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Selecting the Best Exponential Populations in Terms of Reliability: Empirical Bayes Approach*

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Abstract

Suppose that there are $k(k \geq 2)$ populations (equipments) $\pi_1, \cdots, \pi_k$. Of them, $\pi_i (i = 1, \cdots, k)$ is exponentially distributed with unknown parameter θ_i. Suppose R_0 is a scheduled minimum reliability (survival rate). From the qualified subsets, we intend to select which has the maximum reliability, the so-called best population. The qualified subsets refer to the standard deviation is no greater than σ_0 in the k populations, and its reliability is no less than R_0. Apparently, this study is to explore the selection issue of multiple criteria.

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Based on the Bayes framework, we adopt conjugate prior distribution. Then we employ empirical Bayes approach to solve the issue of selection. So, what we concern mainly is: from the *k* populations taking gamma distribution as its conjugate prior. Related empirical Bayes selection rule has been submitted, and its asymptotic optimality has also been proven. At the same time, a simulation study is carried out for the performance of the proposed procedure and it is found satisfactory.

Key words and phrases: *reliability, ranking and selection, loss function, Bayes risk, empirical Bayes selection rule, asymptotic optimality.*

1. Introduction

Issues on ranking and selection derive from the issues on the test of homogeneity. The so-called test of homogeneity is to test k populations (process and equipment) to see whether their means are all equal. When we abandon the assumption of homogeneity, we would not give it up for good. Instead, we want to keep exploring their pros and cons with a view to selecting an ideal target. This is so-called issues on ranking and selection.

Speaking of the methods in solving the problem related to ranking and selection, both Bechhofer (1954) and Gupta (1956) explored this problem by using indifference zone approach and subset selection approach respectively. As to the related study on ranking and selection over the past years, Gupta and Panchapakesan (1979) once had a detailed illustration on it. We tend not to explain it in detail here.

In this article, we use empirical Bayes approach to explore the selection issue. The so-called empirical Bayes approach refers to the construction of some unknown parameters included in the decision-making model under the Bayes framework. First of all, we need to use past data to estimate unknown parameters. Then we adopt current data to make a decision. The empirical Bayes approach was originally submitted by Robbins (1956, 1964). This approach was widely explored and used. For example, by utilizing this approach, Deely (1965) selected the optimal one from normal populations. Then Gupta and Hsiao (1983), Gupta and Leu (1991) also employed empirical Bayes approach to explore the issue of selection. Most of the empirical Bayes procedures are proved to have the feature of asymptotic optimality. This means that the risk of the empirical Bayes procedure will converge into the minimum Bayes risk.

Concerning the issues on ranking and selection, most literatures have simply focused on one criterion for a long time. For example, within some limited populations, if there is one population with either maximum or minimum parameter, we call this population an optimal one. However, on many occasions, this fails to satisfy experimenters' need. For example, in the field of industrial statistics, we not only demand the maximum target (e.g. life), but also hope that the variance between the quality of the products can be made possible within the range of being reasonable. Therefore, when we use one criterion to define the so-called optimal one, we often find that one criterion fails to meet our needs. Recently, Gputa, Liang and Rau (1994) discussed how to select a optimal population under certain control value in order that the related parameter θ_i (i.e. mean) is more than θ_0, and it is the maximum one among the populations. Obviously, the selection issue stated above has covered two criteria. Huang and Lai (1999, 2001) had ever discussed how to select an optimal population under two control values θ_0 and σ_0^2, so that its average θ_i is more than θ_0, and its variance σ_i^2 is less than σ_0^2. Additionally, among the populations, either θ_i is maximum or is closest to θ_0. Apparently, the above-mentioned two selection issues include two parameters and involve three criteria. Huang and Lai (2009) used process capability index to explore similar selection issues. To come to a conclusion, the above-mentioned multiple criteria employed in the issue of selection are actually getting closer to the practical need in terms of the definition of the best population. However, the aforementioned issues on selection all base their research subjects on the populations with normal distributions. Liang (2000) once had a discussion on the selection of the exponential family. However, we know that the mean and variance are correlated when we talk about the population with exponential distribution. It is not easy to find out the so-called best population with maximum mean and minimum variance. This is the motivation which inspired us to study the issue on selection. This study will adopt reliability to explore the selection issue in the populations with exponential distribution.

