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Growth of Harmonic Functions in R^4 *

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Abstract

The paper deals with the study of growth of harmonic functions H in R^4 using the LaPlacian type integral operator (and inverse). Moreover, we have characterized the order and type of H in terms of spherical harmonic coefficients occurring in spherical harmonic expansions which have not been studied so far. Our results apply satisfactorily for studing the time dependent problems in R^3.

Keywords and Phrases: *Spherical harmonic coefficients, Integral transform, Order and type, Analytic function associate and harmonic functions in* R^4.

1. Introduction

The theory in R^3 is well-developed. Several authors studied boundary value problems [7], value distribution and growth and approximation [14,15,17,18,20] of harmonic functions in R^3. Some times it is reasonable and become interesting when we restrict the time dependent problems in R^3, it leds to the study of harmonic functions in R^4. There are only a few integral transforms available for harmonic functions in R^4 (see[6,8,12]). In this paper we study the

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growth of harmonic functions in R^4 using the LaPlacian type integral operator (and inverse) which is an isometry between analytic function of three complex variables as an associate of harmonic function in R^4 on suitable domains of definition[19].

A function which is harmonic in a neighborhood of origin in R^4 can be expanded as a compactly convergent series

$$H(r, z, \zeta, \eta) = \sum_{k=0}^{\infty} H_k(r, z, \zeta, \eta) \tag{1.1}$$

in the sphere $S(r_0) : |X| < r_0$ whose radius is the distance from the origin to the nearest singularity. Here X be a point in R^4 whose spherical coordinates $(r, \theta, \phi, \varphi)$,are defined as [7, p.98] with r=$|X|$ and angles $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, $-2\pi \leq \varphi < 2\pi$.

The series (1.1) has an expansion in terms of complete set

$$H_k(r, z, \zeta, \eta) = \sum_{m=-k}^{m=k} \sum_{n=-k}^{n=k} W_{mn}^k a_{mn}^k H_{mn}^k(r, z, \zeta, \eta),$$

k=0,1,2,3,..., of spherical harmonic polynomials [7,p.161].It is given [23,p.123]

$$H_{mn}^k(r, z, \zeta, \eta) = r^{2k} Y_{mn}^k(z, \zeta, \eta)$$

$$Y_{mn}^k(z, \zeta, \eta) = P_{mn}^k(z) \zeta^m \eta^n,$$

$$z = cos\theta, \zeta = exp(i\phi), \eta = exp(i\varphi),$$

with (m,n,k)$\in \Omega := \{(p, q, r) : -r \leq p, q \leq r, r = 0, 1, 2, 3, ...\}. The P_{mn}^k$ correspond to Jacobi polynomials

$$P_j^{\mu,\nu}(z) = 2^m i^{n-m} (w_{mn}^k)^{-1} (1-z)^{(n-m)/2} (1+z)^{(-n-m)/2} P_{mn}^k(z)$$

$$(w_{mn}^k)^{-1} = [(k-m)!(k+m)!/(k-n)!(k+n)!]^{1/2},$$

where

$$j = k + (\mu + \nu)/2, m = (\mu + \nu)/2, \eta = (\nu - \mu)/2$$

with integer parameters $\mu = m + n$ and $\nu = m - n[1; 7, p.125]$.

Let D be a simply connected domain about the origin in R^4. A harmonic function$H \in C^2(D)$ is a solution of Laplace's equation [7, p.494]

$$\left[1/r^3\partial_r \left(r^3\partial_r + 4/r^2 \left[\partial_{\theta\theta} + ctg(\theta)\partial_\theta + 1/\sin^2\theta(\partial_{\theta\theta} - 2cos(\theta)\partial_{\phi\varphi} + \partial_{\varphi\varphi})\right]\right)\right] H(X) = 0. \tag{1.2}$$

Each harmonic function H is associated with a unique analytic function

$$h(z_1, z_2, z_3) = \sum_{k=0}^{\infty} h_k(z_1, z_2, z_3)$$

of three complex variables called the associate of H, where

$$h_k(z_1, z_2, z_3) = \sum_{m=-k}^{m=k}\sum_{n=-k}^{n=k} a^k_{mn} h^k_{mn}(z_1, z_2, z_3)$$

and

$$h^k_{mn}(z_1, z_2, z_3) = z_3^m(z_1^{k-n} z_2^{k+n})$$

for (m,n,k) $\in \Omega$. The initial domain of definition of $h \in C^3$ is

$$D_{\epsilon,r_0} : |z_1| < r_0^2/2, |z_2| < r_0^2/2, 1 - \epsilon < |z_3| < 1 + \epsilon,$$

for some sufficiently small positive ϵ.

