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Solution of a Time-Space Fractional Diffusion Equation by Integral Transform Method *

V. B. L. Chaurasia†

Department of Mathematics, University of Rajasthan,
Jaipur - 302055, Rajasthan, India

and

Jagdev Singh‡

Department of Mathematics, Jagan Nath University, Village-Rampura,
Tehsil-Chaksu, Jaipur-303901, Rajasthan, India

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Abstract

In this paper, we obtain the solution of a time-space fractional diffusion equation by integral transform method. The method of integral transform based on using a fractional generalization of the Fourier transform and the classical Laplace transform. The solution is derived in a closed and computational form in terms of the Mittag-Leffler function. It provides an elegant extension of a result given earlier by Nikolova and Boyadjiev [20].

Keywords and Phrases: *Fractional diffusion equation, Caputo derivative, Mittag-Leffler function, Laplace transform, Fractional Fourier transform.*

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†E-mail: vblchaurasia@gmail.com

‡Corresponding author. E-mail: jagdevsinghrathore@gmail.com

1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. In recent years, considerable amount of research in fractional calculus was published in engineering and mathematical physics literature. Indeed, recent advances of fractional calculus are dominated by modern examples of applications in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, non-linear control theory, image processing, non-linear biological systems, astrophysics and electrochemistry. There is no doubt that fractional calculus has become an exciting new mathematical method of solution of diverse problems in science, engineering and applied mathematics. One of the main applications of the fractional calculus is modeling of the intermediate physical process. A very important model is the fractional diffusion and wave equations. A space-time fractional diffusion equation, obtain from the standard diffusion equation by replacing the second order space-derivative by a fractional Riesz derivative and the first order time-derivative by a Caputo fractional derivative, has been treated by Saiche and Zaslavsky [25], Gorenflo et al. [6], Uchajkin and Zototarev [29], Scalas et al. [27], Metzler and Klafter [14], Mainardi et al. [10]. The results obtained in [6] are complemented in [10], where the space-time fractional diffusion equation expressed by the Riesz-Feller space-fractional derivative and the Caputo time fractional derivative is considered. The fundamental solution of the corresponding Cauchy problem is found in the cited paper by means of Fourier-Laplace transform. Namias [17] introduced the fractional Fourier transform (FRFT) as a way to solve certain classes of ordinary and partial differential equations appearing in quantum mechanics. For further detail and properties of FRFT (see [1], [3], [12], [20], [22], [23], [31] and [32]). In this article we derive the solution of a time-space fractional diffusion equation by the method of integral transform based on FRFT and the Laplace transform. The Laplace transform of a function f(t) is defined as

$$\mathrm{L}\,[\mathrm{f(t)}\ ;\ \mathrm{s}] \;=\; \int_0^\infty \mathrm{e}^{-\mathrm{st}}\,\mathrm{f(t)}\,\mathrm{dt}. \qquad (1)$$

2. Mathematics Prerequisites

For a function u of the class S of a rapidly decreasing test functions on the real axis R, the Fourier transform is defined as

$$u^*(k) \;=\; F\,[u(x)\;;\;k] \;=\; \int_{-\infty}^{+\infty} e^{ikx}\,u(x)\,dx, \quad k \;\in\; R \tag{2}$$

and the inverse Fourier transform has the form

$$u(x) \;=\; F^{-1}\,[u^*(k)\;;x] \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx}u^*(k)\,dk, \quad x \;\in\; R. \tag{3}$$

In this paper we adopt the following FRFT as introduced in [8].

Definition 2.1. For a function $u \in \phi(R)$, the FRFT $u^*_\alpha$of the order α ($0 < \alpha \leq 1$) is defined as

$$u^*_\alpha(k) \;=\; F_\alpha[u(x)\;;\;k] \;=\; \int_{-\infty}^{+\infty} e_\alpha(k,x)\,u(x)\,dx, \quad k \;\in\; R \tag{4}$$

where

$$e_\alpha(k,x) \;=\; \begin{cases} e^{-i\,|\,k\,|^{\frac{1}{\alpha}}x}, & k \;\leq\; 0 \\ e^{i\,|\,k\,|^{\frac{1}{\alpha}}x}, & k \;>\; 0\;. \end{cases} \tag{5}$$

Evidently if $\alpha = 1$ the kernel (5) reduces to the kernel of (2). The relation between the FRFT (4) and the classical Fourier transform (2) is given by the equality

$$u^*_\alpha(k) \;=\; F_\alpha[u(x)\;;\;k] \;=\; F_1[u(x)\;;\omega] \;=\; u^*(\omega), \tag{6}$$

where

$$\omega \;=\; \begin{cases} -\,|\,k\,|^{\frac{1}{\alpha}}, & k \;\leq\; 0 \\ |\,k\,|^{\frac{1}{\alpha}}, & k \;>\; 0\;. \end{cases} \tag{7}$$