This paper is divided into six sections. The first section is about the study background and motivation. The second section is to introduce the question framework and related Bayes selection rule. Section three is to base on the issue raised in the previous section and to put forward related empirical Bayes selection rules. Section four is to explore the character of the proposed empirical Bayes selection rules employed in large samples and to prove its asymptotic optimality. The fifth section deals with related simulation work and has a finding that the submitted empirical Bayes selection rule in this paper takes on excellent performance. The last section is the conclusion.

2. Formulation of Problem

In this article, we are going to use reliability to access the effectiveness of equipment. The equipments which are often used to test the reliability are survival rate, mean residual life, and hazard rate so on. A detailed illustration can be found in chapter four of the book *Bayesian Reliability Analysis* written by Martz et al. (1982). Suppose random variable X stands for the life of certain equipment. Its probability density function is $f(x)$; T_0 is a time point; and the survival rate $R(T_0)$, mean residual life $M(T_0)$ and hazard rate $H(T_0)$ are sequentially defined by (2.1), (2.2) and (2.3), respectively.

$$R(T_0) = \int_{T_0}^{\infty} f(x)dx, \tag{2.1}$$

$$M(T_0) = \frac{\int_0^{\infty} xf(x+T_0)dx}{R(T_0)}, \tag{2.2}$$

$$H(T_0) = \frac{f(T_0)}{R(T_0)}. \tag{2.3}$$

According to reliability introduced by (2.1), we define the best population as follows.

Definition 2.1. Let $\pi_1, \cdots, \pi_k$ be k exponential populations such that π_i has mean θ_i, $i = 1, \cdots, k$, and let T_0 be an arbitrary, but fixed positive real value. For a time point T_0, the reliability of π_i is $R_i(T_0), i = 1, \cdots, k.$ Let σ_0 and R_0 are two control values (prefixed). Define $S = \left\{ \pi_i \,\middle|\, R_i(T_0) \geq R_0 \,,\, \theta_i \geq \sigma_0^{-1} \right\}$. A population π_i is considered as the best population, if it simultaneously satisfies the following conditions:

(i) $\pi_i \in$ S, and

(ii) $R_i(T_0) = \underset{\pi_j \in s}{Max} R_j(T_0)$.

Let $\underset{\sim}{\theta} = (\theta_1, \cdots, \theta_k)$ and $\Omega = \{\underset{\sim}{\theta} = (\theta_1, \cdots, \theta_k) \mid \theta_i \in (0, \infty), i = 1, \cdots, k \}$ be the parameter space. Let $\underset{\sim}{a} = (a_0, a_1, \cdots, a_k)$ denote an action vector such that a_i takes value 0 or 1 with $\sum_{i=0}^{k} a_i = 1$. When $a_i = 1$, for some $i = 1, \cdots, k$, it means that population π_i is selected as the best population. When $a_0 = 1$, it means that no population is considered as the best population, i.e. none in k populations satisfies both conditions (i) and (ii) in Definition 2.1. Therefore, the best population does not always exist in a given finite set of populations.

For the sake of convenience, corresponding to $\sigma_0, \forall i = 0,1,\cdots,k$, we define a new quantity $R_i'(T_0)$ as follows. For a given small enough positive number δ and for any $i = 0,1,\cdots,k$, define

$$R_i'(T_0) = R_i(T_0)\, I_{\{\theta_i \ge \sigma_0^{-1}\}} + (R_0(T_0) - \delta)\, I_{\{\theta_i < \sigma_0^{-1}\}} \tag{2.4}$$

It is clear to see that those populations which do not meet the requirement (i) will also fail to meet the requirement (ii) in selection criteria in terms of the associated quantity $R_i'(T_0)$.

