text{ as } i \longrightarrow \infty$$

Let $\{a_1, a_2, \ldots, a_n\}$ be a linearly independent set in X. Then

$$\inf\left\{\rho > 0 : \sup_{k\geq 1,\ z_1,\ldots,z_{n-2}\in X} M\left(\left\|\frac{x_k^i - x_k}{\rho}, z_1, \ldots, z_{n-2}, a_j\right\|_X\right) \leq 1\right\} \longrightarrow 0,$$

as $i \longrightarrow \infty$ and for each $j = 1, 2, \ldots, n$. Hence

$$\inf\left\{\rho > 0 : \sup_{k\geq 1,\ z_1,\ldots,z_{n-2}\in X} M\left(\left\|\frac{x_k^i - x_k}{\rho}, z_1, \ldots, z_{n-2}\right\|_X\right) \leq 1\right\} \longrightarrow 0,$$

as $i \longrightarrow \infty$, using Remark 2.3. Thus $\|x^i - x\|_1 \longrightarrow 0$, as $i \longrightarrow \infty$. Hence (x^i) converges to x in the norm $\|\bullet\|_1$.

Theorem 2.8. *Let X be a standard n-normed space and the derived $(n-1)$-norm on X is with respect to an orthonormal set. Let Y be any one of the spaces $c_0(X, M)$, $c(X, M)$ and $\ell_\infty(X, M)$. Then (x^i) is convergent in Y in the the norm $\| \bullet \|_0$ defined by equation (2.1) if and only if (x^i) is convergent in Y in the derived norm $\| \bullet \|_r$ defined by equation (2.2), for $r = 1$.*

Proof. In view of the above Theorem 2.7, it is enough to prove that (x^i) convergent in the norm $\| \bullet \|_1$ implies (x^i) convergent in the norm $\| \bullet \|_0$. Let (x^i) converges to x in Y in the norm $\| \bullet \|_1$. Then $\|x^i - x\|_1 \longrightarrow 0$, as $i \longrightarrow \infty$. Using norm equation (2.2) for $r = 1$, we get

$$\inf \left\{ \rho > 0 : \sup_{k \geq 1,\ z_1, \ldots, z_{n-1} \in X} M \left(\left\| \frac{x_k^i - x_k}{\rho}, z_1, \ldots, z_{n-2} \right\|_\infty \right) \leq 1 \right\} \longrightarrow 0,$$

as $i \longrightarrow \infty$. Now one can observe that

$$\|x_k^i - x_k, z_1, \ldots, z_{n-1}\|_S \leq \|x_k^i - x_k, z_1, \ldots, z_{n-2}\|_S \ \|z_{n-1}\|_S,$$

where $\|\bullet, \bullet, \ldots, \bullet\|_S$ and $\| \bullet \|_S$ on the right hand side denote the standard (n−1)-norm and the usual norm on X respectively (see for instance [7]). Since derived (n−1)-norm on X is with respect to an orthonormal set, using Lemma 1.3, we have

$$\|x_k^i - x_k, z_1, \ldots, z_{n-1}\|_S \leq \sqrt{n} \|x_k^i - x_k, z_1, \ldots, z_{n-2}\|_\infty \ \|z_{n-1}\|_S$$

and in this case $\|\bullet, \bullet, \ldots, \bullet\|_\infty$ on the right hand side is the derived (n−1)-norm which we used to define the norm $\| \bullet \|_1$. Therefore

$$\inf \left\{ \rho > 0 : \sup_{k \geq 1,\ z_1, \ldots, z_{n-1} \in X} M \left(\left\| \frac{x_k^i - x_k}{\rho}, z_1, \ldots, z_{n-1} \right\|_S \right) \leq 1 \right\} \leq$$

$$\inf \left\{ \rho > 0 : \sup_{k \geq 1,\ z_1, \ldots, z_{n-1} \in X} M \left(\sqrt{n} \left\| \frac{x_k^i - x_k}{\rho}, z_1, \ldots, z_{n-2} \right\|_\infty \|z_{n-1}\|_S \right) \leq 1 \right\}$$

Since, n is arbitrarily fixed, let $\lambda = \sqrt{n} \sup_{z_{n-1} \in X} \|z_{n-1}\|_S > 0$ be fixed, then right hand side of above inequality can be written as

$$\lambda \inf \left\{ t > 0 : \sup_{k \geq 1,\ z_1, \ldots, z_{n-2} \in X} M \left(\left\| \frac{x_k^i - x_k}{t}, z_1, \ldots, z_{n-2} \right\|_\infty \right) \leq 1 \right\}, \text{ where } t = \frac{\rho}{\lambda}.$$

Thus $\inf\left\{\rho>0:\sup_{k\geq 1,\ z_1,\ldots,z_{n-1}\in X} M\left(\left\|\frac{x_k^i-x_k}{\rho},z_1,\ldots,z_{n-1}\right\|_S\right)\leq 1\right\}\longrightarrow 0,$

as $i\longrightarrow\infty$. Hence $\|x^i-x\|_0\longrightarrow 0$ as $i\longrightarrow\infty$. Therefore (x^i) is converges to x in Y in the norm $\|\bullet\|_0$.

