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New Some Hadamard's Type Inequalities for Co-ordinated Convex Functions *

Mehmet Zeki Sarıkaya[†]and Erhan. Set[‡]
Department of Mathematics, Faculty of Science and Arts,
Düzce University, Düzce, Turkey

and

M. Emin Ozdemir[§]
Atatürk University, K.K. Education Faculty,
Department of Mathematics, 25240, Campus, Erzurum, Turkey

and

Sever S. Dragomir[¶]
School of Computational and Applied Mathematics,
University of the Witwatersrand, Private Bag-3,
Wits-2050, Johannesburg, South Africa

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Abstract

In this paper, we establish new some Hermite-Hadamard's type inequalities of convex functions of 2−variables on the co-ordinates.

Keywords and Phrases: *convex function, co-ordinated convex mapping, Hermite-Hadamard inequality.*

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†Corresponding author. E-mail: sarikayamz@gmail.com
‡E-mail: erhanset@yahoo.com
§E-mail: emos@atauni.edu.tr
¶E-mail: sever.dragomir@vu.edu.au

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$, the following double inequality is well known in the literature as the Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Let us now consider a bidemensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(tx + (1-t)\,z, ty + (1-t)\,w) \leq tf(x,y) + (1-t)\,f(z,w)$$

holds, for all $(x,y), (z,w) \in \Delta$ and $t \in [0,1]$. A function $f : \Delta \to \mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$ and $y \in [c,d]$ (see [5]).

A formal definition for co-ordinated convex function may be stated as follows (see [5],[7]):

Definition 1. *A function $f : \Delta \to \mathbb{R}$ will be called co-ordinated canvex on Δ, for all $t, s \in [0,1]$ and $(x,y), (u,v) \in \Delta$, if the following inequality holds:*

$$\begin{aligned} & f(tx + (1-t)\,y, su + (1-s)\,v) \\ \leq\ & tsf(x,u) + s(1-t)f(y,u) + t(1-s)f(x,v) + (1-t)(1-s)f(y,v). \end{aligned}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [5]). For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([1]-[9]).

Also, in [5], Dragomir establish the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1. *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \qquad (1.1) \\ \leq \; & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ \leq \; & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\ \leq \; & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx \right. \\ & \left. + \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\ \leq \; & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

The above inequalities are sharp.

The main purpose of this paper is to establish new Hadamard-type inequalities of convex functions of 2-variables on the co-ordinates.

2. Inequalities for Co-ordinated Convex Functions

Lemma 1. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, then the following equality holds:*

$$\begin{aligned} & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \qquad (2.1) \\ & -\frac{1}{2}\left[\frac{1}{b-a}\int_a^b [f(x,c)+f(x,d)]\,dx + \frac{1}{d-c}\int_c^d [f(a,y)+f(b,y)]\,dy\right] \\ = \; & \frac{(b-a)(d-c)}{4}\int_0^1\int_0^1 (1-2t)(1-2s)\frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d)\,dtds. \end{aligned}$$

Proof. By integration by parts, we get

$$\begin{aligned}
&\int_0^1\int_0^1(1-2s)(1-2t)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\,dtds \qquad (2.2)\\
=&\int_0^1(1-2s)\left\{(1-2t)\,\frac{1}{a-b}\frac{\partial f}{\partial s}(ta+(1-t)b,sc+(1-s)d)\Big|_0^1\right.\\
&\left.+\frac{2}{a-b}\int_0^1\frac{\partial f}{\partial s}(ta+(1-t)b,sc+(1-s)d)\,dt\right\}ds\\
=&\int_0^1(1-2s)\left\{-\frac{1}{a-b}\frac{\partial f}{\partial s}(a,sc+(1-s)d)-\frac{1}{a-b}\frac{\partial f}{\partial s}(b,sc+(1-s)d)\right.\\
&\left.+\frac{2}{a-b}\int_0^1\frac{\partial f}{\partial s}(ta+(1-t)b,sc+(1-s)d)\,dt\right\}ds\\
=&\frac{1}{b-a}\left\{\int_0^1(1-2s)\left(\frac{\partial f}{\partial s}(a,sc+(1-s)d)+\frac{\partial f}{\partial s}(b,sc+(1-s)d)\right)ds\right.\\
&\left.-2\int_0^1\int_0^1(1-2s)\frac{\partial f}{\partial s}(ta+(1-t)b,sc+(1-s)d)\,dtds\right\}.
\end{aligned}$$