Construction of the ascending operator mapping W from analytic function $h(z_1, z_2, z_3)$ of three complex variables onto harmonic function $H(r, z, \zeta, \eta) \in R^4$ begins with the Laplace type integral formulation [7, p.147]

$$w^k_{mn} P^k_{mn}(z) = 1/2\pi i \int_{\Gamma(s)} \tau_1^{k-n} \tau_2^{k+n} s^m ds/s. \tag{1.3}$$

The contour $\Gamma(s)$ is positively oriented upper semicircle $|s|= 1$ traversed from s=-1 to s=1 and $(m, n, k) \in \Omega$ with generating variables

$$\tau_1 = \tau_1(z, s) = \phi_+(z)s^{1/2} + i\phi(z)s^{-1/2}$$

$$\tau_2 = \tau_2(z,s) = i\phi(z)s^{1/2} + \phi_+(z)s^{-1/2}, \phi_\pm(z) = [(1 \pm (z))/2]^{1/2}$$

Using (1.3) we can express the spherical harmonics as

$$H_k = W[h_k] = 1/2\pi i \int_{\Gamma(s)} h_k((r^2/\eta)^{1/2}\tau_1, (r^2\eta)^{1/2}\tau_2, s\zeta)ds/s, k = 0,1,2,3....$$

Therefore the ascending operator $W : h \to$ H be defined as

$$H(r,z,\zeta,\eta) = W[h(z_1,z_2,z_3)]$$

$$= 1/2\pi i \int_{\Gamma(s)} h((r^2/\eta)^{1/2}\tau_1(z,s), (r^2\eta)^{1/2}\tau_2(z,s), s\zeta)ds/s.$$

The inverse operator applies orthogonality of the surface harmonics in terms of the functions $P^k_{mn}[7,p,161]$ to define the transform

$$h(z_1,z_2,z_3) = W^{-1}[H(r,z,\zeta,\eta)]$$

$$= 1/(2\pi i)^2 \int_{\Gamma(\eta)} \int_{\Gamma(\zeta)} \int_{\Gamma(z)} H(r,z,\zeta,\eta)[C(\sigma;\tau_1,\tau_2,\eta)]^* dz \frac{d\zeta}{\zeta}\frac{d\eta}{\eta},$$

where

$$[C(\sigma,\tau_1,\tau_2,\eta) = C((z_1,z_2/r^2)^*;\tau_1(z,z_3^*\zeta),\tau_2(z,z_3^*\zeta),(z_2/z_1)^*].$$

The symbol * designates the complex conjugate operator. The contour deformation method [6] produces global reprentations of H and h through reciprocal transform linking the associated harmonic and analytic function elements. These facts are summarized in

Theorem A. *For each function H that is harmonic at origin in sphere $S(r_0)$: $|X| < r_0$ there is a unique W associated function h of three complex variables analytic in the disc $D_{\epsilon,ro}$ and conversely.*

Using various techniques, the characterizations of order and type in terms of the coefficients a^k_{mn} were obtained by Fryant and Shankar [5], Srivastava [20], Temliakow and others. More direct derivations of order and type for harmonic functions in R^3 were given by Fryant [2,3,4] and Kapoor and Nautiyal [10,11].

The growth of entire harmonic functions in R^n was considered by Veselovskaya [22] where order and type were obtained for harmonic functions in terms of spherical harmonic coefficients. Recently, Kumar[12] studied the growth of entire harmonic functions in $R^n, n > 2$ and obtained various characterizations interms of spherical coefficients and approximation error's by harmonic polynomials. However, none of them have considered this method and approach for studying the growth of it in R^4. Although, the order and type of entire harmonic functions was studied in [3] by using Bergmann integral operator as a principal tool. In this paper we obtain the coefficients characterizations of order and type of harmonic function H in R^4. Our results are different from all those authors mentioned above.