In a decision-theoretic approach, we consider the following

Definition 2.2. Let σ_0 and R_0 are two control values (prefixed). For parameter vector $\underset{\sim}{\theta}$, if action $\underset{\sim}{a}$ is taken, a loss $L(\underset{\sim}{a};\underset{\sim}{\theta})$ is incurred and which is defined by

$$L(\underset{\sim}{a};\, \underset{\sim}{\theta}) = R_{[k]}'(T_0) - \sum_{i=0}^{k} a_i R_i'(T_0)\,, \tag{2.5}$$

where $R_{[k]}'(T_0) = \underset{0 \le i \le k}{Max}\, R_i'(T_0)$.

For each $i = 1,\cdots,k$, let $X_{i1},\cdots,X_{iM}$ be an independent random sample of size M from the exponential population π_i with parameter θ_i. The observed value is denoted by $x_{i1},\cdots,x_{iM}$. It is assumed that θ_i is a realization of a random variable Θ_i with a exponential prior distribution $Exp(1/\beta_i)$, (*i.e.* Gamma distribution $Gamma(1, \beta_i)$), where β_i is unknown. The random variables $\Theta_1,\cdots,\Theta_k$ are assumed to be mutually independent.

For convenience, for $i = 1,\cdots,k$, we denote $\underset{\sim}{x_i} = (x_{i1},\cdots,x_{iM})$. The following property is well-known, and can be found in the fourth chapter of 'Theory of Point Estimation' written by Lehmann (1986).

Property 2.1. Let $X_{i1},\cdots,X_{iM}$ be an independent random sample of size M from exponential population π_i with parameter θ_i , and $\underset{\sim}{x_i} = (x_{i1},\cdots,x_{iM})$ be the observed value. Let θ_i be a realization of a random variable Θ_i with exponential

distribution $Exp(1/\beta_i)$, then the posterior distribution of Θ_i given $\underset{\sim}{x}_i$ is gamma distribution $Gamma(M+1, \beta_i')$,
where

$$\beta_i' = 1/(1/\beta_i + \sum_{j=1}^{M} x_{ij}). \tag{2.6}$$

For convenience, let $\underset{\sim}{\beta} = (\beta_1, \cdots, \beta_k)$ and $\underset{\sim}{\beta}' = (\beta_1', \cdots, \beta_k')$, and let $f_i(x_i | \theta_i)$ and $\pi_i(\theta_i \mid \beta_i)$ denote the conditional probability density function of X_i and Θ_i, respectively, $i = 1, \cdots, k$. Let $f(\underset{\sim}{x} \mid \underset{\sim}{\theta}) = \prod_{i=1}^{k} f_i(x_i \mid \theta_i)$ and $\pi(\underset{\sim}{\theta} \mid \underset{\sim}{\beta}) = \prod_{i=1}^{k} \pi_i(\theta_i \mid \beta_i)$ be the joint probability density function of $\underset{\sim}{x} = (x_1, \cdots, x_k)$ and $\underset{\sim}{\Theta} = (\Theta_1, \cdots, \Theta_k)$, respectively. Let χ be the sample space generated by $\underset{\sim}{x}$. A selection rule $\underset{\sim}{d} = (d_0, d_1, \cdots, d_k)$ is a mapping defined on the sample space χ into the k+1 product space $[0,1] \times [0,1] \times \cdots \times [0,1]$ such that $\sum_{i=0}^{k} d_i(\underset{\sim}{x}) = 1$, for all $\underset{\sim}{x} \in \chi$. For every $\underset{\sim}{x} \in \chi$, $d_i(\underset{\sim}{x})$ denotes the probability of selecting population π_i as the best population, $i = 1, \cdots, k$; and $d_0(\underset{\sim}{x})$ denotes the probability that none is selected as the best population.

