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Note on Super-Halley Method and its Variants[*]

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Abstract

In this paper, we propose a new cubically convergent family of super-Halley method based on power means. Some well-known methods can be regarded as particular cases of the proposed family. New classes of higher (third and fourth) order multipoint iterative methods free from second order derivative are derived by semi-discrete modifications of above-mentioned methods. It is shown that super-Halley method is the only method which produces fourth order multipoint iterative methods. Furthermore, these multipoint methods with cubic convergence have also been extended for finding the multiple zeros of non-linear functions. Numerical examples are also presented to demonstrate the performance of proposed multipoint iterative methods.

Keywords and Phrases: *Newton's method, Halley's method, Chebyshev-Halley type methods, Multiple zeros, Power means, Order of convergence*

1. Introduction

The family of Chebyshev-Halley type methods [14] is given by

$$x_{n+1} = x_n - \left[1 + \frac{1}{2}\left\{\frac{L_f(x_n)}{1-\lambda L_f(x_n)}\right\}\right]\frac{f(x_n)}{f'(x_n)}, \tag{1}$$

where $\lambda \in \Re$ and $L_f(x_n) = \dfrac{f''(x_n)\, f(x_n)}{f'^2(x_n)}$.

This family includes the classical Chebyshev's method (when $\lambda = 0$), the famous Halley's method (when $\lambda = 0.5$) and the super-Halley method (when $\lambda = 1$).

The purpose of this work is to provide some alternative derivations through power means and to revisit some well-known zero finding iterative methods. The work is organized as follows. In Section 2, definitions of various means are reviewed.

In Section 3, a family of super-Halley type methods [9, 16] based on power means has been presented which is cubically convergent. In Section 4, a generalization to a family of multipoint iterative methods and their convergence has been presented. In Section 5, extension to the cubically convergent family of iterative methods for multiple zeros of non-linear functional equations has been given. To demonstrate the performance of various proposed multipoint methods, some numerical examples and concluding remarks have been presented in Section 6.

2. Review of Definition of Various Means

For a given finite real number α, the α^{th} - power mean m_α of positive scalars a and b, is defined as (see [11])

$$m_\alpha = \left(\frac{a^\alpha + b^\alpha}{2}\right)^{\frac{1}{\alpha}}. \tag{2}$$

Note that

when $\alpha = -1$, $m_{-1} = \frac{2ab}{a+b}$, (Harmonic mean) (3)

when $\alpha = \frac{1}{2}$, $m_{\frac{1}{2}} = \left\{\frac{\sqrt{a}+\sqrt{b}}{2}\right\}^2$, (4)

when $\alpha = 1$, $m_1 = \frac{a+b}{2}$, (Arithmetic mean) (5)

when $\alpha \to 0$, $m_0 = \lim_{\alpha \to 0} m_\alpha = \sqrt{ab}$, (Geometric mean). (6)

For given positive scalars a and b, some other well-known means are defined as

Heronian mean: $N = \frac{a + \sqrt{ab} + b}{3}$, (7) Contra-harmonic mean: $C = \dfrac{a^2 + b^2}{a + b}$, (8)

Centroidal mean: $T = \dfrac{2\left(a^2 + ab + b^2\right)}{3(a + b)}$, (9)

Logarithmic mean: $L = \dfrac{a - b}{\log(a) - \log(b)}$. (10)

3. Iterative Family of Super-Halley Method Based on Power Means

The well-known Newton's formula for simple zero, and for multiple zeros [3], are given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{11}$$

$$x_{n+1} = x_n - \frac{f(x_n)\, f'(x_n)}{f'^2(x_n) - f(x_n)\, f''(x_n)}. \tag{12}$$

From (11) and (12), one can obtain

$$x_{n+1} = x_n - \frac{1}{2}\left[\left(\frac{f(x_n)}{f'(x_n)}\right) + \left(\frac{f(x_n)\, f'(x_n)}{f'^2(x_n) - f(x_n)\, f''(x_n)}\right)\right], \tag{13}$$

an alternative form of the well-known super-Halley method [9,16,17]. This can be rewritten as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)\left\{f'^2(x_n) - f(x_n)\, f''(x_n)\right\}}\left[\frac{f'^2(x_n) + \left\{f'^2(x_n) - f(x_n)\, f''(x_n)\right\}}{2}\right], \tag{14}$$

which is no different from the formula (1) when $\lambda = 1$.

Let $a = f'^2(x_n)$ and $b = \left\{f'^2(x_n) - f(x_n)\, f''(x_n)\right\}$. For the quantities a and b to be positive and different from zero, we see that

$$f(x_n)\, f''(x_n) < f'^2(x_n).$$

The quantity $a = f'^2(x_n)$ is obviously positive being the square of a nonzero real number. In the case if $x = x_n$ is a very good approximation to the root, then $f(x_n)$ will be sufficiently close to zero and consequently the quantity

$$\left|\frac{f(x_n)f''(x_n)}{f'^2(x_n)}\right|, \tag{15}$$

would be sufficiently small.

We wish to generalize the formula (14) by α^{th} - power mean. For this, we take $a = f'^2(x_n)$ and $b = \{f'^2(x_n) - f(x_n)f''(x_n)\}$. Clearly the quantity a is positive and the quantity b is positive in view of (15). Now approximating the correction factor in (14) by α^{th} - power mean as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)\{f'^2(x_n) - f(x_n)f''(x_n)\}}\left[\frac{a^\alpha + b^\alpha}{2}\right]^{\frac{1}{\alpha}}. \tag{16}$$

This is called the α^{th} - power mean iterative family of super-Halley method. The formula (16) may be considered as the unification of several existing cubically convergent methods. For different values of 'α', these well-known super-Halley formulae have been recovered in the foregoing analysis. It is clear that formula (16) requires three evaluations per iteration and has an efficiency index [2] $\sqrt[3]{3} \approx 1.442$. Therefore, the family (16) of one-point methods does not have optimal order of convergence according to Kung-Traub conjecture [6].