2. Auxiliary Results

In this section we shall prove some auxiliary results which have been used in the sequel.

The maximum modulus of associate $h \in C^3$ is defined in complex function theory

$$M(r_1, r_2, r_3, h) = max_{(|z_i|=r_i, i=1,2,3)} h(z_1, z_2, z_3), r_1 < \frac{r_0^2}{2}, r_2 < \frac{r_0^2}{2}, 1 - \epsilon < r_3 < 1 + \epsilon.$$

The growth of a function h, analytic in D_{ϵ,r_0} as determined by its maximum modulus function $M(r_1, r_2, r_3, h)$ can be studied in several different ways. To measure the growth of h with respect to all the variables simultaneously, the concept of D_{ϵ,r_0} order and D_{ϵ,r_0} type introduced by Juneja and Kapoor[9] have been used.
Set

$$M_{D_{\epsilon,r_0}}(t, h) = max_{(r_1,r_2,r_3) \in tD_{\epsilon,r_0}} M(r_1, r_2, r_3, h), 0 < t < 1.$$

We define the D_{ϵ,r_0} - order $\rho_{D_{\epsilon,r_0}}$(h) of h as

$$\rho_{D_{\epsilon,r_0}}(h) = \limsup_{t \to 1} \frac{\log^+ \log^+ M_{D_{\epsilon,r_0}}(t, h)}{-\log(1-t)} \tag{2.1}$$

if $0 < \rho_{D_{\epsilon,r_0}}(h) < \infty$, the D_{ϵ,r_0} - type $T_{D_{\epsilon,r_0}}$ of h is defined as

$$T_{D_{\epsilon,r_0}}(h) = \limsup_{t\to 1} \frac{\log^+ M_{D_{\epsilon,r_0}}(t,h)}{(1-t)^{-\rho_{D_{\epsilon,r_0}}(h)}}. \tag{2.2}$$

We now prove

Theorem 2.1. *Let $h(z_1,z_2,z_3) = \sum_{k=0}^{\infty}\sum_{m=-k}^{m=k}\sum_{n=-k}^{n=k} a_{mn}^k h_{mn}^k(z)$ be analytic in the polydisc D_{ϵ,r_0} and have D_{ϵ,r_0} order $\rho_{D_{\epsilon,r_0}}(h)$, $0 \le \rho_{D_{\epsilon,r_0}}(h) \le \infty$. Then*

$$\frac{\rho_{D_{\epsilon,r_0}}(h)}{\rho_{D_{\epsilon,r_0}}(h)+1} = \limsup_{k_1+k_2+k_3\to\infty} \{\log^+\log^+ a_{mn}^k r_1^{k_1} r_2^{k_2} r_3^{k_3} / \log(k_1+k_2+k_3)\} \tag{2.3}$$

where $k_1 = k-n, k_2 = k+n, k_3 = m$, and left hand side is interpreted as 1 if $\rho_{D_{\epsilon,r_0}}(h) = \infty$.

Proof. Let us consider the function $\beta(z_1,z_2,z_3,w)$ of four complex variables z_1,z_2,z_3,w defined by

$$\beta(z_1,z_2,z_3,w) = h(w(z_1,z_2,z_3)) = \sum_{k=0}^{\infty}\sum_{m=-k}^{m=k}\sum_{n=-k}^{n=k} a_{mn}^k w^{2k+m} h_{mn}^k(z_1,z_2,z_3)$$

where $|w| < r_0$. Set

$$P_\lambda(z_1,z_2,z_3) = \sum_{k=0}^{\infty}\sum_{m=-k}^{m=k}\sum_{n=-k}^{n=k} a_{mn}^k h_{mn}^k(z_1,z_2,z_3), m+2k=\lambda.$$

Then

$$\beta(z_1,z_2,z_3,w) = \sum_{\lambda=0}^{\infty} P_\lambda(z_1,z_2,z_3) w^\lambda$$

is an analytic function of w in finite disc of radius r_0.In view of Cauchy inequality for the coefficients of a power series of one variable, we get, for $0 < t < 1$ and $(z_1,z_2,z_3) \in D_{\epsilon,r_0}$,

$$|P_\lambda(z_1,z_2,z_3)| \le \max_{|w|=tr_0} |\beta(z_1,z_2,z_3,w)|/t^\lambda r_1^{k_1} r_2^{k_2} r_3^{k_3}, k_1+k_2+k_3=\lambda. \quad (2.4)$$