Under the preceding formulation, the Bayes risk of a selection rule $\underset{\sim}{d}$, denoted by $r(\underset{\sim}{d})$, is given by

$$\begin{aligned}
r(\underset{\sim}{d}) &= E_{\underset{\sim}{\theta}} E_{\underset{\sim}{x}} L(\underset{\sim}{a}; \underset{\sim}{\theta}) \\
&= \int_\Omega \int_\chi L(\underset{\sim}{a}; \underset{\sim}{\theta}) f(\underset{\sim}{x} \mid \underset{\sim}{\theta}) \pi(\underset{\sim}{\theta} \mid \underset{\sim}{\beta}) d\underset{\sim}{x} d\underset{\sim}{\theta} \\
&= \int_\chi \int_\Omega L(\underset{\sim}{a}; \underset{\sim}{\theta}) \pi(\underset{\sim}{\theta} \mid \underset{\sim}{x}, \underset{\sim}{\beta}') f(\underset{\sim}{x}) d\underset{\sim}{\theta} d\underset{\sim}{x} \\
&= \int_\chi \int_\Omega (R_{[k]}'(T_0) - \sum_{i=0}^{k} d_i(\underset{\sim}{x}) R_i'(T_0)) f(\underset{\sim}{x} \mid \underset{\sim}{\theta}) \pi(\underset{\sim}{\theta} \mid \underset{\sim}{x}, \underset{\sim}{\beta}') d\underset{\sim}{x} d\underset{\sim}{\theta} \\
&= \int_\chi \int_\Omega R_{[k]}'(T_0) \pi(\underset{\sim}{\theta} \mid \underset{\sim}{x}, \underset{\sim}{\beta}') f(\underset{\sim}{x}) d\underset{\sim}{\theta} d\underset{\sim}{x} \\
&\qquad - \int_\chi \int_\Omega \sum_{i=0}^{k} d_i(\underset{\sim}{x}) R_i'(T_0) \pi(\underset{\sim}{\theta} \mid \underset{\sim}{x}, \underset{\sim}{\beta}') f(\underset{\sim}{x}) d\underset{\sim}{\theta} d\underset{\sim}{x} \\
&= I_1 - I_2
\end{aligned}$$

Furthermore,

$$I_1 = \int_{\chi} \int_{\Omega} R'_{[k]}(T_0)\pi(\underset{\sim}{\theta} \mid \underset{\sim}{x}, \underset{\sim}{\beta}') f(\underset{\sim}{x}) d\underset{\sim}{\theta} d\underset{\sim}{x} = C,$$

for some constant C, and

$$I_2 = \int_{\chi} \sum_{i=0}^{k} d_i(\underset{\sim}{x}) \left(\int_0^{\sigma_0^{-1}} (R_0(T_0) - \delta)\pi(\theta_i \mid \underset{\sim}{x}_i, \beta_i') d\theta_i \right.$$

$$\left. + \int_{\sigma_0^{-1}}^{\infty} R_i(T_0)\pi(\theta_i \mid \underset{\sim}{x}_i, \beta_i') d\theta_i \right) f(\underset{\sim}{x}) d\underset{\sim}{\theta} d\underset{\sim}{x}$$

$$= \int_{\chi} \sum_{i=0}^{k} d_i(\underset{\sim}{x}) \phi_i(\underset{\sim}{x}_i) f(\underset{\sim}{x}) d\underset{\sim}{x},$$

where

$$\begin{aligned} \phi_i(\underset{\sim}{x}_i) &= (R_0(T_0) - \delta) G\left(\sigma_0^{-1} \middle| M+1, \beta_i'\right) \\ &\quad + (1+\beta_i' T_0)^{-(M+1)} \left(1 - G\left(\sigma_0^{-1} \middle| M+1, \beta_i'/(1+\beta_i' T_0)\right)\right), \end{aligned} \tag{2.7}$$

