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Hyers-Ulam-Rassias Stability of A Pexider Functional Equation In Non-Archimedean Spaces *

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Abstract

Recently the stability of the quadratic functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ was proved in the earlier works. In this article, we prove the generalized Hyers-Ulam stability of the pexiderial functional equation $f(x+y)+f(x-y)=2g(x)+2g(y)$ in non-Archimedean normed spaces.
The concept of Hyers-Ulam-Rassias stability originated from Th. M. Rassias stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297-300.

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1. Introduction

The stability problem of functional equations originated from a question of Ulam [24] concerning the stability of group homomorphisms. Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Th. M. Rassias [18] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1. (Th.M. Rassias) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \le p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2-2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The Hyers-Ulam stability of the quadratic functional equation was proved by Skof [23] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [3] proved the Hyers-Ulam stability of the quadratic functional equation.
The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see also [1], [5],[10]–[22]).

In 1897, Hensel [6] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have

many nice applications (see [4],[8],[9],[15]).
In this paper, we consider the following *Pexider functional equation*

$$f(x+y)+f(x-y)=2g(x)+2g(y) \tag{1.1}$$

and prove the Hyers-Ulam-Rassias stability of the functional equation (1.1) in non-Archimedean normed spaces.

2. Preliminary

Definition 2.1. By a *non-Archimedean field* we mean a field K equipped with a function(valuation) $|.|: K \to [0,\infty)$ such that for all $r,s \in K$, the following conditions hold:
(i) $|r|=0$ if and only if $r=0$
(ii) $|rs|=|r||s|$
(iii) $|r+s| \leq max\{|r|,|s|\}$.

Definition 2.2. Let X be a vector space over a scalar field K with a non-Archimedean non-trivial valuation $|.|$. A function $||.||: X \to R$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:
(i) $||x||=0$ if and only if $x=0$
(ii) $||rx||=|r|||x||$ $(r \in K, x \in X)$
(iii) The strong triangle inequality(ultrametric); namely

$$||x+y|| \leq max\{||x||,||y||\}. \quad x,y \in X$$

Then $(X,||.||)$ is called a non-Archimedean space. Due to the fact that

$$||x_n - x_m|| \leq max\{||x_{j+1}-x_j|| : m \leq j \leq n-1\} \quad (n>m)$$

Definition 2.3. A sequence $\{x_n\}$ is *Cauchy* if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. In [1], the authors investigated stability of approximate additive mappings $f:\mathbb{Q}_p \to \mathbb{R}$. They showed if $f:\mathbb{Q}_p \to \mathbb{R}$ is a continuous mapping for which there exists a fixed ε such that $||f(x+y)-f(x)-f(y)|| \leq \varepsilon$ for all $x,y \in \mathbb{Q}_p$, then there exists a unique additive mapping $T:\mathbb{Q}_p \to \mathbb{R}$ such that $|f(x)-T(x)| \leq \varepsilon$ for

all $x \in \mathbb{Q}_p$.
In this paper, we solve the stability problem for the functional equations

$$f(x+y) + f(x-y) = 2g(x) + 2g(y)$$

when the unknown functions are with values in a non-Archimedean space, in particular in the field of p-adic numbers.

3. Non-Archimedean Stability of Functional Equation (1.1)

Throughout this section, using direct method, we prove the generalized Hyers-Ulam stability of the functional equation (1.1) in non-Archimedean normed spaces.
In the rest of this paper, we assume that H is an additive semigroup and X is a complete non-Archimedean space.

Theorem 3.1. *Let $\psi : H \times H \to [0, +\infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\psi(2^n x, 2^n y)}{|4|^n} = 0 \tag{3.1}$$

for all $x, y \in H$ and let for each $x \in H$ the limit

$$\Psi(x) = \lim_{n\to\infty} max \left\{ max \left\{ \frac{\psi(2^k x, 2^k x)}{|4|^k}, \frac{|2|\psi(2^k x, 0)}{|4|^k} \right\} \;\middle|\; 0 \le k < n \right\} \tag{3.2}$$

exists. Suppose that $f, g : H \to X$ are mappings with $f(0) = g(0) = 0$ and satisfying the following inequality