Now

$$\begin{aligned}\max_{|w|=tr_0} |\beta(z_1,z_2,z_3,w)| &\le \max_{|w|=tr_0} \max_{|z_i|=r_i} |h(w(z_1,z_2,z_3))| \\ &= \max_{|v_i|=tr_i} |h(v_1,v_2,v_3)| \\ &\le \max_{(r_1^*,r_2^*,r_3^*)\varepsilon tD_{\epsilon,r_0}} M(r_1^*,r_2^*,r_3^*,h) = M_{D_{\epsilon,r_0}}(t,h).\end{aligned}$$

Therefore we have

$$|P_\lambda(z_1,z_2,z_3)| \le \frac{M_{D_{\epsilon,r_0}}(t,h)}{t^\lambda r_1^{k_1} r_2^{k_2} r_3^{k_3}}. \quad (2.5)$$

Since (2.5) holds for every $z_1,z_2,z_3 \in D_{\epsilon,r_0}$, we get

$$\begin{aligned}M_{D_{\epsilon,r_0}}(r_0,P_\lambda) &= \max_{(r_1,r_2,r_3)\epsilon tD_{\epsilon,r_0}} M(r_1,r_2,r_3,P_\lambda) \\ &= \max_{(r_1,r_2,r_3)\epsilon tD_{\epsilon,r_0}} \max_{|z_i|=tr_i i=1,2,3} |P_\lambda(z_1,z_2,z_3)| \\ &= \frac{M_{D_{\epsilon,r_0}}(t,h)}{t^\lambda r_1^{k_1} r_2^{k_2} r_3^{k_3}}.\end{aligned}$$

Thus,for all t, $0<t<1$, and every positive integer λ,

$$M_{D\epsilon,r_0}(r_0,P_\lambda) \le \frac{M_{D\epsilon,r_0}(t,h)}{t^\lambda r_1^{k_1} r_2^{k_2} r_3^{k_3}}. \quad (2.6)$$

Now first let$\rho(h) \equiv \rho_{D\epsilon,r_0}(h) < \infty$ by the definition of $\rho(h)$, it follows that for any $\epsilon > 0$ and for all $0<t<1$,

$$\log^+ M_{D\epsilon,r_0}(t,h) < (1-t)^{-\rho(h)-\epsilon}$$

or

$$M_{D\epsilon,r_0}(t,h) < exp\{(1-t)^{-\rho(h)-\epsilon}\}$$

using (2.6), we get

$$M_{D\epsilon,r_0}(r_0,P_\lambda) < exp\{(1-t)^{-\rho(h)-\epsilon}\}t^{-\lambda}r_1^{-k_1}r_2^{-k_2}r_3^{-k_3} \tag{2.7}$$

we now find an upper bound on the coefficients of the polynomial $P_\lambda(z_1,z_2,z_3)$ for $k_1+k_2+k_3=\lambda$ and $(r_1,r_2,r_3)\in tD_{\epsilon,r_0}$

$$a^k_{mn} \le \frac{M(r_1,r_2,r_3,P_\lambda)}{r_1^{k_1},r_2^{k_2},r_3^{k_3}}.$$

Minimizing the right hand side of this inequality for all $(r_1,r_2,r_3)\in D_{\epsilon,r_0}$, it follows that, for any m,n,k with $k_1+k_2+k_3$

$$a^k_{mn} \le M_{D\epsilon,r_0}(r_0,P_\lambda)$$

combining this inequality with(2.7), we get

$$a^k_{mn}r_1^{k_1}r_2^{k_2}r_3^{k_3} \le exp\{(1-t)^{-\rho(h)-\epsilon}\}t^{-\lambda}. \tag{2.8}$$