Special cases:

For $\alpha = 1$ (Arithmetic mean), and $\alpha = -1$ (Harmonic mean), it is easy to see that formula (16) corresponds to the well known super-Halley and Halley's methods respectively. For $\alpha \to 0$ (Geometric mean), formula (16) corresponds to the well-known Ostrowski's square-root formula [2]. While for $\alpha = \frac{1}{2}$, formula (16) reduces to

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)\{f'^2(x_n) - f(x_n)f''(x_n)\}}\left[\frac{\sqrt{f'^2(x_n)} + \sqrt{\{f'^2(x_n) - f(x_n)f''(x_n)\}}}{2}\right]^2, \quad (17)$$

which further reduces to Chebyshev's method after applying binomial theorem.

For $\alpha = 2$ (root mean square), the formula (16) reduces to

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)\{f'^2(x_n) - f(x_n)f''(x_n)\}}\sqrt{\frac{f'^4(x_n) + \{f'^2(x_n) - f(x_n)f''(x_n)\}^2}{2}}. \quad (18)$$

Some other cubically convergent iterative methods based on Heronian mean, contra-harmonic mean, centroidal mean and logarithmic mean can also be obtained from formula (14) respectively.

3.1 Convergence Analysis

Theorem 3.1. *Let r be a simple zero of a sufficiently differentiable function $f : I \subseteq \Re \to \Re$ on an open interval I. If the initial guess x_0 is sufficiently close to r, then for $\alpha \in \Re$, the methods defined by family (16) has cubic convergence with the following error equation*

$$e_{n+1} = -\left\{\left(\frac{\alpha - 1}{2}\right)c_2^{\ 2} + c_3\right\}e_n^{\ 3} + O\left(e_n^{\ 4}\right), \quad (19)$$

where $e_n = x_n - r$ and $c_k = \frac{1}{k!}\frac{f^k(r)}{f'(r)}, k = 2,3,...$

Proof: Since $f(x)$ is sufficiently differentiable function, therefore expanding $f(x_n)$, $f'(x_n)$ and $f''(x_n)$ about $x = r$ by means of Taylor's expansion, we have

$$f(x_n) = f'(r)\left[e_n + c_2 e_n^{\ 2} + c_3 e_n^{\ 3} + O\left(e_n^{\ 4}\right)\right], \quad (20)$$

$$f'(x_n) = f'(r)\left[1 + 2c_2 e_n + 3c_3 e_n^{\ 2} + O\left(e_n^{\ 3}\right)\right], \quad (21)$$

$$f''(x_n) = f'(r)\left[2c_2 + 6c_3 e_n + 12c_4 e_n^{\ 2} + O\left(e_n^{\ 3}\right)\right]. \quad (22)$$

From (20)-(22), we may obtain

$$\frac{f(x_n)}{f'(x_n)} = \left[e_n - c_2 e_n{}^2 + 2\left(c_2^2 - c_3\right)e_n{}^3 + O\left(e_n{}^4\right)\right], \tag{23}$$

and

$$\frac{f(x_n) f''(x_n)}{f'^2(x_n)} = \left[2c_2 e_n + \left\{6c_3 - 6c_2{}^2\right\}e_n{}^2 + \left(16c_2{}^3 + 12c_4 - 28c_2 c_3\right)e_n{}^3 + O\left(e_n{}^4\right)\right]. \tag{24}$$

Case (i) For $\alpha \in \Re \setminus \{0\}$, formula (16) may be written as

$$x_{n+1} = x_n - \left(\frac{f(x_n)}{f'(x_n)}\right)\left(1 - \frac{f(x_n) f''(x_n)}{f'^2(x_n)}\right)^{-1}\left[\frac{1 + \left(1 - \frac{f(x_n) f''(x_n)}{f'^2(x_n)}\right)^{\alpha}}{2}\right]^{1/\alpha}. \tag{25}$$

Using binomial theorem and the formulae (23) and (24), the formula (25) yields

$$e_{n+1} = -\left\{\left(\frac{\alpha - 1}{2}\right)c_2{}^2 + c_3\right\}e_n{}^3 + O\left(e_n{}^4\right). \tag{26}$$

This proves cubic convergence.

Case (ii) For $\alpha \to 0$, we have seen that formula (16) reduces to Ostrowski's square-root method (not quartically convergent Ostrowski's method), which converges cubically and the proof of which is given in Ref. [2].

Therefore, it can be concluded that for all $\alpha \in \Re$, the α^{th} - power mean family (16) of super-Halley method converges cubically.

4. Generalized Family of Multipoint Iterative Methods without Memory

The practical difficulty associated with the above mentioned third-order methods given by (16) may be in the evaluation of second-order derivative. Recently, some new variants of Newton's method free from second-order derivative have been developed in [1, 4, 5, 10, 13, 15] by discretization of second-order derivative or by considering different quadrature formulae for the computation of integral arising from

Newton's theorem [12]. These multipoint methods calculate new approximations to a zero of $f(x)$ by sampling $f(x)$ and possibly its derivatives for a number of values of the independent variable, at each step. Nedzhibov et al. [5] have modified Chebyshev-Halley methods to derive several third and fourth-order multipoint iterative methods free from second-order derivative. Recently, Wang and Li [7] further derived a family of new derivative-free third-order methods for solving non-linear equations numerically. Here, we also intend to develop and unify the general class of multipoint iterative methods free from second- order derivative. The main idea of the proposed generalized family lies in the discretization of second-order derivative involved in the α^{th}- power mean family of super-Halley type methods (16) (similar to Nedzhibov et al. [5]). Therefore, this work can be viewed as the generalization over Nedzhibov et al. [5], Wang and Li [7] families of multipoint iterative methods. We shall derive following three families free from second order derivative involved in the family (16).