and β_i' is defined by (2.6), $G(c|a,b)$ is the cumulative probability of gamma distribution $Gamma(a,b)$ before c. Furthermore, $\phi_0(\underset{\sim}{x}_0) = R_0(T_0)$.
Hence,

$$r(\underset{\sim}{d}) = C - \int_{\chi} \sum_{i=0}^{k} d_i(\underset{\sim}{x}) \phi_i(\underset{\sim}{x}_i) f(\underset{\sim}{x}) d\underset{\sim}{x}. \tag{2.8}$$

Bayes rule:

For each $\underset{\sim}{x} \in \chi$, let

$$Q(\underset{\sim}{x}) = \left\{ i \,\middle|\, \phi_i(\underset{\sim}{x}_i) = \underset{0 \le j \le k}{Max}\, \phi_j(\underset{\sim}{x}_j), i = 0, 1, \cdots, k \right\}. \tag{2.9}$$

Then, define

$$i^* = i^*(\underset{\sim}{x}) = \begin{cases} 0 & , if\ Q(\underset{\sim}{x}) = \{0\}; \\ Min\{i \mid i \in Q(\underset{\sim}{x}), i \neq 0\} & , otherwise. \end{cases} \tag{2.10}$$

Then, according to (2.7), (2.9) and (2.10), it can be derived that a Bayes selection rule $\underset{\sim}{d}^B = (d_0^B, d_1^B, \cdots, d_k^B)$ is given as follows

$$\begin{cases} d_{i^*}^B(\underset{\sim}{x}) = 1, \\ d_j^B(\underset{\sim}{x}) = 0, \quad for\ j \neq i^*. \end{cases} \tag{2.11}$$

3. The Empirical Bayes Selection Rule

Since $\phi_i(\underset{\sim}{x}_i)$ still involves the unknown parameters β_i, $i=1,\cdots,k$, hence, the proposed Bayes selection rule $\underset{\sim}{d}^B$ is not applicable. However, based on the past data, these unknown parameters can be estimated and a decision can be made if one more observation is taken. For $i=1,\cdots,k$, let X_{ijt} denote a sample of size M from π_i with a exponential distribution $Exp(\theta_{it})$ at time t ($t=1,\cdots,n$), $j=1,\cdots,M$ and θ_{it} is a realization of a random vector Θ_{it} which is an independent copy of Θ_i with a exponential distribution $Exp(1/\beta_i)$. It is assumed that Θ_{it}, $i=1,\cdots,k$, $t=1,\cdots,n$, are mutually independent. For our convenience, we denote the current random sample X_{ijn+1} by X_{ij}, for $j=1,\cdots,M$, $i=1,\cdots,k$.

For each population π_i, $i=1,\cdots,k$, we estimate the unknown parameters β_i based on the past data X_{ijt}, $j=1,\cdots,M$, $t=1,\cdots,n$. In this paper, β_i is estimated with moment method. Let $Y_{it}=\sum_{j=1}^{M}X_{ijt}$. Since π_i is exponentially distributed, $Exp(\theta_{it})$, so the distribution of Y_{it} is gamma distribution $Gamma(M,1/\theta_{it})$.

Property 3.1. Let the distribution of Y_{it} is gamma distribution $Gamma(M,1/\theta_{it})$, and θ_{it} is a realization of a random variable Θ_i with exponential distribution $Exp(1/\beta_i)$, then

$$E[1/Y_{it}]=\beta_i/(M-1) \tag{3.1}$$

According to (3.1), we can easily find out that the unbiased estimator of β_i is $\hat{\beta}_{in}$ that is defined as follows.