Minimizing the right hand side of(2.8), we have

$$a^k_{mn}r_1^{k_1}r_2^{k_2}r_3^{k_3} < exp\{(1+\rho(h)+\epsilon)(\frac{k_1+k_2+k_3}{\rho(h)+\epsilon})^{(\rho(h)+\epsilon)/(\rho(h)+1+\epsilon)}\}$$

or

$$\frac{\log^+\log^+(a^k_{mn}r_1^{k_1}r_2^{k_2}r_3^{k_3})}{\log(k_1+k_2+k_3)} \le \frac{\rho(h)+\epsilon}{\rho(h)+1+\epsilon}+o(1).$$

Proceeding to limits in above inequality, we obtain

$$\limsup_{k_1+k_2+k_3\to\infty}\{\frac{\log^+\log^+(a^k_{mn}r_1^{k_1}r_2^{k_2}r_3^{k_3})}{\log(k_1+k_2+k_3)}\} \le \frac{\rho(h)}{\rho(h)+1}. \tag{2.9}$$

If $\rho(h)=\infty$, we produced with an arbitrary large number in place of $\rho(h)+\epsilon$ and get 1 on the right hand side of(2.9).

To prove the reverse inequality, we let

$$\limsup_{k_1+k_2+k_3\to\infty}\{\frac{\log^+\log^+(a^k_{mn}r_1^{k_1}r_2^{k_2}r_3^{k_3})}{\log(k_1+k_2+k_3)}\}=\alpha.$$

For any $\epsilon > 0$, there exists a nonnegative integer $\lambda(\epsilon)$ such that, for $k_1+k_2+k_3 \geq \lambda(\epsilon)$,

$$a^k_{mn}r_1^{k_1}r_2^{k_2}r_3^{k_3} < exp\{(k_1+k_2+k_3)^{\alpha+\epsilon}\}.$$

This inequality implies for $0 < t < 1$,

$$M_{D\epsilon,r_0}(t,h) \leq max_{(r_1,r_2,r_3)\in D\epsilon,r_0}\sum_{k_1+k_2+k_3=0}^{\infty} a^k_{mn}r_1^{k_1}r_2^{k_2}r_3^{k_3}t^{(k_1+k_2+k_3)} \tag{2.10}$$

$$\begin{aligned}
&\leq \sum_{k_1+k_2+k_3\leq\lambda(\epsilon)} a^k_{mn}r_1^{k_1}r_2^{k_2}r_3^{k_3}t^{(k_1+k_2+k_3)} \\
&\quad + \sum_{k_1+k_2+k_3>\lambda(\epsilon)} t^{(k_1+k_2+k_3)}\exp(k_1+k_2+k_3)^{\alpha+\epsilon} \\
&< c_1t^{\lambda(\epsilon)}+c_2+\sum_{\lambda=0}^{\infty}t^{\lambda}(1+\lambda)^3\exp(\lambda)^{(\alpha+\epsilon)}
\end{aligned}$$

where c_1 and c_2 are constants. Now it can be shown that

$$\begin{aligned}
&F(z_1,z_2,z_3,t) \\
= &\sum_{k_1+k_2+k_3=0}^{\infty}(1+k_1+k_2+k_3)^3\exp(k_1+k_2+k_3)^{\alpha+\epsilon}z_1^{k_1}z_2^{k_2}z_3^{k_3}t^{(k_1+k_2+k_3)}
\end{aligned}$$

is analytic in finite disc D_{ϵ,r_0}. The order of F is $(\alpha+\epsilon)/(1-\alpha-\epsilon)$. Therefore, by the definition of order in three complex variables any $\epsilon^{'} > 0$ and t sufficiently close to 1,

$$M_{D\epsilon,r_0}(t,F) < exp\{(1-t)^{-((\alpha+\epsilon)/(1-\alpha-\epsilon))+\epsilon^{'}}\}.$$

By (2.10), we have

$$M_{D\epsilon,r_0}(t,h) < c_1t^{\lambda(\epsilon)}+c_2+M_{D\epsilon,r_0}(t,F)$$

so that we have t sufficiently close to 1,

$$M_{D\epsilon,r_0}(t,h) < c_1 t^{\lambda(\epsilon)} + c_2 + exp\{(1-t)^{-(\alpha+\epsilon/(1-\alpha-\epsilon))+\epsilon'}\}.$$

Since ϵ and ϵ' are arbitrary, the above inequality gives that

$$\rho(h) \leq \frac{\alpha}{1-\alpha}$$

or

$$\frac{\rho(h)}{\rho(h)+1} \leq \alpha. \tag{2.11}$$

Combining (2.9) and (2.11), the proof is completed.