a. *First family*

Expanding the function $f(x_n - \theta u)$, $\theta \in \Re - \{0\}$ but finite, about the point $x = x_n$ with $f(x_n) \neq 0$, we have

$$f(x_n - \theta u) = f(x_n) - \theta u\, f'(x_n) + \frac{\theta^2 u^2}{2} f''(x_n) + O(u^3). \tag{27}$$

Let us take $u = \dfrac{f(x_n)}{f'(x_n)}$, and inserting this into (27), we obtain

$$f(x_n)\, f''(x_n) \approx \frac{2 f'^2(x_n)}{\theta^2 f(x_n)} \{ f(x_n - \theta u) - (1-\theta)\, f(x_n) \}. \tag{28}$$

Using this approximate value of $f(x_n)\, f''(x_n)$ into formula (16), we have

$$x_{n+1} = x_n - \frac{\theta^2 f^2(x_n)}{f'^3(x_n)\left[(\theta^2 - 2\theta + 2) f(x_n) - 2 f(x_n - \theta u)\right]} \left[\frac{a'^{\alpha} + b'^{\alpha}}{2} \right]^{\frac{1}{\alpha}}, \tag{29}$$

where $a' = f'^2(x_n)$ and $b' = \left\{ f'^2(x_n) - \dfrac{2 f'^2(x_n)}{\theta^2 f(x_n)} \{ f(x_n - \theta u) - (1-\theta) f(x_n) \} \right\}$.

This is the modification over Nedzhibov et al. formula (2.1) in [5]. It is seen that this family depends on the real parameters α and θ.

Special cases:

For different specific values of parameters α and θ, the following various families of multipoint iterative methods can be stemmed from (29), e.g.

i. For $(\alpha,\theta)=(1,1)$, we get the formula

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\left\{\frac{f(x_n)-f(x_n-u)}{f(x_n)-2f(x_n-u)}\right\}. \tag{30}$$

This is the well-known fourth-order Traub-Ostrowski formula [5, 8, 15].

ii. For $(\alpha,\theta)=(-1,1)$, we get

$$x_{n+1}=x_n-\frac{f^2(x_n)}{f'(x_n)\{f(x_n)-f(x_n-u)\}}. \tag{31}$$

This is a cubically convergent Newton-Secant formula [5, 8, 15].

iii. For $\alpha\to 0$ and $\theta=1$, we get

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\sqrt{\frac{f(x_n)}{f(x_n)-2f(x_n-u)}}. \tag{32}$$

This formula is a new for multipoint iterative method.

Note that the family (29) can produce many more new multipoint methods by choosing different values of the parameters.

b. *Second family*

Replacing the second order derivative in (16) by the following definition, similar to Nedzhibov et al. [5] as

$$f''(x_n)\approx\frac{f'(x_n)-f'(x_n-\theta u)}{\theta u},\ \theta\in\Re-\{0\}, \tag{33}$$

we get following new generalized family as

$$x_{n+1}=x_n-\frac{\theta f(x_n)}{f'^2(x_n)[(\theta-1)f'(x_n)+f'(x_n-\theta u)]}\left[\frac{a''^{\alpha}+b''^{\alpha}}{2}\right]^{\frac{1}{\alpha}}, \tag{34}$$

where $a'' = f'^2(x_n)$ and $b'' = \left\{\dfrac{(\theta-1)f'^2(x_n) + f'(x_n)f'(x_n-\theta u)}{\theta}\right\}$ respectively.

Special cases:

(I) For $\alpha = 1$ in formula (34), we obtain the family based on arithmetic mean given by

$$x_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)}\left\{\frac{(2\theta-1)f'(x_n) + f'(x_n-\theta u)}{(\theta-1)f'(x_n) + f'(x_n-\theta u)}\right\}. \tag{35}$$

Some interesting particular cases of (35) are:

i. For $\theta = 1$ in (35), we get the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left\{\frac{1}{f'(x_n)} + \frac{1}{f'(x_n-u)}\right\}. \tag{36}$$

This formula is same as derived by Traub [8] independently.

ii. For $\theta = 2/3$, we get the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)}\left\{\frac{3f'\left(x_n - \frac{2}{3}u\right) + f'(x_n)}{3f'\left(x_n - \frac{2}{3}u\right) - f'(x_n)}\right\}. \tag{37}$$

This is a well-known quartically convergent Jarratt's multipoint iterative formula [10].

(II) For $\alpha = -1$ in formula (34), we obtain the family based on harmonic mean given by

$$x_{n+1} = x_n - \frac{2\theta f(x_n)}{(2\theta-1)f'(x_n) + f'(x_n-\theta u)}. \tag{38}$$

This is the modification over the formula (3.8) of Weerakoon and Fernando [12].

Some interesting particular cases of (38) are:

i. For $\theta = 1$ in (38), we get the formula

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n-u)}. \tag{39}$$

This formula was independently obtained by Traub [8] and Weerakoon and Fernando [12]. Some new third-order multipoint methods based on Heronian mean, contra-harmonic mean, centroidal mean etc. can also be obtained from formula (39) as follows:

Cubically convergent multipoint method based on Heronian mean is

$$x_{n+1} = x_n - \frac{3f(x_n)}{\left\{f'(x_n) + \sqrt{f'(x_n)f'(x_n-u)} + f'(x_n-u)\right\}}. \tag{40}$$

Cubically convergent multipoint method based on contra-harmonic mean is

$$x_{n+1} = x_n - \frac{f(x_n)\{f'(x_n) + f'(x_n-u)\}}{f'^2(x_n) + f'^2(x_n-u)}. \tag{41}$$

Cubically convergent multipoint method based on centroidal mean is

$$x_{n+1} = x_n - \frac{3f(x_n)\{f'(x_n) + f'(x_n-u)\}}{2\left\{f'^2(x_n) + f'^2(x_n-u) + f'(x_n)f'(x_n-u)\right\}}. \tag{42}$$

ii. For $\theta = \frac{1}{2}$ in (38), we get the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(x_n - \frac{1}{2}u\right)}. \tag{43}$$

This is a well-known cubically convergent iterative formula [8].