$$||f(x+y) + f(x-y) - 2g(x) - 2g(y)|| \le \psi(x, y) \tag{3.3}$$

for all $x \in X$. Then there exists a mapping $T : H \to X$ such that

$$||f(x) - T(x)|| \le \frac{1}{|4|}\Psi(x) \tag{3.4}$$

and

$$||g(x) - T(x)|| \le max \left\{ \frac{1}{|4|}\Psi(x), \frac{1}{|2|}\psi(x, 0) \right\}. \tag{3.5}$$

for all $x \in X$. Moreover, if

$$\lim_{j\to\infty}\lim_{n\to\infty} max\left\{max\left\{\frac{\psi(2^k x, 2^k x)}{|4|^k}\ ,\ \frac{|2|\psi(2^k x, 0)}{|4|^k}\right\}\ ,\ j \le k < n+j\right\} = 0 \tag{3.6}$$

then T is the unique mapping satisfying (3.4) and (3.5).

Proof. Putting $y = 0$ in (3.3), we get

$$||f(x) - g(x)|| \le \frac{\psi(x, 0)}{|2|}. \tag{3.7}$$

Substituting $y = x$ in (3.3), we have

$$\left\|\frac{f(2x)}{4} - g(x)\right\| \le \frac{\psi(x, x)}{|4|}. \tag{3.8}$$

So

$$\left\|\frac{f(2x)}{4} - f(x)\right\| \le max\left\{\frac{\psi(x, 0)}{|2|}\ ,\ \frac{\psi(x, x)}{|4|}\right\} \tag{3.9}$$

for all $x \in H$. Replacing x by $2^{n-1}x$ in (3.9) and dividing both sides by 4^{n-1}, we get

$$\left\|\frac{f(2^n x)}{4^n} - \frac{f(2^{n-1}x)}{4^{n-1}}\right\| \le max\left\{\frac{\psi(2^{n-1}x, 2^{n-1}x)}{|4|^n}\ ,\frac{|2|\psi(2^{n-1}x, 0)}{|4|^n}\right\} \tag{3.10}$$

It follows from (3.1) and (3.10) that the sequence $\left\{\frac{f(2^n x)}{4^n}\right\}$ is a Cauchy sequence. Since X is complete, so $\left\{\frac{f(2^n x)}{4^n}\right\}$ is convergent. Set $T(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$. Using induction we see that

$$\left\|\frac{f(2^n x)}{4^n} - f(x)\right\| \le \frac{1}{|4|} max\left\{max\left\{\frac{\psi(2^k x, 2^k x)}{|4|^k}\ ,\frac{|2|\psi(2^k x, 0)}{|4|^k}\right\}\ \Big|\ 0 \le k < n\right\}. \tag{3.11}$$

It's clear that (3.11) holds for $n = 1$ by (3.9). Now, if (3.11) holds for every

$0 \leq k < n-1$, we obtain

$$\begin{aligned}
&\left\|\frac{f(2^n x)}{4^n} - f(x)\right\| \\
&= \left\|\frac{f(2^n x)}{4^n} \pm \frac{f(2^{n-1}x)}{4^{n-1}} - f(x)\right\| \\
&\leq max\left\{\left\|\frac{f(2^n x)}{4^n} - \frac{f(2^{n-1}x)}{4^{n-1}}\right\| \ , \ \left\|\frac{f(2^{n-1}x)}{4^{n-1}} - f(x)\right\|\right\} \\
&\leq max\Bigg\{max\left\{\frac{\psi(2^{n-1}x, 2^{n-1}x)}{|4|^n} \ , \ \frac{|2|\psi(2^{n-1}x, 0)}{|4|^n}\right\} \\
&, \frac{1}{|4|}max\left\{max\left\{\frac{\psi(2^k x, 2^k x)}{|4|^k} \ , \frac{|2|\psi(2^k x, 0)}{|4|^k}\Big| 0 \leq k < n-1\right\}\right\}\Bigg\} \\
&\leq \frac{1}{|4|}max\left\{max\left\{\frac{\psi(2^k x, 2^k x)}{|4|^k} \ , \ \frac{|2|\psi(2^k x, 0)}{|4|^k}\right\} \ \Big| \ 0 \leq k < n\right\}.
\end{aligned}$$

So for all $n \in N$ and all $x \in H$, (3.11) holds. By taking n to approach infinity in (3.11) and using (3.2) one obtains (3.4). On the other hand, by (3.7), we obtain

$$\begin{aligned}
||g(x) - T(x)|| &\leq max\{||g(x) - f(x)||, ||f(x) - T(x)||\} \\
&\leq max\left\{\frac{1}{|4|}\Psi(x), \frac{1}{|2|}\psi(x, 0)\right\},
\end{aligned}$$