Theorem 2.2. *Let* $h(z_1,z_2,z_3) = \sum_{k=0}^{\infty}\sum_{m=-k}^{k}\sum_{n=-k}^{k} a_{mn}^k h_{mn}^k(z_1,z_2,z_3)$ *by analytic in the polydisc* D_{ϵ,r_0}, *have order* $\rho(h)(0 < \rho(h) < \infty)$ *and type* $T_{D_{\epsilon,r_0}}(h)(0 \leq T_{D_{\epsilon,r_0}}(h) \leq \infty)$, *then*

$$\frac{(\rho(h)+1)^{\rho(h)+1}}{\rho(h)^{\rho(h)}} T_{D_{\epsilon,r_0}}(h) = \limsup_{k_1+k_2+k_3 \longrightarrow \infty} \{\frac{(\log^+ a_{mn}^k r_1^{k_1} r_2^{k_2} r_3^{k_3})^{\rho(h)+1}}{(k_1+k_2+k_3)^{\rho(h)}}\}. \tag{2.12}$$

Proof. Following the techniques employed in the proof of the Theorem 2.1, we get

$$a_{mn}^k r_1^{k_1} r_2^{k_2} r_3^{k_3} \leq \exp\{(T(h)+\epsilon)(1-t)^{-\rho(h)}\}t^{-\lambda}, T(h) \equiv T_{D_{\epsilon,r_0}}(h).$$

Minimizing the right hand side of above inequality, we get

$$a_{mn}^k r_1^{k_1} r_2^{k_2} r_3^{k_3} < \exp\{(\frac{T(h)+\epsilon}{\rho(h)})^{1/(\rho(h)+\epsilon)}\}(\rho(h)+1)(k_1+k_2+k_3)^{\rho(h)/(\rho(h)+1)}$$

which gives

$$\frac{(\rho(h)+1)^{\rho(h)+1}}{(\rho(h))^{\rho(h)}} T(h) \geq \limsup_{k_1+k_2+k_3 \to \infty} \{\frac{(\log^+(a_{mn}^k r_1^{k_1} r_2^{k_2} r_3^{k_3}))^{\rho(h)+1}}{(k_1+k_2+k_3)^{\rho(h)}}\}. \tag{2.13}$$

To prove the reverse inequality, let

$$\limsup_{k_1+k_2+k_3\to\infty}\{\frac{(\log^+(a^k_{mn}r_1^{k_1}r_2^{k_2}r_3^{k_3}))^{\rho(h)+1}}{(k_1+k_2+k_3)^{\rho(h)}}\}=\theta.$$

By Considering

$$\begin{aligned}&F^{'}(z_1,z_2,z_3,t)\\ =&\sum_{k_1+k_2+k_3=0}^{\infty}\exp\{(\theta+\varepsilon)^{1/(\rho(h)+1)}(k_1+k_2+k_3)^{\rho(h)/(\rho(h)+1)}\}z_1^{k_1}z_2^{k_2}z_3^{k_3}t^{(k_1+k_2+k_3)}\end{aligned}$$

in place of $F(z_1,z_2,z_3,t)$in Theorem 2.1. Now using [9, Lemma 2.2] with $D=\rho(h)/1+\rho(h)$ and $c=(\theta+\epsilon)^{1/(\rho(h)+1)}$, we have for all t sufficient close to 1,

$$\log M_{D_{\epsilon,r_0}}(t,F^{'})<\frac{(\rho(h)^{\rho(h)})}{(\rho(h)+1)^{\rho(h)+1}}(\theta+\varepsilon)+0(1)(1-t)^{-\rho(h)}.$$