$$\hat{\beta}_{in}=\frac{M-1}{n}\sum_{t=1}^{n}\frac{1}{Y_{it}}. \tag{3.2}$$

From (3.2), we define

$$\hat{\beta}'_{in}=1/(1/\hat{\beta}_{in}+\sum_{t=1}^{n}X_{ij}), \tag{3.3}$$

and

$$\hat{\phi}_{in}(\underset{\sim}{x}_i) = (R_0(T_0) - \delta) G\left(\sigma_0^{-1} \middle| M+1,\ \hat{\beta}'_{in}\right)$$
$$+ (1+\hat{\beta}'_{in}T_0)^{-(M+1)} \left(1 - G\left(\sigma_0^{-1} \middle| M+1,\ \hat{\beta}'_{in} / (1+\hat{\beta}'_{in}T_0)\right)\right), \qquad (3.4)$$

for all $i = 1, \cdots, k$. Note that $\hat{\phi}_{in}(\underset{\sim}{x}_0) = R_0(T_0)$.

We consider $\hat{\phi}_{in}(\underset{\sim}{x}_i)$ to be estimator of $\phi_i(\underset{\sim}{x}_i)$. Properties of these estimators previously defined will be discussed in Section 4.

Empirical Bayes rule:

For each $\underset{\sim}{x} \in \chi$, let

$$Q_n(\underset{\sim}{x}) = \left\{ i \middle| \hat{\phi}_{in}(\underset{\sim}{x}_i) = \underset{0 \le j \le k}{Max}\, \hat{\phi}_{jn}(\underset{\sim}{x}_j), i = 0, 1, \cdots, k \right\}. \qquad (3.5)$$

Again, define

$$i_n^* = i_n^*(\underset{\sim}{x}) = \begin{cases} 0 & \text{if } Q_n(\underset{\sim}{x}) = \{0\}, \\ Min\{\, i \mid i \in Q_n(\underset{\sim}{x}), i \ne 0 \,\} & \text{otherwise.} \end{cases} \qquad (3.6)$$

Then, according to (3.3), (3.4), (3.5) and (3.6), we propose an empirical Bayes selection rule $\underset{\sim}{d}^{*n} = (d_0^{*n}, d_1^{*n}, \cdots, d_k^{*n})$ as follows

$$\begin{cases} d_{i_n^*}^{*n}(\underset{\sim}{x}) = 1, \\ d_j^{*n}(\underset{\sim}{x}) = 0, \quad \text{for } j \ne i_n^*. \end{cases} \qquad (3.7)$$

4. Some large Sample Properties

In this section, we study the asymptotic optimality of the proposed empirical Bayes selection rule. Before we proceed, we discuss the consistency of the estimators defined in (3.2)~(3.4) for $M \ge 2$. Lai (2004) has showed that $\hat{\beta}_{in}$, $\hat{\beta}'_{in}$ and $\phi_{in}(\underset{\sim}{x}_i)$ defined by (3.2), (3.3) and (3.4) are consistent estimators of β_i, β'_i and $\phi_i(\underset{\sim}{x}_i)$ respectively, $i = 1, \cdots, k$.

Considering an empirical Bayes selection rule $\underset{\sim}{d}^n = (d_0^n, d_1^n, \cdots, d_k^n)$ and denoting its Bayes risk by $r(\underset{\sim}{d}^n)$, from (2.8), (2.11) and (3.6), we have

$$r(\underset{\sim}{d}^n) - r(\underset{\sim}{d}^B) = \int_\chi \sum_{i=0}^{k} [d_i^B(\underset{\sim}{x}) - d_i^n(\underset{\sim}{x})] \phi_i(\underset{\sim}{x}_i) f(\underset{\sim}{x}) d\underset{\sim}{x}$$

$$= \int_{\chi} \sum_{i=0}^{k} \sum_{j=0}^{k} I_{\{i^*=i,\, i_n=j\}} [\phi_j(\underset{\sim}{x}_j) - \phi_i(\underset{\sim}{x}_i)] f(\underset{\sim}{x}) d\underset{\sim}{x} \qquad (4.1)$$