Thus, again by integration by parts in the right hand side of (2.2), it follows that

$$\begin{aligned}
&\int_0^1(1-2s)\left(\frac{\partial f}{\partial s}(a,sc+(1-s)d)+\frac{\partial f}{\partial s}(b,sc+(1-s)d)\right)ds \qquad (2.3)\\
&-2\int_0^1\int_0^1(1-2s)\frac{\partial f}{\partial s}(ta+(1-t)b,sc+(1-s)d)\,dtds\\
=&(1-2s)\,\frac{(f(a,sc+(1-s)d)+f(b,sc+(1-s)d))}{c-d}\Big|_0^1\\
&+\frac{2}{c-d}\int_0^1(f(a,sc+(1-s)d)+f(b,sc+(1-s)d))\,ds\\
&-2\int_0^1\left\{(1-2s)\,\frac{f(ta+(1-t)b,sc+(1-s)d)}{c-d}\Big|_0^1\right.\\
&\left.+\frac{2}{c-d}\int_0^1 f(ta+(1-t)b,sc+(1-s)d)\,ds\right\}dt
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{f(a,c)+f(b,c)}{c-d}-\frac{f(a,d)+f(b,d)}{c-d} \\
&\quad +\frac{2}{c-d}\int_0^1 \left(f(a,sc+(1-s)d)+f(b,sc+(1-s)d)\right)ds \\
&\quad -2\int_0^1 \left\{-\frac{f(ta+(1-t)b,c)}{c-d}-\frac{f(ta+(1-t)b,d)}{c-d}\right. \\
&\quad \left. +\frac{2}{c-d}\int_0^1 f(ta+(1-t)b,sc+(1-s)d)\,ds\right\}dt \\
&= \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{(d-c)} \\
&\quad +\frac{4}{(d-c)}\int_0^1\int_0^1 f(ta+(1-t)b,sc+(1-s)d)\,dsdt \\
&\quad -\frac{2}{(d-c)}\left\{\int_0^1 \left(f(a,sc+(1-s)d)+f(b,sc+(1-s)d)\right)ds\right. \\
&\quad \left. +\int_0^1 \left(f(ta+(1-t)b,c)+f(ta+(1-t)b,d)\right)dt\right\}.
\end{aligned}
$$

Writing (2.3) in (2.2), using the change of the variable $x = ta + (1-t)b$ and $y = sc + (1-s)d$ for $t, s \in [0,1]^2$, and multiplying the both sides by $\frac{(b-a)(d-c)}{4}$, we obtain (2.1), which completes the proof.

Theorem 2. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left|\frac{\partial^2 f}{\partial t \partial s}\right|$ is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$
\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
&\quad \left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A\right| \\
&\leq \frac{(b-a)(d-c)}{16}\left(\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|}{4}\right)
\end{aligned}
\tag{2.4}
$$

where

$$
A = \frac{1}{2}\left[\frac{1}{b-a}\int_a^b \left[f(x,c)+f(x,d)\right]dx + \frac{1}{d-c}\int_c^d \left[f(a,y)+f(b,y)\right]dy\right].
$$

Proof. From Lemma 1, we have

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right|\\\leq\ &\frac{(b-a)(d-c)}{4}\\&\times\int_0^1\int_0^1|(1-2t)(1-2s)|\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|dtds.\end{aligned}$$

Since $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ, then one has:

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right.\\&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right|\\\leq\ &\frac{(b-a)(d-c)}{4}\\&\times\int_0^1\left[\int_0^1|(1-2t)(1-2s)|\left\{t\left|\frac{\partial^2 f}{\partial t\partial s}(a,sc+(1-s)d)\right|\right.\right.\\&\left.\left.+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,sc+(1-s)d)\right|\right\}dt\right]ds.\end{aligned}$$