Thus, if

$$F_\alpha[u(x);\;k] \;=\; F_1[u(x)\;;\omega] \;=\phi(\omega),$$

then

$$u(x) \;=\; F_\alpha^{-1}[u^*_\alpha(k)\;;\;x] \;=\; F_1^{-1}\,[\phi(\omega);\,x]\,. \tag{8}$$

Some properties of FRFT

Theorem 2.1 [20]. *If* $0 \leq \alpha < 1$ and $u^{(n)}(x) \in \phi(R)$, *then*

$$F_\alpha[u^{(n)}(x);\ k] = (-i \,\mathrm{sign}\, k \,|\, k\,|^{1/\alpha})^n\, u^*_\alpha(k)\ , \quad k \in R.$$

Theorem 2.2 (Convolution theorem) [20]. *If* $0 < \alpha \leq 1$ *and* $u(x),\ v(x) \in \phi(R)$, *then*

$$F_\alpha[(u * v)\ (x)\ ;\ k] = u^*_\alpha(k)\ v^*_\alpha(k),$$

where

$$(u\ *\ v)\ (x) = \int_{-\infty}^{+\infty} u\ (x-\xi)\, v(\xi) d\xi,$$

and

$$F_\alpha[u(x)\ ;\ k] = u^*_\alpha(k)\ ,\ F_\alpha[v(x)\ ;\ k] = v^*_\alpha(k).$$

The right-sided Riemann-Liouville fractional integral of order α is defined by Miller and Ross [15, p.45], Samko et al. [26]:

$$ {}^{RL}_{\ a}D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau,\ (t > a) \tag{9}$$

where $R(\alpha) > 0$.
The right-sided Riemann-Liouville fractional derivative of order α is defined as

$$ {}^{RL}_{\ a}D_t^{\alpha} f(t) = \left(\frac{d}{dt}\right)^n (I_a^{n-\alpha} f(t)) \quad (Re(\alpha) > 0,\ n = [Re(\alpha)] + 1), \tag{10}$$

where $[\alpha]$ represents the integral part of the number α.

The following fractional derivative of order $\alpha > 0$ is introduced by Caputo [4] in the form (if $m - 1 < \alpha \leq m$, $Re(\alpha) > 0$, $m \in N$):

$$\begin{aligned} {}^{c}_{0}D_t^{\alpha} f(t) &= \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)\, d\tau}{(t-\tau)^{\alpha+1-m}} \\ &= \frac{d^m f(t)}{dt^m}, \quad \text{if } \alpha = m \end{aligned} \tag{11}$$

where $\frac{d^m f(t)}{dt^m}$ is the m-th derivative of order m of the function f(t) with respect to t. The Laplace transform of this derivative given in [24] in the form

$$L\,\{{}_0^cD_t^\alpha\, f(t)\,;\,s\} \;=\; s^\alpha\,\bar{f}(s)\;-\;\sum_{r=0}^{m-1} s^{\alpha-r-1}\, f^{(r)}(\,0+\,), (m-1\;<\;\alpha\;\leq\;m)\,. \quad (12)$$

A generalization of the Riemann-Liouville fractional derivative operator (10) and Caputo fractional derivative operator (11) is given by Hilfer [7], by introducing a right-sided fractional derivative operator of two parameters of order $0 < \alpha < 1$ and $0 \leq \beta \leq 1$ in the form

$${}_0D_{a+}^{\alpha,\beta}\, f(t) \;=\; \left(I_{a+}^{\beta(1-\alpha)}\,\frac{d}{dt}\,\left(I_{a+}^{(1-\beta)\,(1-\alpha)}\, f(t)\right)\right). \quad (13)$$

