Tamsui Oxford Journal of Information and Mathematical Sciences **28(2)** (2012) 217-226
Aletheia University

A Sharp General Ostrowski Type Inequality for Double Integrals *

Zheng Liu†

Institute of Applied Mathematics, School of Science
University of Science and Technology Liaoning
Anshan 114051, Liaoning, China

Received January 17, 2011, Accepted December 2, 2011.

Abstract

A new sharp general Ostrowski type inequality for double integrals is established. Some special cases are discussed.

Keywords and Phrases: *Ostrowski type inequality, Quasi-midpoint inequality, Quasi-trapezoid inequality, Sharp inequality, Double integrals*

1. Introduction

In [1], Barnett and Dragomir proved the following inequality of Ostrowski type for double integrals:

Theorem 1. *Let $f : [a, b]\times[c, d] \to \mathbf{R}$ be a continuous function and $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y}$ exists on $(a, b) \times (c, d)$ and is bounded, that is,*

$$\|f''_{x,y}\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)} |\frac{\partial^2 f}{\partial x \partial y}| < \infty,$$

*2010 *Mathematics Subject Classification.* Primary 26D15.

†E-mail: lewzheng@163.net

then for any $x \in [a,b]$ *and* $y \in [c,d]$

$$
\begin{aligned}
&|\int_a^b \int_c^d f(s,t)\,ds\,dt - (b-a)\int_c^d f(x,t)\,dt - (d-c)\int_a^b f(s,y)\,ds \\
&\qquad +(b-a)(d-c)f(x,y)| \qquad (1)\\
&\le [\tfrac{1}{4}(b-a)^2 + (x-\tfrac{a+b}{2})^2][\tfrac{1}{4}(d-c)^2 + (y-\tfrac{c+d}{2})^2]\|f''_{x,y}\|_\infty .
\end{aligned}
$$

In particular is the quasi-midpoint inequality

$$
\begin{aligned}
&|\int_a^b \int_c^d f(s,t)\,ds\,dt - (b-a)\int_c^d f(\tfrac{a+b}{2},t)\,dt - (d-c)\int_a^b f(s,\tfrac{c+d}{2})\,ds \\
&\qquad +(b-a)(d-c)f(\tfrac{a+b}{2},\tfrac{c+d}{2})| \qquad (2)\\
&\le \tfrac{1}{16}(b-a)^2(d-c)^2\|f''_{x,y}\|_\infty .
\end{aligned}
$$

In [2], Barnett, Dragomir and Pearce derived a quasi-trapezoid inequality for double integrals as follows:

Theorem 2. *Under the assumptions of Theorem 1, then for any* $x \in [a,b]$ *and* $y \in [c,d]$

$$
\begin{aligned}
&|\int_a^b \int_c^d f(s,t)\,ds\,dt + \tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}(b-a)(d-c) \\
&\qquad -(d-c)\int_a^b \tfrac{f(s,c)+f(s,d)}{2}\,ds - (b-a)\int_c^d \tfrac{f(a,t)+f(b,t)}{2}\,dt| \qquad (3)\\
&\le \tfrac{1}{16}(b-a)^2(d-c)^2\|f''_{x,y}\|_\infty .
\end{aligned}
$$

In this work, we will derive a new sharp inequality for double integrals with a parameter for absolutely continuous functions whose partial derivative of order 2 is $f''_{x,y} \in L_\infty((a,b)\times(c,d))$, which will not only provides a sharp generalization of inequalities (1), (2) and (3), but also gives some other interesting sharp inequalities as special cases.