Thus, we have

$$M_{D_{\epsilon,r_0}}(t,h)<c_1t^{\lambda(\epsilon)}+c_2+\exp\{(\frac{(\rho(h))^{\rho(h)}}{(\rho(h)+1)^{\rho(h)+1}}(\theta+\varepsilon)+0(1))(1-t)^{-\rho(h)}\}$$

or

$$T(h)=\limsup_{t\to1}\frac{\log^+M_{D_{\epsilon,r_0}}(t,h)}{(1-t)^{\rho(h)}}\le\frac{(\rho(h))^{\rho(h)}}{(\rho(h)+1)^{\rho(h)+1}}(\theta+\varepsilon).$$

Since ϵ is arbitrary, it follows that

$$\theta\ge\frac{(\rho(h)+1)^{\rho(h)+1}}{(\rho(h))^{\rho(h)}}=T(h).\tag{2.14}$$

On combining (2.13) and (2.14) we get the required result i.e.,(2.12).

Now if T(h) = 0, then $h(z_1,z_2,z_3)$ is of order at most $\rho(h)$ its growth $(\rho(h),0)$. Similarly if T(h) = ∞, its growth $(\rho(h),\infty)$ This completes the proof of the theorem.

3. Main Results

Theorem 3.1. *Let H is harmonic function in R^4 with W associate $h(z_1, z_2, z_3)$. Then orders and types of H and h respectively are equal.*

Proof. The nonnegativity and normalization of the measure lead directly from relation (1.4) to the bound

$$M(r, H) \leq M(r_1, r_2, r_3, h) \tag{3.1}$$

where

$$M(r, H) = \max_{\theta,\phi,\varphi} H(r, \theta, \phi, \varphi), r = |X|, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi, -2\pi < \varphi < 2\pi.$$

The inverse relation

$$h(z_1, z_2, z_3) = W^{-1}[H(r, z, \zeta, \eta)]$$

leads to the bound

$$|h(z_1, z_2, z_3)| \leq M(r, H)N(\sigma), \sigma = (\frac{(z_1, z_2)^2}{r})^*$$

and

$$N(\sigma) = \max\{C(\sigma; \tau_1, \tau_2, \eta) : |\sigma\tau_1^2/\eta| < 1, |\sigma\tau_2^2\eta| < 1\}.$$

It gives

$$M(r_1, r_2, r_3, h) =\leq M(r, H)N(\sigma).$$

For

$$z_1 = \varepsilon r e^{i\epsilon}, z_2 = \varepsilon r e^{i\epsilon},$$

we have

$$M(r_1, r_2, r_3, h) \leq M(\varepsilon^{-1} r, H)N(\varepsilon^2). \tag{3.2}$$

From the inequalities (3.1) and (3.2) with the definition of order and type of analytic function h of three complex variables we get the requisite conclusions to complete the proof of the Theorem 3.1.

Theorem 3.2. *Suppose*

$$H(r,z,\zeta,\eta)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\sum_{n=-k}^{k} w_{mn}^{k}a_{mn}^{k}H_{mn}^{k}(r,z,\zeta,\eta)$$

where

$$\limsup_{k_1+k_2+k_3\to\infty}[w_{mn}^{k}a_{mn}^{k}]^{\frac{1}{k_1+k_2+k_3}}=\frac{1}{r_0}.$$

Then the order $\rho(H)$ of H in the sphere $S(r_0):|X|<r_0$ is given

$$\frac{\rho(H)}{\rho(H)+1}=\limsup_{k_1+k_2+k_3\to\infty}\{\frac{\log^{+}\log^{+}(w_{mn}^{k}a_{mn}^{k}r_0^{k_1+k_2+k_3})}{\log(k_1+k_2+k_3)}\}$$

and the type T(H) of H in the sphere $S(r_0)$ is given by

$$T(H)=\frac{\rho(H)^{\rho(H)}}{(\rho(H)+1)^{\rho(H)+1}}\limsup_{k_1+k_2+k_3\to\infty}\{\frac{\log^{+}(w_{mn}^{k}a_{mn}^{k}r_0^{k_1+k_2+k_3})}{(k_1+k_2+k_3)^{\rho(H)}}^{\rho(H)}\}^{\rho(H)}.$$

Here k_1,k_2,k_3 are defined as earlier.

Proof. By Theorem A, the harmonic function H in R^4 is analytic if the associate $h\in C^3$ is analytic. Now applying Theorem 3.1 with Theorem 2.1 and 2.2 the proof of the theorem is completed.

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