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$\mathcal{S}$-Antipodal Signed Graphs *

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Abstract

In this paper we introduced a new notion $\mathcal{S}$-antipodal signed graph of a signed graph and its properties are obtained. Also we give the relation between antipodal signed graphs and $\mathcal{S}$-antipodal signed graphs. Further, we discuss structural characterization of $\mathcal{S}$-antipodal signed graphs.

Keywords and Phrases: *Signed graphs, Marked graphs, Balance, Switching, Antipodal signed graphs, $\mathcal{S}$-Antipodal signed graphs, Complement, Negation.*

1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [4]; the non-standard will be given in this paper when required. We treat only finite simple graphs without self loops and isolates.

In 1953, Harary published "On the notion of balance of a signed graph", [5], the first paper to introduce signed graphs. In this paper, Harary defined a signed graph as a graph whose edge set has been partitioned into positive and

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negative edges. He called a cycle positive if it has an even number of negative edges, and he called a signed graph balanced if every cycle is positive. Then he gave both necessary and sufficient conditions for balance.

Since then, mathematicians have written numerous papers on the topic of signed graphs. Many of these papers demonstrate the connection between signed graphs and different subjects: circuit design (Barahona [3]), coding theory (Solé and Zaslavsky [23]), physics (Toulouse [25]) and social psychology (Abelson and Rosenberg [1]). While these subjects seem unrelated, balance plays an important role in each of these fields.

Four years after Harary's paper, Abelson and Rosenberg [1], wrote a paper in which they discuss algebraic methods to detect balance in a signed graphs. It was one of the first papers to propose a measure of imbalance, the "complexity" (which Harary called the "line index of balance"). Abelson and Rosenberg introduced an operation that changes a signed graph while preserving balance and they proved that this does not change the line index of imbalance. For more new notions on signed graphs refer the papers ([8, 9, 11, 12], [14]-[22]).

A *signed graph* is an ordered pair $S = (G, \sigma)$, where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \to \{+, -\}$ is a function. A *marking* of S is a function $\mu : V(G) \to \{+, -\}$; A signed graph S together with a marking μ is denoted by S_μ. Given a signed graph S one can easily define a marking μ of S as follows: For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking μ of S is called *canonical marking* of S. In a signed graph $S = (G, \sigma)$, for any $A \subseteq E(G)$ the *sign* $\sigma(A)$ is the product of the signs on the edges of A.

The following characterization of balanced signed graphs is well known.

Proposition 1. (**E. Sampathkumar [10]**) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

Let $S = (G, \sigma)$ be a signed graph. *Complement* of S is a signed graph $\overline{S} = (\overline{G}, \sigma')$, where for any edge $e = uv \in \overline{G}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S}$ as defined here is a balanced signed graph due to Proposition 1.

The idea of switching a signed graph was introduced in [1] in connection with structural analysis of social behavior and also its deeper mathematical aspects, significance and connections may be found in [27].

Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in E(G)$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [24]) or *cycle isomorphic* (see [26]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [26]):

Proposition 2. (**T. Zaslavsky** [26]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

2. $\mathcal{S}$-Antipodal Signed Graphs

The *eccentricity* $e(v)$ of a vertex v in the graph G is the distance to a vertex farthest from v. The maximum eccentricity is the *diameter* of G and the minimum is the *radius* of G. The *center* $C(G)$ of a graph G is the set of vertices with minimum eccentricity. A graph G is *self centered*, if all its vertices lie in the center. Equivalently, a self centered graph is a graph whose diameter equals its radius.

Radhakrishnan Nair and Vijayakumar [7] introduced the concept of $\mathcal{S}$-antipodal graph of a graph G as the graph $A^*(G)$ has the vertices in G with maximum eccentricity and two vertices of $A^*(G)$ are adjacent if they are at a distance of $diam(G)$ in G.