Since $r(\underset{\sim}{d}^B)$ is the minimum Bayes risk, it implies $r(\underset{\sim}{d}^n) - r(\underset{\sim}{d}^B) \geq 0$. Thus $\mathrm{E}_n[r(\underset{\sim}{d}^n)] - r(\underset{\sim}{d}^B) \geq 0$, where the expectation E_n is taken with respect to the past observations X_{ijt}, $i = 1, \cdots, k$, $j = 1, \cdots, M$, and $t = 1, \cdots, n$, The non-negative difference $\mathrm{E}_n[r(\underset{\sim}{d}^n)] - r(\underset{\sim}{d}^B)$ can be used to measure the performance of the selection rule $\underset{\sim}{d}^n$.

Definition 4.1. A sequence of empirical Bayes selection rule $\{\underset{\sim}{d}^n\}_{n=1}^{\infty}$ is said to be asymptotically optimal, if $\lim_{n\to\infty} [\mathrm{E}_n[r(\underset{\sim}{d}^n)] - r(\underset{\sim}{d}^B)] = 0$.

According to Lemma 4.2, checking conditions (C) and (D) of Robbins (1964) and applying the Dominated Convergence Theorem, we can conclude the following

Theorem 4.1. *The empirical Bayes selection rule* $\underset{\sim}{d}^{*n}(\underset{\sim}{x})$*, defined by (3.3)~(3.7), is asymptotically optimal.*

5. Simulation Study

In order to investigate the performance of proposed empirical Bayes selection rule $\underset{\sim}{d}^{*n}(\underset{\sim}{x})$ defined in Section 3, we carried out a simulation study which is summarized in this section. The quantity $E_n[r(\underset{\sim}{d}^{*n})] - r(\underset{\sim}{d}^B)$, mentioned in Definition 4.1, is used as a measure of performance of the empirical Bayes selection rule $\underset{\sim}{d}^{*n}(\underset{\sim}{x})$.

For a given current observation $\underset{\sim}{x}$ and given past observation x_{ijt}, let

$$\begin{aligned} D_n(\underset{\sim}{x}) &= \sum_{i=0}^{k} [\, d_i^{*n}(\underset{\sim}{x}) - d_i^B(\underset{\sim}{x}) \,] \phi_i(\underset{\sim}{x}_i) \\ &= \phi_{i^*}(\underset{\sim}{x}_{i^*}) - \phi_{i_n^*}(\underset{\sim}{x}_{i_n^*}). \end{aligned}$$

Then

$$E_n[r(\underset{\sim}{d}^{*n})] - r(\underset{\sim}{d}^B) = E[E_n[D_n(\underset{\sim}{X})]].$$

Therefore, the sample mean of $D_n(\underset{\sim}{x})$ based on the observations of $\underset{\sim}{x}$ and x_{ijt}, $i = 1, \cdots, k,\ j = i, \cdots, M, t = 1, \cdots, n,$ can be used as an estimator of $E_n[r(\underset{\sim}{d}^{*n})] - r(\underset{\sim}{d}^B)$.

We briefly explain the simulation scheme as follows:

(1) For each time, $t=1,\cdots,n,$ and each population $\pi_i, i=1,\cdots,k,$ generate observations $x_{i1t},\cdots,x_{iMt},$ by the following way.

a. Take a value θ_{it} according to distribution $Exp(1/\beta_i)$, i.e. $Gamma(1, \beta_i)$.

b. For given θ_{it}, generate random samples $x_{i1t},\cdots,x_{iMt},$ according to distribution $Exp(\theta_{it})$.

(2) Based on the observations x_{ijt}, $i=1,\cdots,k, j=i,\cdots,M, t=1,\cdots,n,$ estimate the known parameters β_i according to (3.2) and they are denoted by $\hat{\beta}_{in}$, respectively.