iii. For $\theta = 2$ in (38), we get another new formula given by

$$x_{n+1} = x_n - \frac{4f(x_n)}{3f'(x_n) + f'(x_n-2u)}. \tag{44}$$

(III) For $\alpha \to 0$ in formula (34), we obtain the family based on geometric mean as

$$x_{n+1} = x_n - sign(f'(x_0))f(x_n)\sqrt{\frac{\theta}{(\theta-1)f'^2(x_n) + f'(x_n)f'(x_n-\theta u)}}. \tag{45}$$

Some interesting particular cases of family (45) are:

i. For $\theta = 1$ and $\theta = -1$, we get the formulas

$$x_{n+1} = x_n - \frac{f(x_n)}{sign(f'(x_0))\sqrt{f'(x_n)f'(x_n-u)}}, \tag{46}$$

and

$$x_{n+1} = x_n - \frac{f(x_n)}{sign(f'(x_0))\sqrt{2f'^2(x_n) - f'(x_n)f'(x_n+u)}}, \tag{47}$$

where the positive sign is taken if $x_n < r$ and the negative sign is taken if $x_n > r$.

These are cubically convergent multipoint iterative formulae. Formula (46) is also derived by Lukić and Ralević [13] independently.

Other modifications can directly be obtained from formula (14) by replacing the second derivative in (14) by a finite difference (33). This gives

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left[\frac{1}{f'(x_n)} + \frac{\theta}{(\theta-1)f'(x_n) + f'(x_n-\theta u)}\right]. \tag{48}$$

If we rewrite the formula (48) as

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left\{\frac{1}{f'(x_n)} + \frac{\theta}{2\left[\frac{f'(x_n-\theta u) + f'(x_n)}{2}\right] + (\theta-2)f'(x_n)}\right\}, \tag{49}$$

and replace the arithmetic mean $\frac{1}{2}(f'(x_n-\theta u) + f'(x_n))$ with the midpoint value $f'\{((x_n-\theta u)+x_n)/2\}$ in (49), we obtain a new family of methods given by

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left\{\frac{1}{f'(x_n)} + \frac{\theta}{2f'(x_n-0.5\theta u) + (\theta-2)f'(x_n)}\right\}. \tag{50}$$

For $\theta = 1$, the formula (50), yields a new formula given by

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left\{\frac{1}{f'(x_n)} + \frac{1}{2f'(x_n-0.5u) - f'(x_n)}\right\}. \tag{51}$$

This formula can be obtained from family (48), if $\theta = 1/2$.

(ii) For $\theta = 1/4$ and $\theta = -1$ in formula (48), we get the formulas

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left\{\frac{1}{f'(x_n)} + \frac{1}{4f'(x_n - 0.25u) - 3f'(x_n)}\right\}, \tag{52}$$

and

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left\{\frac{1}{f'(x_n)} + \frac{1}{2f'(x_n) - f'(x_n + u)}\right\}. \tag{53}$$

These are other new cubically convergent iterative formulas.

c. *Third family*

Replacing the second derivative in (16) by a finite difference similar to Nedzhibov [5] as

$$f''(x_n) \approx \frac{5f'(x_n) - 4f'\left(x_n - \frac{\theta u}{2}\right) - f'(x_n - \theta u)}{3\theta u}, \theta \in \Re - \{0\}, \tag{54}$$

we get new generalized family as

$$x_{n+1} = x_n - \frac{3\theta f(x_n)}{f'^2(x_n)\left\{(3\theta - 5)f'(x_n) + 4f'\left(x_n - \frac{\theta u}{2}\right) + f'(x_n - \theta u)\right\}}\left(\frac{a'''^{\alpha} + b'''^{\alpha}}{2}\right)^{\frac{1}{\alpha}}, \tag{55}$$

where $a''' = f'^2(x_n)$ and

$$b''' = \left\{\frac{(3\theta - 5)f'^2(x_n) + 4f'\left(x_n - \frac{\theta u}{2}\right)f'(x_n) + f'(x_n - \theta u)f'(x_n)}{3\theta}\right\}.$$

For particular values of α and θ, some interesting particular cases of this family are:

i. For $(\alpha, \theta) = (1,1)$, we get the formula

$$x_{n+1}=x_n-\frac{f(x_n)}{2f'(x_n)}\left\{\frac{f'(x_n)+4f'\left(x_n-\frac{u}{2}\right)+f'(x_n-u)}{f'(x_n-u)-2f'(x_n)+4f'\left(x_n-\frac{u}{2}\right)}\right\}. \tag{56}$$

This is an order four new multipoint iterative formula. It is clear that formula (56) requires four evaluations per iteration and has an efficiency index $4^{\frac{1}{4}}\approx 1.414$, which is the same as the classical Newton's method.

ii. For $(\alpha,\theta)=(-1,1)$, we get the formula

$$x_{n+1}=x_n-\frac{6f(x_n)}{f'(x_n)+4f'\left(x_n-\frac{u}{2}\right)+f'(x_n-u)}. \tag{57}$$

This is a cubically convergent formula explored by Hasanov et al. [5, 14]. It is clear that formula (57) requires four evaluations per iteration and has an efficiency index $3^{\frac{1}{4}}\approx 1.316$. Therefore, efficiency index of method (56) is better than Hasnov et al. method (57).

The order of convergence of family (29), (34) and (55) will be studied in Theorems (4.1), in the subsequent section.

4.1. Analysis of Convergence of Multipoint Methods

Theorem 4.1. *Let $f: I \rightarrow \Re$ be continuous and sufficiently differentiable function defined in the open interval I.* If $f(x)$ has a simple root $r \in I$, then for sufficiently *close initial guess x_0 to r,*

(i) the family (29) has 3^rd^ order of convergence, for

$$\alpha \neq 1 \,\&\, \theta = 1,$$
$$\alpha = 1 \,\&\, \theta \neq 1,$$
$$\alpha \neq 1 \,\&\, \theta \neq 1,$$
$$\& \quad \alpha \rightarrow 0 \;\&\; \theta \in \Re,$$

and 4th order of convergence for $(\alpha,\theta)=(1,1)$.