Motivated by the existing definition of complement of a signed graph, we extend the notion of $\mathcal{S}$-antipodal graphs to signed graphs as follows: The *$\mathcal{S}$-antipodal signed graph* $A^*(S)$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $A^*(G)$ and sign of any edge uv in $A^*(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called $\mathcal{S}$-antipodal signed graph, if $S \cong A^*(S')$ for some signed graph S'. The following result indicates the limitations of the notion $A^*(S)$ introduced above, since the entire class of unbalanced signed graphs is forbidden to be $\mathcal{S}$-antipodal signed graphs.

Proposition 3. *For any signed graph $S = (G, \sigma)$, its $\mathcal{S}$-antipodal signed graph $A^*(S)$ is balanced.*

Proof. Since sign of any edge uv in $A^*(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S, by Proposition 1, $A^*(S)$ is balanced.

For any positive integer k, the k^{th} iterated $\mathcal{S}$-antipodal signed graph $A^*(S)$ of S is defined as follows:

$$(A^*)^0(S) = S,\ (A^*)^k(S) = A^*((A^*)^{k-1}(S))$$

Corollary 4. *For any signed graph $S = (G, \sigma)$ and any positive integer k, $(A^*)^k(S)$ is balanced.*

In [7], the authors characterized those graphs that are isomorphic to their $\mathcal{S}$-antipodal graphs.

Proposition 5. (**Radhakrishnan Nair and Vijayakumar [7]**) *For a graph $G = (V, E)$, $G \cong A^*(G)$ if, and only if, G is a regular self-complementary graph.*

We now characterize the signed graphs that are switching equivalent to their $\mathcal{S}$-antipodal signed graphs.

Proposition 6. *For any signed graph $S = (G, \sigma)$, $S \sim A^*(S)$ if, and only if, G is a regular self-complementary graph and S is a balanced signed graph.*

Proof. Suppose $S \sim A^*(S)$. This implies, $G \cong A^*(G)$ and hence G is a regular self-complementary graph. Now, if S is any signed graph with underlying graph as regular self-complementary graph, Proposition 3 implies that $A^*(S)$ is balanced and hence if S is unbalanced and its $A^*(S)$ being balanced can not be switching equivalent to S in accordance with Proposition 2. Therefore, S must be balanced.

Conversely, suppose that S is a balanced signed graph and G is regular self-complementary. Then, since $A^*(S)$ is balanced as per Proposition 3 and since $G \cong A^*(G)$, the result follows from Proposition 2 again.

Proposition 7. *For any two signed graphs S_1 and S_2 with the same underlying graph, their $\mathcal{S}$-antipodal signed graphs are switching equivalent.*

Proof. Suppose $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ be two signed graphs with $G \cong G'$. By Proposition 3, $A^*(S_1)$ and $A^*(S_2)$ are balanced and hence, the result follows from Proposition 2.

Remark 8. *If G is regular self-complementary graph, then $G \cong \overline{G}$. The above result is holds good for $\overline{S} \sim A^*(S)$.*

Singleton [13] introduced the concept of antipodal graph of a graph G as the graph $A(G)$ having the same vertex set as that of G and two vertices are adjacent if they are at a distance of $diam(G)$ in G.

In [22], Siva Kota Reddy et al. introduced antipodal signed graph of a signed graph as follows:
The *antipodal signed graph* $A(S)$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $A(G)$ and sign of any edge uv is $A(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called antipodal signed graph, if $S \cong A(S')$ for some signed graph S'.

Proposition 9. (**Siva Kota Reddy et al. [22]**)
For any signed graph $S = (G, \sigma)$, its antipodal signed graph $A(S)$ is balanced.

We now characterize signed graphs whose $\mathcal{S}$-antipodal signed graphs and antipodal signed graphs are switching equivalent. In case of graphs the following result is due to Radhakrishnan Nair and Vijayakumar [7].