(3) For each population $\pi_i, i=1,\cdots,k,$ repeat step (1) with $t = n+1$ and take its sample mean as our current sample x_{ij}, $j=1,\cdots,M$. Thus the current sample vector is given by $\underset{\sim}{x}=(x_1,\cdots,x_k)$.

(4) For given values of T_0, δ and two control values $R_0(T_0)$ and σ_0, based on the current sample vector, determine the Bayes selection rule $\underset{\sim}{d}^B$ and the empirical Bayes selection rule $\underset{\sim}{d}^{*n}$ according to (2.11) and (3.6), respectively. Then compute $D_n(\underset{\sim}{x})$.

(5) Repeat step (1) through step (4) ten thousand times, and then take its average denoted by $\overline{D}_n$ which is used as an estimate of $E[r(\underset{\sim}{d}^{*n})]-r(\underset{\sim}{d}^B)$. In addition, $SE(\overline{D}_n)$, the estimated standard error and $n\overline{D}_n$ are computed. Simultaneously, f_n, the relative frequency of correct selection are also computed.

To Table 1, we take k = 4, $\beta_1 =$ 0.015, $\beta_2 = 0.0175$, $\beta_3 = 0.020$, $\beta_4 =$ 0.0225 M=5, $\sigma_0 = 65$, $\delta = 0.0001, T_0 = 10$, and $R_0(T_0) = 0.8$. The relative frequency that the proposed empirical Bayes selection rule coincides with that of the Bayes selection rule is computed and denoted by f_n. It can be seen from Table 1 that values of $\overline{D}_n$ decrease rapidly as n increases, and the values of f_n increase rapidly as n increases. The performance of the proposed empirical Bayes rule behaves satisfactorily.

Table 1. Behavior of empirical Bayes rules based on different sample sizes

n	f_n	$\bar{D}_n$	$n\bar{D}_n$	$SE(\bar{D}_n)$
20	0.94367	9.6331×10^{-5}	1.9266×10^{-3}	6.8233×10^{-7}
40	0.96159	3.8256×10^{-5}	1.5303×10^{-3}	1.1126×10^{-7}
60	0.96913	2.2526×10^{-5}	1.3515×10^{-3}	4.6693×10^{-8}
80	0.97475	1.5157×10^{-5}	1.2126×10^{-3}	2.2701×10^{-8}
100	0.97542	1.2607×10^{-5}	1.2607×10^{-3}	1.7115×10^{-8}
200	0.98341	5.4795×10^{-6}	1.0959×10^{-3}	4.3202×10^{-9}
300	0.98610	3.7953×10^{-6}	1.1386×10^{-3}	2.4125×10^{-9}
400	0.98822	2.9404×10^{-6}	1.1762×10^{-3}	1.6953×10^{-9}
500	0.98981	2.2427×10^{-6}	1.1213×10^{-3}	1.1927×10^{-9}
600	0.99047	1.9927×10^{-6}	1.1956×10^{-3}	9.6635×10^{-10}
700	0.99136	1.6752×10^{-6}	1.1726×10^{-3}	7.6688×10^{-10}
800	0.99111	1.4497×10^{-6}	1.1597×10^{-3}	5.8219×10^{-10}
900	0.99239	1.1995×10^{-6}	1.0795×10^{-3}	4.7058×10^{-10}
1000	0.99280	1.0105×10^{-6}	1.0105×10^{-3}	3.4554×10^{-10}

6. Conclusion

Dealing with the issue on selection and ranking, there are two viewpoints worthy of being mentioned in this paper. First of all, the mean and variance derived from the exponential population are correlated. It is an interesting issue that how to define the so-called best population in order that its variance is not too big to be expected and its mean is large enough. In this paper, we put forward the reliability to define the best population, which takes on the character of being reasonable and practical. For sure, those decision makers may modify the definition of the best population based on the actual problems they are facing. Moreover, applying the reliability to the selection issue of other distribution population is an issue which is worth spending time on.

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