Using Lemma 1.3, we get the following Corollary.

Corollary 2.9. *Let X be a standard n-normed space and the derived $(n-r)$-norm on X are with respect to an orthonormal set. Let Y be any one of the spaces $c_0(X,M)$, $c(X,M)$ and $\ell_\infty(X,M)$. Then a sequence in Y is convergent in the the norm $\|\bullet\|_0$ defined by equation (2.1) if and only if it is convergent in the derived norm $\|\bullet\|_1$ and by induction, in the derived norm $\|\bullet\|_r$ defined by equation (2.2), for all $r=1,2,\ldots,n-1$. In particular, a sequence in Y is convergent in the norm $\|\bullet\|_0$ if and only if it is convergent in the derived norm $\|\bullet\|_{n-1}$, defined by*

$$\|x\|_{n-1}=\inf\left\{\rho>0:\sup_k M\left(\left\|\frac{x_k}{\rho}\right\|_\infty\right)\leq 1\right\} \tag{2.3}$$

Theorem 2.10. *Let X be a standard n-normed space and derived $(n-r)$-norm on X for all $r=1,2,\ldots,n-1$ are with respect to an orthonormal set. Let Y be any one of the spaces $c_0(X,M)$, $c(X,M)$ and $\ell_\infty(X,M)$. Then Y is complete with respect to the norm $\|\bullet\|_0$ defined by equation (2.1) if and only if it is complete with respect to the derived norm $\|\bullet\|_1$ defined by equation (2.2). By induction, Y is complete with respect to the norm $\|\bullet\|_0$ if and only if it is complete with respect to the derived norm $\|\bullet\|_{n-1}$, defined by equation (2.3).*

Proof. By replacing the phrase ‘ (x^i) converges to x ’ with ‘ (x^i) is Cauchy sequence ’ and ‘ x^i-x ’ with ‘ x^i-x^j ’, we see that the analogues of Theorem 2.7, Theorem 2.8 and Corollary 2.9 hold for Cauchy sequences. This completes the proof.

Remark 2.11. *Associated to the derived norm $\|\bullet\|_{n-1}$, we can defined open balls $S(x,\varepsilon)$ centered at x and radius ε as $S(x,\varepsilon)=\{y:\|x-y\|<\varepsilon\}$.*

Using these balls, Corollary 2.9, becomes:

Lemma 2.12. *Let Y be any one of the spaces $c_0(X,M)$, $c(X,M)$ and $\ell_\infty(X,M)$. A sequence (x_k) is convergent to x in Y if and only if for every $\varepsilon>0$, there exists $n_0\in N$ such that $x_k\in S(x,\varepsilon)$ for all $k\geq n_0$.*

Hence we have the following result.

Theorem 2.13. *The spaces* $c_0(X, M)$*,* $c(X, M)$ *and* $\ell_\infty(X, M)$ *are normed spaces and their topology agrees with that generated by the derived norm* $\|\bullet\|_{n-1}$ *defined by equation (2.3).*

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References

[1] S. Gähler, 2-metrische Räume und ihre topologische Struktur, *Math. Nachr.*, **28** (1963), 115-148.

[2] S. Gähler, Lineare 2-normierte Räume, *Math. Nachr.*, **28** (1964), 1-43.

[3] S. Gähler, Uber der Uniformisierbarkeit 2-metrische Räume, *Math. Nachr.*, **28** (1965), 235-244.

[4] H. Gunawan, On n-inner product, n-norms, and The Cauchy-Schwarz Inequality, *Scientiae Mathematicae Japonicae Online*, **5** (2001), 47-54.

[5] H. Gunawan, The space of p-summable sequences and its natural n-norm, *Bull. Aust. Math. Soc.*, **64(1)** (2001), 137-147.

[6] H. Gunawan and Mashadi, On finite dimensional 2-normed spaces, *Soochow J. Math.*, **27(3)** (2001), 631-639.

[7] H. Gunawan and Mashadi, On n-normed spaces, *Int. J. Math. and Math. Sci.*, **27(10)** (2001), 631-639.

[8] P. K. Kamthan and M. Gupta, Sequence Spaces and Series, *Marcel Dekker Inc, New York* (1981).

[9] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379-390.

[10] A. Misiak, n-inner product spaces, *Math. Nachr.*, **140** (1989), 299-319.

[11] S. D. Parasar and B. Choudhary, Sequence spaces defined by Orlicz functions, *Indian J. Pure and Appl.* **25(4)** (1994), 419-428.