(ii) the family (34) has 3rd order of convergence, for

$$\alpha \neq 1 \,\&\, \theta = 2/3,$$
$$\alpha = 1 \,\&\, \theta \neq 2/3,$$
$$\alpha \neq 1 \,\&\, \theta \neq 2/3,$$
$$\&\ \ \alpha \to 0 \ \&\ \theta \in \Re,$$

and 4th order of convergence $(\alpha, \theta) = (1, 2/3)$.

(iii) the family (55) has 3rd order of convergence for

$$\alpha \neq 1 \,\&\, \theta = 1,$$
$$\alpha = 1 \,\&\, \theta \neq 1,$$
$$\alpha \neq 1 \,\&\, \theta \neq 1,$$
$$\&\ \ \alpha \to 0 \ \&\ \theta \in \Re,$$

and 4th order convergence for $(\alpha, \theta) = (1, 1)$.

Proof: Since $f(x)$ is sufficiently differentiable, expanding $f(x_n)$ and $f'(x_n)$ about $x = r$ by Taylor's expansion, we have

$$f(x_n) = f'(r)\left[e_n + c_2 e_n{}^2 + c_3 e_n{}^3 + c_4 e_n{}^4 + c_5 e_n{}^5 + O\left(e_n{}^6\right)\right], \tag{58}$$

and

$$f'(x_n) = f'(r)\left[1 + 2c_2 e_n + 3c_3 e_n{}^2 + 4c_4 e_n{}^3 + 5c_5 e_n{}^4 + O\left(e_n{}^5\right)\right], \tag{59}$$

where c_k, e_n are defined earlier.

Using (58) and (59), we have

$$u(x_n) = \frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n{}^2 - 2\left(c_3 - c_2^2\right)e_n{}^3 - \left(3c_4 - 7c_2 c_3 + 4c_2^3\right)e_n{}^4 + O\left(e_n{}^5\right). \tag{60}$$

Also using (60), we have

$$e_n - \theta u = e_n - \theta e_n + \theta c_2 e_n{}^2 + \theta(2c_3 - 2c_2^2)e_n{}^3 + \theta\left(3c_4 - 7c_2 c_3 + 4c_2^3\right)e_n{}^4 + O\left(e_n{}^5\right), \tag{61}$$

and

$$f'(x_n - \theta u)$$
$$= f'(r)\left[1 + 2(1-\theta)c_2 e_n + \left(2\theta c_2^2 + 3(1-\theta)^2 c_3\right)e_n{}^2 + \left(10\theta c_2 c_3 - 6\theta^2 c_2 c_3 - 4\theta c_2^3 + 4c_4(1-\theta)^3\right)e_n{}^3 + O\left(e_n{}^4\right)\right]. \tag{62}$$

Upon using (59) and (62), we get

$$\frac{f'(x_n-\theta u)}{f'(x_n)}$$
$$=1-2\theta c_2 e_n+\left(6\theta c_2^2-3\left(2\theta-\theta^2\right)c_3\right)e_n{}^2+\left(4c_2c_3\theta(7-3\theta)-16c_2^3\theta+4c_4\left((1-\theta)^3-1\right)\right)e_n{}^3+O\left(e_n{}^4\right),$$
(63)

and

$$\frac{f'(x_n-\theta u/2)}{f'(x_n)}$$

$$=1-\theta c_2 e_n+\left(3\theta c_2^2-3\left(\theta-\frac{\theta^2}{4}\right)c_3\right)e_n{}^2+\left(2c_2\,c_3\theta\left(7-\frac{3\theta}{2}\right)-8c_2^3\theta+4c_4\left(\left(1-\frac{\theta}{2}\right)^3-1\right)\right)e_n{}^3+O\left(e_n{}^4\right).$$

(64)

Case (i) For $\alpha \in \Re \setminus \{0\}$, using binomial theorem in (55) and making use of (60), (63), (64), we obtain

$$e_{n+1}$$
$$=-\left\{\left(\frac{\alpha-1}{2}\right)c_2^2-c_3(\theta-1)\right\}e_n{}^3+\left\{(2\alpha-3)c_2^3-[(2-\alpha)\theta+3(\alpha-1)]c_2c_3+(\theta-1)(\theta-3)c_4\right\}e_n{}^4+O\left(e_n{}^5\right). \quad (65)$$

For the cases: $\alpha \neq 1 \,\&\, \theta = 1$, $\alpha = 1 \,\&\, \theta \neq 1$ and $\alpha \neq 1 \,\&\, \theta \neq 1$, we see that from (65) that the family (55) has 3rd order convergence. While for $(\alpha,\theta)=(1,1)$, we have

$$e_{n+1}=\left(c_2{}^3-c_2c_3\right)e_n{}^4+O\left(e_n{}^5\right),$$

which proves the 4th order convergence of family (55).

Case (ii) For $\alpha \to 0$, formula (55) can be written as

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}\sqrt{\frac{3\theta}{(3\theta-5)+4\dfrac{f'\left(x_n-\dfrac{\theta u}{2}\right)}{f'(x_n)}+\dfrac{f'(x_n-\theta u)}{f'(x_n)}}}\,. \quad (66)$$

Upon using equations (60), (63) and (64) and making use of binomial theorem in (66), we obtain

$$sign(f'(x_0))\left(\frac{f(x_n)}{f'(x_n)}\right)\sqrt{\frac{3\theta}{(3\theta-5)+4\dfrac{f'\left(x_n-\dfrac{\theta u}{2}\right)}{f'(x_n)}+\dfrac{f'(x_n-\theta u)}{f'(x_n)}}}$$

$$=e_n+\left[-\frac{1}{2}c_2^2-(\theta-1)c_3\right]e_n^{\ 3}+O\left(e_n^{\ 4}\right). \tag{67}$$

Using (67) in (66), we have

$$e_{n+1}=\left[\frac{1}{2}c_2^2+(\theta-1)c_3\right]e_n^{\ 3}+O\left(e_n^{\ 4}\right), \tag{68}$$

which proves 3rd order convergence of family (55) for $\alpha\to 0$.