It is interesting to observe that for $\beta = 0$, (13) reduces to the classical Riemann-Liouville fractional derivative operator (10). On the other hand, for $\beta = 1$ it yields the Caputo fractional derivative operator defined by (11). The Laplace transform formula for this operator is given by Hilfer [7]

$$L\,\{{}_0D_{0+}^{\alpha,\beta}\, f(t)\,;\,s\} \;=\; s^\alpha\bar{f}(s)\;-\;s^{\beta(\alpha-1)}\, I_{0+}^{(1-\beta)\,(1-\alpha)}\, f(0+\,),\quad (0\;<\;\alpha\;<\;1),\; (14)$$

where the initial value term

$$I_{0+}^{(1-\beta)\,(1-\alpha)}\, f(0+\,), \quad (15)$$

involves the Riemann-Liouville fractional integral operator of order $(1-\beta)$ $(1-\alpha)$ evaluated in the limit as $t \to 0+$. For more details and properties of this operator see Tomovski et al. [28]. For generalization of the time-space diffusion equation, we use the fractional derivative operator of the form

$$D_\beta^\alpha\, u(x) \;=\; (1-\beta)\, D_+^\alpha\, u(x) -\; \beta\, D_-^\alpha\, u(x),\;\; 0\;<\;\alpha\;\leq\;1,\;\; \beta\;\in\;\; R \quad (16)$$

where D_+^α and D_-^α are the Riemann-Liouville fractional derivatives on the real axis given as

$$D_+^\alpha\, u(x) \;=\; \frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dx}\int_{-\infty}^{x} (x-\tau)^{\alpha-1}\, u(\tau)\, d\tau$$

and

$$D_-^\alpha\, u(x) \;=\; -\;\frac{1}{\Gamma(1-\alpha)}\,\frac{d}{dx}\int_{x}^{\infty} (\tau-x)^{\alpha-1}\, u(\tau)\, d\tau.$$

A key role in our consideration is given to a relation established in [8] according to which for $0 < \alpha \le 1$, any value of β and a function $u(x) \in \phi(R)$,

$$F_\alpha[D_\beta^\alpha u(x); k] = (-i\,C_\alpha k)\,F_\alpha[u(x)\,;\,k]\,,\quad k \in R, \tag{17}$$

where

$$C_\alpha = \sin(\alpha\pi/2) + i\,\mathrm{sign}k(1-2\beta)\cos(\alpha\pi/2).$$

3. Generalized Fractional Diffusion Equation

In this section we apply the FRFT (4) for the Cauchy-type problem for the fractional diffusion equation

$$_0D_t^{\alpha,\beta}\,u(x,t) = \mu\,D_\eta^{\gamma+1}\,u\,(x,t),\quad x \in R,\quad t > 0 \tag{18}$$

subject to the initial condition

$$I_{0+}^{(1-\beta)(1-\alpha)}\,u\,(x,0+) = f(x), \tag{19}$$

where $_0D_t^{\alpha,\beta}$ is the generalized Riemann-Liouville fractional derivative operator, defined by (13),

$$I_{0+}^{(1-\beta)(1-\alpha)}\,u(x,0+),$$

involves the Riemann-Liouville fractional integral operator of order $(1-\beta)(1-\alpha)$ evaluated in the limit as $t \to 0+$ and $D_\eta^{\gamma+1}$ is the space fractional derivative (16) we can refer to as generalized Riemann-Liouville space fractional derivative.

Theorem 3.1. *If $f(x) \in \phi(R)$, $0 < \alpha < 1$, $0 < \beta \le 1$, $0 < \gamma \le 1$ and for every value of $\eta \in R$, the Cauchy type problem (18) – (19) is solvable and its solution $u(x,t)$ is given by the integral*

$$u(x,t) = \int_{-\infty}^{+\infty} G(x-\xi,t)\,f(\xi)\,d\xi, \tag{20}$$

where

$$G\,(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega x}\,t^{\alpha-\beta(\alpha-1)-1}\,E_{\alpha,\alpha-\beta(\alpha-1)}(-i\mu\,C_{\gamma+1}\omega\,t^\alpha)\,d\omega.$$

Proof. According to (17) it is clear that the application of the FRFT $F_{\gamma+1}$to the equation (18) and the initial condition (19) results to the equation

$$ {}_0D_t^{\alpha,\beta}\, u^*_{\gamma+1}(k,t) \;=\; (-\,i\,\mu\, C_{\gamma+1}k)\; u^*_{\gamma+1}\,(k,t), \tag{21}$$

subject to the condition

$$ I_{0+}^{(1-\beta)(1-\alpha)}\; u^*_{\gamma+1}(k, 0+) \;=\; f^*_{\gamma+1}(k). \tag{22}$$

The Laplace transform (1) applied then to (21) and (22) implies

$$ L\,[u^*_{\gamma+1}\,(k,t)\;;\;s] \;=\; \frac{s^{\beta(\alpha-1)}\, f^*_{\gamma+1}(k)}{s^{\alpha} + i\,\mu\, C_{\gamma+1}k}. \tag{23}$$