Firstly, by calculating the integral in above inequality, we have

$$\begin{aligned}&\int_0^1|1-2t|\left\{t\left|\frac{\partial^2 f}{\partial t\partial s}(a,sc+(1-s)d)\right|+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,sc+(1-s)d)\right|\right\}dt\\=\ &\int_0^{\frac{1}{2}}(1-2t)\left\{t\left|\frac{\partial^2 f}{\partial t\partial s}(a,sc+(1-s)d)\right|+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,sc+(1-s)d)\right|\right\}dt\\&+\int_{\frac{1}{2}}^1(2t-1)\left\{t\left|\frac{\partial^2 f}{\partial t\partial s}(a,sc+(1-s)d)\right|+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,sc+(1-s)d)\right|\right\}dt\\&=\frac{1}{4}\left(\left|\frac{\partial^2 f}{\partial t\partial s}(a,sc+(1-s)d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,sc+(1-s)d)\right|\right).\end{aligned}$$

Thus, we obtain

$$\begin{aligned} &\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\ &\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A\right| \\ \leq\ & \frac{(b-a)(d-c)}{16} \\ &\times\int_0^1 |1-2s|\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,sc+(1-s)d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,sc+(1-s)d)\right|\right\}ds. \end{aligned} \tag{2.5}$$

A similar way for other integral, since $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ, we get

$$\begin{aligned} &\int_0^1 |1-2s|\left\{\left|\frac{\partial^2 f}{\partial t\partial s}(a,sc+(1-s)d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,sc+(1-s)d)\right|\right\}ds \\ =\ & \int_0^{\frac{1}{2}}(1-2s)\left\{s\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\right\}ds \\ &+\int_0^{\frac{1}{2}}(1-2s)\left\{s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}ds \\ =\ & \int_{\frac{1}{2}}^1(2s-1)\left\{s\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\right\}ds \\ &+\int_{\frac{1}{2}}^1(2s-1)\left\{s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}ds \\ =\ & \frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{4}. \end{aligned} \tag{2.6}$$

By the (2.5) and (2.6), we get the inequality (2.4).

Theorem 3. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}^2$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is a convex* *function on the co-ordinates on* Δ, *then one has the inequalities:*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right| \\
\leq\ &\frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}
\end{aligned}$$

where

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b[f(x,c)+f(x,d)]\,dx+\frac{1}{d-c}\int_c^d[f(a,y)+f(b,y)]\,dy\right].$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1, we have

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right| \\
\leq\ &\frac{(b-a)(d-c)}{4} \\
&\times\int_0^1\int_0^1|(1-2t)(1-2s)|\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|dtds.
\end{aligned}$$

By using the well known Hölder inequality for double integrals, $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ, then one has:

$$\begin{aligned} & \left| \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} \right. \\ & \left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\,dydx - A \right| \\ \leq\; & \frac{(b-a)(d-c)}{4} \left(\int_0^1 \int_0^1 |(1-2t)(1-2s)|^p\, dtds \right)^{\frac{1}{p}} \\ & \times \left(\int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d) \right|^q dtds \right)^{\frac{1}{q}}. \end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t \partial s}\right|^q$ is convex function on the co-ordinates on Δ, we know that for $t \in [0,1]$

$$\begin{aligned} & \left| \frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d) \right|^q \\ \leq\; & t \left| \frac{\partial^2 f}{\partial t \partial s}(a, sc+(1-s)d) \right|^q + (1-t) \left| \frac{\partial^2 f}{\partial t \partial s}(b, sc+(1-s)d) \right|^q \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d) \right|^q \\ \leq\; & ts \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q + t(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q \\ & +(1-t)s \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q + (1-t)(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q \end{aligned}$$

hence, it follows that

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A\right| \\
\leq\ & \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \\
&\times\left(\int_0^1\int_0^1\left\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right.\right. \\
&\left.\left.+(1-t)s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\} dtds\right)^{\frac{1}{q}} \\
=\ & \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 4. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q \geq 1$, is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \qquad (2.7)\\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A\right| \\
\leq\ & \frac{(b-a)(d-c)}{16} \\
&\times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}
\end{aligned}$$

where

$$A=\frac{1}{2}\left[\frac{1}{b-a}\int_a^b\left[f\left(x,c\right)+f\left(x,d\right)\right]dx+\frac{1}{d-c}\int_c^d\left[f\left(a,y\right)+f\left(b,y\right)\right]dy\right].$$