2. The Results

Theorem 3. *Let* $f : [a,b] \times [c,d] \to \mathbf{R}$ *be an absolutely continuous function and* $f''_{x,y} = \frac{\partial^2 f}{\partial x \partial y} \in L_\infty((a,b)\times(c,d))$. *Then for any* $\theta \in [0,1]$ *with*

$$a + \theta\frac{b-a}{2} \le x \le b - \theta\frac{b-a}{2}$$

and

$$c + \theta\frac{d-c}{2} \le y \le d - \theta\frac{d-c}{2}$$

we have

$$\begin{aligned}
&|\textstyle\int_a^b\int_c^d f(s,t)\,ds\,dt + (b-a)(d-c)\{(1-\theta)^2 f(x,y)\\
&+\tfrac{\theta(1-\theta)}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)]\\
&+\tfrac{\theta^2}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\}\\
&-\tfrac{d-c}{2}\textstyle\int_a^b[\theta f(s,c)+2(1-\theta)f(s,y)+\theta f(s,d)]\,ds \qquad (4)\\
&-\tfrac{b-a}{2}\textstyle\int_c^d[\theta f(a,t)+2(1-\theta)f(x,t)+\theta f(b,t)]\,dt|\\
&\le \{(x-\tfrac{a+b}{2})^2(y-\tfrac{c+d}{2})^2 + \tfrac{1-2\theta+2\theta^2}{4}[(d-c)^2(x-\tfrac{a+b}{2})^2\\
&+(b-a)^2(y-\tfrac{c+d}{2})^2] + \tfrac{(1-2\theta+2\theta^2)^2}{16}(b-a)^2(d-c)^2\}\|f''_{x,y}\|_\infty.
\end{aligned}$$

The inequality (4) is sharp in the sense that the constant $\frac{1}{16}$ of the right-hand side cannot be replaced by a smaller one.

Proof. For any fixed $\theta \in [0,1]$ with $a + \theta\frac{b-a}{2} \le x \le b - \theta\frac{b-a}{2}$ and $c + \theta\frac{d-c}{2} \le y \le d - \theta\frac{d-c}{2}$, we put

$$K(s,t) := \begin{cases} [s-(a+\theta\frac{b-a}{2})][t-(c+\theta\frac{d-c}{2})], & (s,t)\in[a,x]\times[c,y],\\ [s-(a+\theta\frac{b-a}{2})][t-(d-\theta\frac{d-c}{2})], & (s,t)\in[a,x]\times(y,d],\\ [s-(b-\theta\frac{b-a}{2})][t-(c+\theta\frac{d-c}{2})], & (s,t)\in(x,b]\times[c,y],\\ [s-(b-\theta\frac{b-a}{2})][t-(d-\theta\frac{d-c}{2})], & (s,t)\in(x,b]\times(y,d]. \end{cases} \qquad (5)$$

By (5), we have

$$\begin{aligned}\int_a^b\int_c^d K(s,t)f''(s,t)\,ds\,dt &= \int_a^x\int_c^y[s-(a+\theta\tfrac{b-a}{2})][t-(c+\theta\tfrac{d-c}{2})]f''(s,t)\,ds\,dt\\ &+ \int_a^x\int_y^d[s-(a+\theta\tfrac{b-a}{2})][t-(d-\theta\tfrac{d-c}{2})]f''(s,t)\,ds\,dt\\ &+ \int_x^b\int_c^y[s-(b-\theta\tfrac{b-a}{2})][t-(c+\theta\tfrac{d-c}{2})]f''(s,t)\,ds\,dt\\ &+ \int_x^b\int_y^d[s-(b-\theta\tfrac{b-a}{2})][t-(d-\theta\tfrac{d-c}{2})]f''(s,t)\,ds\,dt.\end{aligned}$$

Integrating by parts, we obtain

$$\begin{gathered}\int_a^x\int_c^y[s-(a+\theta\tfrac{b-a}{2})][t-(c+\theta\tfrac{d-c}{2})]f''(s,t)\,ds\,dt\\ =[y-(c+\theta\tfrac{d-c}{2})][x-(a+\theta\tfrac{b-a}{2})]f(x,y)+\theta\tfrac{b-a}{2}[y-(c+\theta\tfrac{d-c}{2})]f(a,y)\\ +\theta\tfrac{d-c}{2}[x-(a+\theta\tfrac{b-a}{2})]f(x,c)+\tfrac{\theta^2}{4}(b-a)(d-c)f(a,c)\\ -[y-(c+\theta\tfrac{d-c}{2})]\int_a^x f(s,y)\,ds-[x-(a+\theta\tfrac{b-a}{2})]\int_c^y f(x,t)\,dt\\ -\theta\tfrac{d-c}{2}\int_a^x f(s,c)\,ds-\theta\tfrac{b-a}{2}\int_c^y f(a,t)\,dt+\int_a^x\int_c^y f(s,t)\,ds\,dt,\end{gathered}$$