The formula

$$ L^{-1}\left[\frac{s^{\alpha-1}}{s^{\beta}+i\,\mu\, C_{\gamma+1}k}\;;\,t\right] \;=\; t^{\beta-\alpha}\, E_{\beta,\beta-\alpha+1}\,(-i\,\mu\, C_{\gamma+1}k\; t^{\beta}), \tag{24}$$

enables us to conclude from (23) that

$$ u^*_{\gamma+1}(k,t) \;=\; f^*_{\gamma+1}\,(k)\; t^{\alpha-\beta(\alpha-1)-1}\, E_{\alpha,\alpha-\beta(\alpha-1)}\,(-i\mu\, C_{\gamma+1}\; k\; t^{\alpha}).$$

Because of (6) and (7), the latest quantity gives

$$ u^*(\omega,t) \;=\; f^*(\omega)\, t^{\alpha-\beta(\alpha-1)-1}\, E_{\alpha,\alpha-\beta(\alpha-1)}(-i\,\mu\, C_{\gamma+1}\omega\; t^{\alpha}). \tag{25}$$

By Theorem 2.2, we obtain from the equation (25) that the solution desired is indeed given by (20), where

$$ G(x,t) \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega x}\, t^{\alpha-\beta(\alpha-1)-1}\, E_{\alpha,\alpha-\beta(\alpha-1)}\,(-i\mu\, C_{\gamma+1}\omega\, t^{\alpha})\, d\omega.$$

4. Special Cases

If we set $\beta = 0$, then the Hilfer fractional derivative (13) reduces to a Riemann-Liouville fractional derivative (10) and the theorem yields the following:

Corollary 4.1. *Consider the Cauchy-type problem for the fractional diffusion equation*

$$ {}_0^{RL}D_t^{\alpha}\, u(x,t) \;=\; \mu\, D_\eta^{\gamma+1}\, u(x,t),\; x \in R,\; t > 0,\; 0 < \alpha < 1,\, 0 < \gamma \le 1, \tag{26} $$

subject to the initial condition

$$ {}_0^{RL}D_t^{\alpha-1}\, u(x,0) \;=\; f(x), \tag{27} $$

where ${}_0^{RL}D_t^{\alpha}$ *is the Riemann-Liouville fractional derivative operator of order* α *defined by (10),* $[{}_0^{RL}D_t^{\alpha-1}\, u(x,0)]$ *means the Riemann-Liouville fractional partial derivative of u(x,t) with respect to t of order* $\alpha-1$ *evaluated at t = 0 and* $D_\eta^{\gamma+1}$ *is the space fractional derivative (16) we can refer to as generalized Riemann-Liouville space fractional derivative. Then for the solution of (26) with initial condition (27), there holds the formula*

$$ u(x,t) \;=\; \int_{-\infty}^{+\infty} G\,(x-\xi,t)\, f(\xi)\, d\xi, \tag{28} $$

where

$$ G(x,t) \;=\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega t}\, t^{\alpha-1}\, E_{\alpha,\alpha}\,(-i\,\mu\, C_{\gamma+1}\omega\, t^{\alpha}) d\omega. \tag{29} $$

Finally, we take $\beta = 1$, then the Hilfer fractional derivative (13) reduces to a Caputo fractional derivative operator (11) and it yields the following result recently obtained by Nikolova and Boyadjiev [20]:

Corollary 4.2. *Consider the Cauchy type problem for the fractional diffusion equation*

$$ {}_0^{c}D_t^{\alpha}\, u(x,t) \;=\; \mu\, D_\eta^{\gamma+1}\, u(x,t),\; x \in R,\; t > 0,\; 0 < \alpha < 1,\, 0 < \gamma \le 1, \tag{30} $$

subject to the initial condition

$$ u(x,0) \;=\; f(x), \tag{31} $$

where ${}_0^{c}D_t^{\alpha}$ *is the Caputo fractional derivative operator of order* α *and* $D_\eta^{\gamma+1}$ *is the space fractional derivative (16) we can refer to as generalized Riemann-Liouville space fractional derivative. Then for the solution of (30) with initial condition (31), there holds the formula*

$$ u(x,t) \;=\; \int_{-\infty}^{+\infty} G\,(x-\xi,t)\phi(\xi)\, d\xi, \tag{32} $$

where

$$\mathrm{G}\,(\mathrm{x},\mathrm{t}) \;=\; \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\omega\mathrm{x}}\,\mathrm{E}_{\alpha}(-\mathrm{i}\mu\,\mathrm{C}_{\gamma+1}\omega\,\mathrm{t}^{\alpha})\mathrm{d}\omega. \tag{33}$$

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