The proofs of said convergence of families (29) and (34) can be proved on similar lines.

5. Extension for Multiple Zeros

These multipoint iterative schemes for simple zero can further be extended to the case of multiple zeros with cubic convergence. These results are rather interesting in view of the fact that usual third and fourth-order multipoint iterative methods e.g., Jarratt's method, Newton-secant method, Traub-Orstrowski's method etc. show linear convergence in case of multiple roots. For the purpose of demonstration, the iteration scheme (48) meant for simple zero can be extended for multiple zeros of nonlinear functions by introducing unknown auxiliary functions ϕ_1 , ϕ_2 and may be described through the following theorem:

Theorem 5.1 *Let* $f: D\to\Re$ *be a function for an open interval* $D\subset\Re$. *Let* $f(x)$ *has a multiple root, say* $x=r_m\in D$ *with multiplicity* $m>1$ *and* x_0 *be the initial guess to the multiple root. Assume that* f *is sufficiently differentiable in* D , *then the iterative scheme*

$$x_{n+1} = x_n - \phi_1 \frac{1}{2}\frac{f(x_n)}{f'(x_n)} - \phi_2 \frac{1}{2}\frac{\theta f(x_n)}{(\theta-1)f'(x_n) + f'(x_n - \theta u)}, \tag{69}$$

where $\theta \neq m$ *and* $\theta \neq \dfrac{2m}{m+1}$

(i) will have third-order convergence in the vicinity of r_m, *if*

$$\phi_1 = 2m\left(1 + \frac{(m-\theta)\left(1-\frac{\theta}{m}\right)^{-m}\left((m-\theta)(\theta-1) + m\left(1-\frac{\theta}{m}\right)^{m}\right)}{\theta(\theta(m+1)-2m)}\right), \tag{70}$$

and

$$\phi_2 = -\frac{2m\left(1-\frac{\theta}{m}\right)^{-m}\left((m-\theta)(\theta-1) + m\left(1-\frac{\theta}{m}\right)^{m}\right)^2}{\theta^2(\theta(m+1)-2m)}. \tag{71}$$

(ii) will satisfy the following error equation:

$$e_{n+1} = \frac{1}{2m^2(m-\theta)(A_0-\theta)}\left\{-2A_0(m-\theta)^2 C_3 + \frac{(m-\theta)(\theta-1)A_1 + \left(1-\frac{\theta}{m}\right)^{m} A_2}{(m-\theta)(\theta-1) + m\left(1-\frac{\theta}{m}\right)^{m}} C_2^{\,2}\right\} e_n^{\,3} + O(e_n^{\,4}), \tag{72}$$

where

$$A_0 = (m+2)\theta - 2m, \tag{73}$$

$$A_1 = -2m^2(3+m) + m\theta(6+m(9+m)) - 2\theta^2(1+m)^2, \tag{74}$$

$$A_2 = -2m^3(3+m) + m^2(2+m)(7+m)\theta - 2m(1+m)(5+m)\theta^2 + 2\theta^3(1+m)^2. \tag{75}$$

Proof: Since $f(x)$ is sufficiently differentiable function, therefore expanding $f(x_n)$ about $x = r_m$ by Taylor's expansion and using $f(r_m) = f'(r_m) = \ldots f^{(m-1)}(r_m) = 0$ and $f^{(m)}(r_m) \neq 0$ (a condition for $x = r_m$ to be a root of multiplicity m), we have

$$f(x_n) = \frac{f^{(m)}(r_m)}{m!}\, e_n^m \left[1 + C_2 e_n + C_3 e_n^2 + O\left(e_n^3\right)\right], \tag{76}$$

where $e_n = x_n - r_m$ and $C_k = \dfrac{f^{(m+k-1)}(r_m)}{(m+1)(m+2)\cdots(m+k-1) f^{(m)}(r_m)}, k = 2, 3, \cdots$.

Similarly, for $f'(x_n)$, it may be shown that

$$f'(x_n) = \frac{f^{(m)}(r_m)}{(m-1)!} e_n^{m-1}\left[1 + \frac{(m+1)}{m} C_2 e_n + \frac{(m+2)}{m} C_3 e_n^2 + O\left(e_n^3\right)\right]. \tag{77}$$

The expressions of Taylor's polynomials in terms of e_n for the different functions involved in formula (69) are cumbersome and lead to tedious calculations. Therefore, we shall make use of symbolic computation in programming package Mathematica7 to find out ϕ_1, ϕ_2 and to derive the corresponding error equation. Let us introduce the following abbreviations used in this program:

$$Ck = C_k,\ \ fx = f(x_n),\ \ f1x = f'(x_n),\ \ f1a = f^{(m)}(r_m),\ \ e = e_n = x_n - r_m,$$

$$f1y = f'(x_n - \theta u),\ \ \phi 1 = \phi_1,\ \ \phi 2 = \phi_2,\ u = \frac{f(x_n)}{f'(x_n)},\ fz = \frac{\theta\, f(x_n)}{(\theta - 1) f'(x_n) + f'(x_n - \theta u)}$$

and $e1 = e_{n+1} = x_{n+1} - r_m$.