$$\begin{gathered}\int_a^x\int_y^d[s-(a+\theta\tfrac{b-a}{2})][t-(d-\theta\tfrac{d-c}{2})]f''(s,t)\,ds\,dt\\ =-[y-(d-\theta\tfrac{d-c}{2})][x-(a+\theta\tfrac{b-a}{2})]f(x,y)-\theta\tfrac{b-a}{2}[y-(d-\theta\tfrac{d-c}{2})]f(a,y)\\ +\theta\tfrac{d-c}{2}[x-(a+\theta\tfrac{b-a}{2})]f(x,d)+\tfrac{\theta^2}{4}(b-a)(d-c)f(a,d)\\ +[y-(d-\theta\tfrac{d-c}{2})]\int_a^x f(s,y)\,ds-[x-(a+\theta\tfrac{b-a}{2})]\int_y^d f(x,t)\,dt\\ -\theta\tfrac{d-c}{2}\int_a^x f(s,d)\,ds-\theta\tfrac{b-a}{2}\int_y^d f(a,t)\,dt+\int_a^x\int_y^d f(s,t)\,ds\,dt,\end{gathered}$$

$$\int_x^b \int_c^y [s-(b-\theta\tfrac{b-a}{2})][t-(c+\theta\tfrac{d-c}{2})]f''(s,t)\,ds\,dt$$
$$= -[y-(c+\theta\tfrac{d-c}{2})][x-(b-\theta\tfrac{b-a}{2})]f(x,y)+\theta\tfrac{b-a}{2}[y-(c+\theta\tfrac{d-c}{2})]f(b,y)$$
$$-\theta\tfrac{d-c}{2}[x-(b-\theta\tfrac{b-a}{2})]f(x,c)+\tfrac{\theta^2}{4}(b-a)(d-c)f(b,c)$$
$$-[y-(c+\theta\tfrac{d-c}{2})]\int_x^b f(s,y)\,ds+[x-(b-\theta\tfrac{b-a}{2})]\int_c^y f(x,t)\,dt$$
$$-\theta\tfrac{d-c}{2}\int_x^b f(s,c)\,ds-\theta\tfrac{b-a}{2}\int_c^y f(b,t)\,dt+\int_x^b\int_c^y f(s,t)\,ds\,dt,$$

and

$$\int_x^b \int_y^d [s-(b-\theta\tfrac{b-a}{2})][t-(d-\theta\tfrac{d-c}{2})]f''(s,t)\,ds\,dt$$
$$= [y-(d-\theta\tfrac{d-c}{2})][x-(b-\theta\tfrac{b-a}{2})]f(x,y)-\theta\tfrac{b-a}{2}[y-(d-\theta\tfrac{d-c}{2})]f(b,y)$$
$$-\theta\tfrac{d-c}{2}[x-(b-\theta\tfrac{b-a}{2})]f(x,d)+\tfrac{\theta^2}{4}(b-a)(d-c)f(b,d)$$
$$+[y-(d-\theta\tfrac{d-c}{2})]\int_x^b f(s,y)\,ds+[x-(b-\theta\tfrac{b-a}{2})]\int_y^d f(x,t)\,dt$$
$$-\theta\tfrac{d-c}{2}\int_x^b f(s,d)\,ds-\theta\tfrac{b-a}{2}\int_y^d f(b,t)\,dt+\int_x^b\int_y^d f(s,t)\,ds\,dt.$$