Proposition 10. *For a graph $G = (V, E)$, $A^*(G) \cong A(G)$ if, and only if, G is self centered.*

Proposition 11. *For any signed graph $S = (G, \sigma)$, $A^*(S) \sim A(S)$ if, and only if, G is self centered.*

Proof. Suppose $A^*(S) \sim A(S)$. This implies, $A^*(G) \cong A(G)$ and hence by Proposition 10, we see that the graph G must be self centered.

Conversely, suppose that G is self centered. Then $A^*(G) \cong A(G)$ by Proposition 10. Now, if S is a signed graph with underlying graph as self centered, by Propositions 3 and 9, $A^*(S)$ and $A(S)$ are balanced and hence, the result follows from Proposition 2.

In [7], the authors shown that $A^*(G) \cong A^*(\overline{G})$ if G is either complete or totally disconnected. We now characterize signed graphs whose $A^*(S)$ and $A^*(\overline{S})$ are switching equivalent.

Proposition 12. *For any signed graph $S = (G, \sigma)$, $A^*(S) \sim A^*(\overline{S})$ if, and only if, G is either complete or totally disconnected.*

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [6] as follows: $\eta(S)$ has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

Proposition 6, 11 & 12 provides easy solutions to other signed graph switching equivalence relations, which are given in the following results.

Corollary 13. *For any signed graph $S = (G, \sigma)$, $S \sim A^*(\eta(S))$ if, and only if, G is a regular self-complementary graph and S is a balanced signed graph.*

Corollary 14. *For any signed graph $S = (G, \sigma)$, $A^*(S) \sim A(\eta(S))$ (or $A^*(\eta(S)) \sim A(S)$) if, and only if, G is self centered.*

Corollary 15. *For any signed graph $S = (G, \sigma)$, $A^*(S) \sim A^*(\eta(\overline{S}))$ (or $A^*(\eta(S)) \sim A^*(\overline{S})$) if, and only if, G is either complete or totally disconnected.*

Problem 16. *Characterize signed graphs for which*
i). $\eta(S) \sim A^*(S)$
ii). $\eta(\overline{S}) \sim A(S)$
iii). $\eta(A^*(S)) \sim A(S)$
iv). $A^*(S) \sim \eta(A(S))$
v). $\eta(A^*(S)) \sim A^*(\overline{S})$
vi). $A^*(S) \sim \eta(A^*(\overline{S}))$

For a signed graph $S = (G, \sigma)$, the $A^*(S)$ is balanced (Proposition 3). We now examine, the conditions under which negation $\eta(S)$ of $A^*(S)$ is balanced.

Proposition 17. *Let $S = (G, \sigma)$ be a signed graph. If $A^*(G)$ is bipartite then $\eta(A^*(S))$ is balanced.*

Proof. Since, by Proposition 3, $A^*(S)$ is balanced, each cycle C in $A^*(S)$ contains even number of negative edges. Also, since $A^*(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $A^*(S)$ is also even. Hence $\eta(A^*(S))$ is balanced.

2.1. Characterization of $\mathcal{S}$-Antipodal Signed Graphs

The following result characterize signed graphs which are $\mathcal{S}$-antipodal signed graphs.

Proposition 18. *A signed graph $S = (G, \sigma)$ is a $\mathcal{S}$-antipodal signed graph if, and only if, S is balanced signed graph and its underlying graph G is a $\mathcal{S}$-antipodal graph.*

Proof. Suppose that S is balanced and its underlying graph G is a $\mathcal{S}$-antipodal graph. Then there exists a graph H such that $A^*(H) \cong G$. Since S is balanced, by Proposition 1, there exists a marking μ of G such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $S' = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $A^*(S') \cong S$. Hence S is a $\mathcal{S}$-antipodal signed graph.

Conversely, suppose that $S = (G, \sigma)$ is a $\mathcal{S}$-antipodal signed graph. Then there exists a signed graph $S' = (H, \sigma')$ such that $A^*(S') \cong S$. Hence by

Proposition 3, S is balanced.

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