a) Program code in Mathematica7 for finding the auxiliary functions ϕ_1 and ϕ_2

```
In[1]:=   fx = (f1a/m!)*e^m*(1 + C2*e + C3*e^2);
In[2]:=   f1x
          = (f1a/(m - 1)!)*e^(m - 1)*(1 + ((m + 1)/m)*C2*e + ((m + 2)/m)*C3*e^2);
In[3]:=   u = Series[fx/f1x, {e, 0, 3}] // FullSimplify;
In[4]:=   Clear[φ1,φ2 ];
In[5]:=   v = e - θ*u;
In[6]:=   fy = (f1a/m!)*v^m*(1 + C2*v + C3*v^2);
In[7]:=   f1y
```

```
= (f1a/(m - 1)!)*v^(m - 1)*(1 + ((m + 1)/m)*C2*v + ((m + 2)/m)*C3*v^2);
In[8]:=   fz = Series[θ*fx/((θ - 1)*f1x + f1y), {e, 0, 3}] // FullSimplify;
In[9]:=   e1 = e - (ϕ1/2)*u - (ϕ2/2)*fz // Simplify;
In[10]:=  a1 = Coefficient[e1, e] // Simplify;
In[11]:=  a2 = Coefficient[e1, e^2] // Simplify;
In[12]:=  Solve[{a1 == 0, a2 == 0}, {ϕ1, ϕ2}] // Simplify
Out[12]:=
```

$$\left\{\left\{\phi_1 \to 2m\left(1+\frac{(m-\theta)\left(1-\frac{\theta}{m}\right)^{-m}\left(-(-1+\theta)\theta+m\left(-1+\theta+\left(1-\frac{\theta}{m}\right)^{m}\right)\right)}{\theta(m(-2+\theta)+\theta)}\right),\right.\right.$$

$$\left.\left.\phi_2 \to -\frac{2m\left(1-\frac{\theta}{m}\right)^{-m}\left((-1+\theta)\theta-m\left(-1+\theta+\left(1-\frac{\theta}{m}\right)^{m}\right)\right)^{2}}{\theta^{2}(m(-2+\theta)+\theta)}\right\}\right\}$$

After some simplification, we obtain (70) and (71). This completes the proof of the first part of the theorem.

For second part of the theorem, we substitute the values of ϕ_1 , ϕ_2 and u (obtained from In[3]) and fz (obtained from In[8]) into scheme equation (69).

b) Program code in Mathematica7 for proving the second part of the theorem

```
In[13]:=  fx=(f1a/m!)*e^m*(1+C2*e+C3*e^2);
In[14]:=  f1x=(f1a/(m-1)!)*e^(m-1)*(1+((m+1)/m)*C2*e
              +((m+2)/m)*C3*e^2);
In[15]:=  u=Series[fx/f1x,{e,0,3}]//FullSimplify;
In[16]:=  Clear[ϕ1,ϕ2];
```

$$In[17]:=\ \phi_1 = 2m\left(1+\frac{(m-\theta)\left(1-\frac{\theta}{m}\right)^{-m}\left(-(-1+\theta)\theta+m\left(-1+\theta+\left(1-\frac{\theta}{m}\right)^{m}\right)\right)}{\theta(m(-2+\theta)+\theta)}\right);$$

$$In[18]:=\ \phi_2 = -\frac{2m\left(1-\frac{\theta}{m}\right)^{-m}\left((-1+\theta)\theta-m\left(-1+\theta+\left(1-\frac{\theta}{m}\right)^{m}\right)\right)^{2}}{\theta^{2}(\theta(m+1)-2m)};$$

```
In[19]:=  v=e-θ*u;
In[20]:=  f1y=(f1a/(m-1)!)*v^(m-1)*(1+((m+1)/m)*C2*v
   +((m+2)/m)*C3*v^2);
In[21]:=fz
      =Series[θ*fx/((θ-1)*f1x+f1y),{e,0,3}]//Simplify;
In[22]:=  e1=e-(ϕ1/2)*u-(ϕ2/2)*fz//Simplify;
In[23]:=  a1=Coefficient[e1,e]//Simplify
In[24]:=  a2=Coefficient[e1,e^2]//FullSimplify
In[25]:=  a3=Coefficient[e1,e^3]//FullSimplify
```

Out[23]:= 0
Out[24]:= 0
Out[25]:=

$$\frac{1}{2m^2(m-\theta)(A_0-\theta)}\left\{-2\,A_0\,(m-\theta)^2 C_3+\frac{(m-\theta)(\theta-1)A_1+\left(1-\frac{\theta}{m}\right)^{m}A_2}{(m-\theta)(\theta-1)+m\left(1-\frac{\theta}{m}\right)^{m}}C_2{}^2\right\}$$

Out[25] gives the required coefficient of $e_n{}^3$. Therefore, the final error equation (72) for the formula (69) is fully achieved. This completes the proof of the first part of the theorem.

For $m=1$ (simple root), ϕ_1 and ϕ_2 given by (70) and (71) are both unity respectively.

Following example shows the working of the procedure (69) for $\theta=-2$.

Example: Consider the following example

$$(x-2)^3(x+2)^4=0.$$

This equation has multiple roots at $x=2$ and $x=-2$ with multiplicity $m=3$ and $m=4$.

For $\theta=-2$, let $x_0=1$ be the initial guess for the triple positive root. To apply method (69), ϕ_1 and ϕ_2 are required and they can be computed from (70), (71) respectively. We obtain the required root 2.000000000000000 after five iterations. For an initial guess $x_0=4$, the formula (69) gives the root after six iterations. On the other hand, it is seen in Ref. [10] that using geometric mean Newton's method, the required root of this problem is obtained after twenty one iterations when $x_0=1$ and after twenty four iterations when $x_0=4$.

For the negative root, let $x_0=-1$ be the initial guess. We obtain the required multiple root -2.000000000000000 only after six iterations. For initial guess $x_0=-3$, formula (69) gives the root only after four iterations while, Lukić and Ralević formula [10] gives the required root after thirty one and thirty iterations respectively.

It is seen that on applying the same idea to Nedzhibov et al. formula (2.1) in [4] , we can obtain other cubically convergent family of multipoint iterative methods for the case of multiple roots.