If S be another mapping satisfies (3.4) and (3.5), then for all $x \in H$, we get

$$\begin{aligned}
&||T(x) - S(x)|| \\
&= \lim_{j\to\infty}\left\|\frac{T(2^j x)}{4^j} - \frac{S(2^j x)}{4^j}\right\| \\
&\leq \lim_{j\to\infty} max\left\{\left\|\frac{T(2^j x)}{4^j} - \frac{f(2^j x)}{4^j}\right\| \ , \ \left\|\frac{f(2^j x)}{4^j} - \frac{S(2^j x)}{4^j}\right\|\right\} \\
&\leq \frac{1}{|4|}\lim_{j\to\infty}\lim_{n\to\infty} max\left\{\max\left\{\frac{\psi(2^k x, 2^k x)}{|4|^k} \ , \ \frac{|2|\psi(2^k x, 0)}{|4|^k}\right\} \ \Big| \ j \leq k < n+j\right\} \\
&= 0.
\end{aligned}$$

Therefore $T = S$. This completes the proof.

Corollary 3.1. *Let $\lambda : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\lambda(|2|t) \leq \lambda(|2|)\lambda(t) \quad (t \geq 0), \quad \lambda(|2|) < |4|$$

Let $\delta > 0$, H be a normed space and let $f, g : H \to X$ are mappings with $f(0) = g(0) = 0$ and satisfying

$$||f(x+y) + f(x-y) - 2g(x) - 2g(y)|| \leq \delta(\lambda(||x||) + \lambda(||y||))$$

for all $x, y \in H$. Then there exists a unique mapping $T : H \to X$ such that

$$||f(x) - T(x)|| \leq \frac{2\delta\lambda(||x||)}{|4|}$$

and

$$||g(x) - T(x)|| \leq max\left\{\frac{2\delta\lambda(||x||)}{|4|}, \frac{\delta\lambda(||x||)}{|2|}\right\}$$

for all $x \in H$.

Proof. Defining $\psi : H^2 \to [0, \infty)$ by $\psi(x, y) := \delta(\lambda(||x||) + \lambda(||y||))$, then we have

$$\lim_{n\to\infty} \frac{\psi(2^n x, 2^n y)}{|4|^n} \leq \lim_{n\to\infty} \left[\frac{\lambda(|2|)}{|4|}\right]^n \psi(x, y) = 0 \tag{3.12}$$

for all $x, y \in H$. On the other hand

$$\Psi(x) = \lim_{n\to\infty} max\left\{max\left\{\frac{\psi(2^k x, 2^k x)}{|4|^k}, \frac{|2|\psi(2^k x, 0)}{|4|^k}\right\} \Big| 0 \leq k < n\right\}$$

$$= max\{\psi(x, x), |2|\psi(x, 0)\}$$

exists for all $x \in H$.
Also

$$\lim_{j\to\infty} \lim_{n\to\infty} max\left\{max\left\{\frac{\psi(2^k x, 2^k y)}{|4|^k}, \frac{\psi(2^k x, 0)}{|4|^k}\right\} \Big| \ j \leq k < n + j\right\} = 0. \tag{3.13}$$

Applying Theorem 3.1, we conclude desired result.

Corollary 3.2. *Let $\lambda : [0, \infty) \to [0, \infty)$ be a function satisfying*

$$\lambda(|2|t) \leq \lambda(|2|)\lambda(t) \quad (t \geq 0), \quad \lambda(|2|) < |4|$$

Let $\delta > 0$, H be a normed space and let $f, g : H \to X$ be mappings with $f(0) = g(0) = 0$ and satisfying

$$||f(x+y) + f(x-y) - 2g(x) - 2g(y)|| \leq \delta\{\lambda(||x||).\lambda(||y||)\}$$

for all $x, y \in H$. Then there exists a unique mapping $T : H \to X$ such that

$$||f(x) - T(x)|| \leq \frac{\delta(\lambda(||x||))^2}{|4|} \tag{3.14}$$

and

$$||g(x) - T(x)|| \leq \frac{\delta(\lambda(||x||))^2}{|4|} \tag{3.15}$$

for all $x \in H$.

Proof. Defining $\psi : H \times H \to [0, +\infty)$ by $\psi(x, y) := \delta\{\lambda(||x||).\lambda(||y||)\}$

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