Proof. From Lemma 1, we have

$$\begin{aligned}&\left|\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}\right.\\&\left.+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f\left(x,y\right)dydx-A\right|\\ \leq\;&\frac{\left(b-a\right)\left(d-c\right)}{4}\\&\times\int_0^1\int_0^1\left|(1-2t)(1-2s)\right|\left|\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b,sc+(1-s)d\right)\right|dtds.\end{aligned}$$

By using the well known power mean inequality for double integrals, $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ, then one has:

$$\begin{aligned}&\left|\frac{f\left(a,c\right)+f\left(a,d\right)+f\left(b,c\right)+f\left(b,d\right)}{4}\right.\\&\left.+\frac{1}{\left(b-a\right)\left(d-c\right)}\int_a^b\int_c^d f\left(x,y\right)dydx-A\right|\\ \leq\;&\frac{\left(b-a\right)\left(d-c\right)}{4}\left(\int_0^1\int_0^1\left|(1-2t)(1-2s)\right|dtds\right)^{1-\frac{1}{q}}\\&\times\left(\int_0^1\int_0^1\left|(1-2t)(1-2s)\right|\left|\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b,sc+(1-s)d\right)\right|^q dtds\right)^{\frac{1}{q}}.\end{aligned}$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is convex function on the co-ordinates on Δ, we know that for $t\in[0,1]$

$$\begin{aligned}&\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|^q\\ \leq\;& t\left|\frac{\partial^2 f}{\partial t\partial s}(a,sc+(1-s)d)\right|^q+(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,sc+(1-s)d)\right|^q\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d)\right|^q \\
\leq\ & ts\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + t(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q \\
&+(1-t)s\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q
\end{aligned}$$

hence, it follows that

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx - A\right|
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{(b-a)(d-c)}{4}\left(\frac{1}{4}\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1\int_0^1 |(1-2t)(1-2s)|\left\{ts\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + t(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q\right.\right. \\
&\left.\left.+(1-t)s\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q\right\} dtds\right)^{\frac{1}{q}}.
\end{aligned}$$

Firstly, by calculating the integral in above inequality, we have

$$\begin{aligned}
&\int_0^1 |1-2t| \left(ts \left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + t(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q \right.\\
&\left. +(1-t)s\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q \right) dt \\
=\ &\int_0^{\frac{1}{2}} (1-2t) \left(ts \left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + t(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q \right.\\
&\left. +(1-t)s\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q \right) dt \\
&+\int_{\frac{1}{2}}^1 (2t-1) \left(ts \left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + t(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q \right.\\
&\left. +(1-t)s\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q \right) dt \\
=\ &\frac{s\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q}{24} + \frac{(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q}{24} \\
&+\frac{5s\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q}{24} + \frac{5(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{24} \\
&+\frac{5s\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q}{24} + \frac{5(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q}{24} \\
&+\frac{s\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q}{24} + \frac{(1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{24} \\
=\ &\frac{s\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q + (1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q}{4} \\
&+\frac{s\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q + (1-s)\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{4}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}\right. \\
&\left.+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx-A\right| \\
\leq\; &\frac{(b-a)(d-c)}{16}\left[\int_0^1|1-2s|\left(s\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right.\right. \\
&\left.\left.+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)ds\right]^{\frac{1}{q}}.
\end{aligned} \tag{2.8}$$

A similar way for other integral, since $f:\Delta\to\mathbb{R}$ is co-ordinated convex on Δ, we get

$$\begin{aligned}
&\int_0^1|1-2s|\left(s\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right. \\
&\left.+s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)ds \\
=\; &\int_0^{\frac{1}{2}}(1-2s)\left(s\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right. \\
&\left.+s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)ds \\
&+\int_{\frac{1}{2}}^1(2s-1)\left(s\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\right. \\
&\left.+s\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)ds \\
=\; &\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q}{24}+\frac{5\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q}{24}+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q}{24}+\frac{5\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{24} \\
&+\frac{5\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q}{24}+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q}{24}+\frac{5\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q}{24}+\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{24} \\
=\; &\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}.
\end{aligned} \tag{2.9}$$

By the (2.8) and (2.9), we get the inequality (2.7).

Remark 1. *Since* $\frac{1}{4} < \frac{1}{(p+1)^{\frac{2}{p}}} < 1$, *if* $p > 1$, *the estimation given in Theorem 4 is better than the one given in Theorem 3.*

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