6. Numerical Results

In this section, we shall present the numerical results obtained by employing the methods namely Newton's method (NM), formula (32), Traub's formula (36) (TM), formula (51), formula (52), Weerakoon and Fernando method (39) (WFM), formula (40) based on Heronian mean, formula (46), formula (56) and Hasnov et al. method (57) (HM^*) respectively to solve nonlinear equations given in Table 1. The results are summarized in Table 2. Computations have been performed using C^{++} in double precision arithmetic. We use $\in = 10^{-15}$. The following stopping criteria are used for computer programs:

$$(i) |x_{n+1} - x_n| < \in, \qquad (ii) |f(x_{n+1})| < \in.$$

Table 1: Test problems

No	Problem	$[a, b]$	Initial guess x_0	Root (r)
1.	$(x-1)^6 - 1 = 0$	$[1,3]$	1.1 3.0	2.000000000000000
2.	$x^3 + 4x^2 - 10 = 0$	$[0,2]$	0.1 2.0	1.3652300134140969
3.	$\cos x - x = 0$	$[0,2]$	0.0 2.0	0 .7390851332151600
4.	$\tan^{-1} x = 0$	$[-1,2]$	-1.0 2.0	0.000000000000000
5.	$x^3 + 4x^2 + \cos(x-1) - 6 = 0$	$[0.5,3]$	1.8 3.0	1.000000000000000

Table 2: Results of problems (D below-stands for divergent)

	Number of iterations									
Problem	NM	Formula (32)	TM (36)	Formula (51)	Formula (52)	WFM (39)	Formula (40)	Formula (46)	Formula (56)	HM^* (57)
1.	58	D	30	23	19	D	D	D	25	D
	8	4	4	4	4	5	5	5	4	5
2.	9	4	5	4	4	7	7	4	5	7
	4	3	3	2	2	3	3	3	2	3
3.	3	3	3	3	3	3	3	2	2	2
	3	2	3	3	3	3	3	3	2	3
4.	5	3	4	4	4	4	4	4	3	3
	D	4	D	5	5	D	D	D	5	4
5.	6	3	3	3	2	3	3	3	2	3
	6	4	3	3	3	4	4	4	3	4

7. Conclusions

This work proposes a family of super-Halley type methods based on non-linear means. Proposed family (16) unifies some of the most known third-order iterative methods for solving non-linear equations and they also provide many more unknown processes. Further, we have also presented many new third and fourth order multipoint iterative methods free from second order derivative by dicretization. Super-Halley method is the only method which produces the multipoint iterative methods of 4^{th} order. Numerical examples presented here prove that the proposed multipoint iterative methods can compete with any of the existing methods including classical Newton's method. A reasonably close initial guess is necessary for the multipoint methods to converge. This condition, however, applies to practically all the iterative methods for solving equations. Third-order multipoint iterative methods of first and second family have the efficiency indices equal to $\sqrt[3]{3} \cong 1.442$ (see [2]), which is better than the one of Newton's method having efficiency index $\sqrt[2]{2} \cong 1.414$. In third family (55), we have obtained a new quartically convergent iterative method (56). Of course, the efficiency index of this method is the same as that of Newton's method but it is better than Hasnov et al. method (57). Furthermore, family (48) has also been extended for finding multiple zeros of non-linear equations.

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References

[1] A. Lliev and N. Kyurkchiev, *Nontrivial Methods in Numerical Analysis (Selected Topics in Numerical Analysis)*, Lambert Acad. Publishing, 2010.

[2] A. M. Ostrowski, *Solution of Equations in Euclidean and Banach Space. 3rd ed*, Academic Press, New York, 1973.

[3] E. Schröder, *On infinitely many algorithms for solving equations*, Math. Ann, **2** (1870) 317-365. (Translated by G.W. Stewart, 1992).

[4] F. Soleymani, M. Sharifi, and B. S. Mousavi, An improvement of Ostrowski's and King's techniques with optimal convergence order eight, *Journal of Optimization Theory and Applications*, (2011), doi:10.1007/s10957-011-9929-9.

[5] G. H. Nedzhibov, V. I. Hasnov, and M. G. Petkov, *On some families of multipoint iterative methods for solving nonlinear equations,* Numer Algors, **42** (2006), 127-136.

[6] H. T. Kung and J. F. Traub, Optimal order of one-point and multipoint iteration, *Journal of Association for Computing Machinery*, **21** (1974), 643-551.

[7] H. Wang and S. Li, A family of derivative-free methods for nonlinear equations, *Rev. Mat. Complut.*, **24** no. 2 (2010), 375-389, DOI: 10.1007/s13163-010-0044-5.

[8] J. F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[9] M. A. Hernández, Newton-Raphson's method and convexity, *Univ. u Novom Sadu Zb. Rad. Period.-Mat. Fak. Ser. Mat.*, **22** (1992), 159-166.

[10] P. Jarratt, Some fourth order multipoint iterative methods for solving equations, *Math. Comput.*, **20** (1966), 434-437.

[11] P. S. Bullen, *The Power Means. Hand book of Means and Their Inequalities*, Kluwer Dordrecht, Netherlands, 2003.

[12] S. Weerakoon and T. G. I. Fernando, A variant of Newton's method with accelerated third-order convergence, *Appl. Math. Lett.*, **13** (2000), 87-93.

[13] T. Lukić and Nebojša M. Ralević, Geometric mean Newton's method for simple and multiple roots, *Appl. Math. Lett.*, **21** (2008), 30-36.

[14] V. I. Hasanov, I. G. Ivanov, and G. H. Nedzhibov, A new modification of Newton's method, *Application of Mathematics in Engineering'27*, Heron Sofia (2002), 278-286.

[15] V. Kanwar and S. K. Tomar, Modified families of multi-point iterative methods for solving nonlinear equations, *Numer Algors*, **44** (2007), 381-389.

[16] V. Kanwar, S. Singh, and S. Bakshi, Simple geometric constructions of quadratically and cubically convergent iterative functions to solve nonlinear equations, *Numer Algors*, **47** (2008), 95-107.

[17] W. Werner, Some improvement of classical methods for the solution of nonlinear equations, in Numerical Solution of Nonlinear Equations, *Lecture Notes in Math.*, **878** (1981), 426-440.