Thus

$$\int_a^b\int_c^d K(s,t)f''(s,t)\,ds\,dt=\int_a^b\int_c^d f(s,t)\,ds\,dt$$
$$+(b-a)(d-c)\{(1-\theta)^2 f(x,y)+\tfrac{\theta(1-\theta)}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)]$$
$$+\tfrac{\theta^2}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\}$$
$$-\tfrac{d-c}{2}\int_a^b[\theta f(s,c)+2(1-\theta)f(s,y)+\theta f(s,d)]\,ds$$
$$-\tfrac{b-a}{2}\int_c^d[\theta f(a,t)+2(1-\theta)f(x,t)+\theta f(b,t)]\,dt. \tag{6}$$

Observe that

$$\int_p^r |u-q|du = \int_p^q (q-u)\,du + \int_q^r (u-q)\,du$$

$$= \tfrac{1}{2}[(q-p)^2 + (r-q)^2] = \tfrac{1}{4}(r-p)^2 + (q-\tfrac{p+r}{2})^2$$

for all p, q, r such that $p \le q \le r$, it is easy to find that

$$\int_a^x |s-(a+\theta\frac{b-a}{2})|\,ds = \frac{1}{4}(x-a)^2 + [(a+\theta\frac{b-a}{2}) - \frac{a+x}{2}]^2,$$

$$\int_x^b |s-(b-\theta\frac{b-a}{2})|\,ds = \frac{1}{4}(b-x)^2 + [(b-\theta\frac{b-a}{2}) - \frac{x+b}{2}]^2,$$

$$\int_c^y |t-(c+\theta\frac{d-c}{2})|\,dt = \frac{1}{4}(y-c)^2 + [(c+\theta\frac{d-c}{2}) - \frac{c+y}{2}]^2,$$

and

$$\int_y^d |t-(d-\theta\frac{d-c}{2})|\,dt = \frac{1}{4}(d-y)^2 + [(d-\theta\frac{d-c}{2}) - \frac{y+d}{2}]^2.$$

Hence,

$$\int_a^b \int_c^d |K(s,t)|\,ds\,dt$$

$$= (x-\tfrac{a+b}{2})^2(y-\tfrac{c+d}{2})^2 + \tfrac{1-2\theta+2\theta^2}{4}[(d-c)^2(x-\tfrac{a+b}{2})^2 \tag{7}$$

$$+(b-a)^2(y-\tfrac{c+d}{2})^2] + \tfrac{(1-2\theta+2\theta^2)^2}{16}(b-a)^2(d-c)^2$$

From (6) and (7), we can easily get (4).

Now we prove that the inequality (4) is sharp in the sense that the constant $\frac{1}{16}$ of the right-hand side cannot be replaced by a smaller one. In fact, we may suppose that (4) holds with a constant $C > 0$ as

$$\begin{aligned}
&|\int_a^b \int_c^d f(s,t)\,ds\,dt + (b-a)(d-c)\{(1-\theta)^2 f(x,y) + \\
&\tfrac{\theta(1-\theta)}{2}[f(a,y)+f(b,y)+f(x,c)+f(x,d)] \\
&+\tfrac{\theta^2}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\} \\
&-\tfrac{d-c}{2}\int_a^b [\theta f(s,c)+2(1-\theta)f(s,y)+\theta f(s,d)]\,ds \qquad (8) \\
&-\tfrac{b-a}{2}\int_c^d [\theta f(a,t)+2(1-\theta)f(x,t)+\theta f(b,t)]\,dt| \\
&\le \{(x-\tfrac{a+b}{2})^2(y-\tfrac{c+d}{2})^2 + \tfrac{1-2\theta+2\theta^2}{4}[(d-c)^2(x-\tfrac{a+b}{2})^2 \\
&+(b-a)^2(y-\tfrac{c+d}{2})^2] + C(1-2\theta+2\theta^2)^2(b-a)^2(d-c)^2\}\|f''_{x,y}\|_\infty .
\end{aligned}$$

Take $x=\frac{a+b}{2}$, $y=\frac{c+d}{2}$ and an absolutely continuous function

$$f:[a,b]\times[c,d]\to \mathbf{R}$$

as

$$f(s,t) := \begin{cases} |s-(a+\theta\frac{b-a}{2})||t-(c+\theta\frac{d-c}{2})|, & (s,t)\in[a,\frac{a+b}{2}]\times[c,\frac{c+d}{2}], \\ |s-(a+\theta\frac{b-a}{2})||t-(d-\theta\frac{d-c}{2})|, & (s,t)\in[a,\frac{a+b}{2}]\times(\frac{c+d}{2},d], \\ |s-(b-\theta\frac{b-a}{2})||t-(c+\theta\frac{d-c}{2})|, & (s,t)\in(\frac{a+b}{2},b]\times[c,\frac{c+d}{2}], \\ |s-(b-\theta\frac{b-a}{2})||t-(d-\theta\frac{d-c}{2})|, & (s,t)\in(\frac{a+b}{2},b]\times(\frac{c+d}{2},d]. \end{cases}$$

It follows that

$$f''_{s,t}(s,t) = \begin{cases} 1, (s,t) \\ \in (a, a+\theta\frac{b-a}{2}) \times (c, c+\theta\frac{d-c}{2}) \cup (a, a+\theta\frac{b-a}{2}) \times (\frac{c+d}{2}, d-\theta\frac{d-c}{2}), \\ -1, (s,t) \\ \in (a, a+\theta\frac{b-a}{2}) \times (c+\theta\frac{d-c}{2}, \frac{c+d}{2}) \cup (a, a+\theta\frac{b-a}{2}) \times (d-\theta\frac{d-c}{2}, d), \\ -1, (s,t) \\ \in (a+\theta\frac{b-a}{2}, \frac{a+b}{2}) \times (c, c+\theta\frac{d-c}{2}) \cup (a+\theta\frac{b-a}{2}, \frac{a+b}{2}) \times (\frac{c+d}{2}, d-\theta\frac{d-c}{2}), \\ 1, (s,t) \\ \in (a+\theta\frac{b-a}{2}, \frac{a+b}{2}) \times (c+\theta\frac{d-c}{2}, \frac{c+d}{2}) \cup (a+\theta\frac{b-a}{2}, \frac{a+b}{2}) \times (d-\theta\frac{d-c}{2}, d), \\ 1, (s,t) \\ \in (\frac{a+b}{2}, b-\theta\frac{b-a}{2}) \times (c, c+\theta\frac{d-c}{2}) \cup (\frac{a+b}{2}, b-\theta\frac{b-a}{2}) \times (\frac{c+d}{2}, d-\theta\frac{d-c}{2}), \\ -1, (s,t) \\ \in (\frac{a+b}{2}, b-\theta\frac{b-a}{2}) \times (c+\theta\frac{d-c}{2}, \frac{c+d}{2}) \cup (\frac{a+b}{2}, b-\theta\frac{b-a}{2}) \times (d-\theta\frac{d-c}{2}, d), \\ -1, (s,t) \\ \in (b-\theta\frac{b-a}{2}, b) \times (c, c+\theta\frac{d-c}{2}) \cup (b-\theta\frac{b-a}{2}, b) \times (\frac{c+d}{2}, d-\theta\frac{d-c}{2}), \\ 1, (s,t) \\ \in (b-\theta\frac{b-a}{2}, b) \times (c+\theta\frac{d-c}{2}, \frac{c+d}{2}) \cup (b-\theta\frac{b-a}{2}, b) \times (d-\theta\frac{d-c}{2}, d). \end{cases} \tag{9}$$

By (5)-(9), it is not difficult to find that the left-hand side of the inequality (8) becomes

$$L.H.S.(8) = \frac{1}{16}(1-2\theta+2\theta^2)^2(b-a)^2(d-c)^2 \tag{10}$$

and the right-hand side of the inequality (8) is

$$R.H.S.(8) = C(1-2\theta+2\theta^2)^2(b-a)^2(d-c)^2. \tag{11}$$

From (8), (10) and (11), we find that $C \geq \frac{1}{16}$, proving that the constant $\frac{1}{16}$ is the best possible in (4).

Corollary 1. *Let the assumptions of Theorem 3 hold. Then for any* $\theta \in [0,1]$ *we have*

$$
\begin{aligned}
&|\int_a^b\int_c^d f(s,t)\,ds\,dt+(b-a)(d-c)\{(1-\theta)^2 f(\tfrac{a+b}{2},\tfrac{c+d}{2})\\
&+\tfrac{\theta(1-\theta)}{2}[f(a,\tfrac{c+d}{2})+f(b,\tfrac{c+d}{2})+f(\tfrac{a+b}{2},c)+f(\tfrac{a+b}{2},d)]\\
&+\tfrac{\theta^2}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\}\\
&-\tfrac{d-c}{2}\int_a^b[\theta f(s,c)+2(1-\theta)f(s,\tfrac{c+d}{2})+\theta f(s,d)]\,ds\\
&-\tfrac{b-a}{2}\int_c^d[\theta f(a,t)+2(1-\theta)f(\tfrac{a+b}{2},t)+\theta f(b,t)]\,dt|\\
&\le \tfrac{(1-2\theta+2\theta^2)^2}{16}(b-a)^2(d-c)^2\|f''_{x,y}\|_\infty .
\end{aligned}
\tag{12}
$$

Proof. We set $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (4) to get (12).

Remark 1. If we take $\theta=0$ and $\theta=1$ in (12), we recapture the quasi-midpoint inequality (2) and the quasi-trapezoid inequality (3) with the sharpness which have not been proved in [1] and [2], respectively.

If we take $\theta=\frac{1}{3}$ in (12), we get a sharp quasi-Simpson type inequality as

$$
\begin{aligned}
&|\int_a^b\int_c^d f(s,t)\,ds\,dt+\tfrac{f(a,\frac{c+d}{2})+f(\frac{a+b}{2},c)+f(\frac{a+b}{2},d)+f(b,\frac{c+d}{2})+4f(\frac{a+c}{2},\frac{b+d}{2})}{9}\\
&+\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{36}-\tfrac{d-c}{6}\int_a^b[f(s,c)+4f(s,\tfrac{c+d}{2})+f(s,d)]\,ds\\
&-\tfrac{b-a}{6}\int_c^d[f(a,t)+4f(\tfrac{a+b}{2},t)+f(b,t)]\,dt|\\
&\le \tfrac{25}{1296}(b-a)^2(d-c)^2\|f''_{s,t}\|_\infty .
\end{aligned}
\tag{13}
$$

and if we take $\theta=\frac{1}{2}$ in (12), we get a sharp quasi averaged midpoint-trapezoid inequality as

$$
\begin{aligned}
&|\int_a^b\int_c^d f(s,t)\,ds\,dt + \tfrac{f(a,\frac{c+d}{2})+f(\frac{a+b}{2},c)+f(\frac{a+b}{2},d)+f(b,\frac{c+d}{2})+2f(\frac{a+b}{2},\frac{c+d}{2})}{8} \\
&+\tfrac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{16} - \tfrac{d-c}{4}\int_a^b[f(s,c)+2f(s,\tfrac{c+d}{2})+f(s,d)]\,ds \\
&-\tfrac{b-a}{4}\int_c^d[f(a,t)+2f(\tfrac{a+b}{2},t)+f(b,t)]\,dt| \\
&\leq \tfrac{1}{64}(b-a)^2(d-c)^2\|f''_{s,t}\|_\infty.
\end{aligned} \tag{14}
$$

It is interesting to notice that the smallest bound for (4) is obtained at $x = \frac{a+b}{2}, y = \frac{c+d}{2}$ and $\theta = \frac{1}{2}$. Thus the sharp quasi averaged midpoint-trapezoid inequality (14) is optimal.

Remark 2. If we take $\theta = 0$ in (4), then the Ostrowski type inequality (1) is recaptured with the sharpness which has not been proved in [1].

References

[1] N. S. Barnett and S. S. Dragomir, An Ostrowski type inequality for double integrals and applications to cubature formulae, *Soochow J. Math.*, **27(1)** (2001), 1-10.

[2] N. S. Barnett, S. S. Dragomir , and C. E. M. Pearce, A quasi-trapezoid inequality for double integrals, *ANZIAM J.*, **44** (2003), 355-364.