Tamsui Oxford Journal of Information and Mathematical Sciences **28(3)** (2012) 303-326
Aletheia University

# Some Types of Separation Axioms in Topological Spaces *

Alias Barakat Khalaf†

*Department of Mathematics, College of Science,*
*Sulaimani University, Duhok, Kurdistan Region, Iraq*

Halgwrd Mohammed Darwesh‡

*Department of Mathematics, Kinki University,*
*Sulaimani, Kurdistan Region, Iraq*

and

K. Kannan§

*Department of Mathematics, Srinivasa Ramanujan Centre,*
*SASTRA University, Kumbakonam - 612001 Tamil Nadu, India*

Received February 23, 2011, Accepted March 13, 2012.

**Abstract**

In this paper, we introduce some types of separation axioms via $\omega$-open sets, namely $\omega$-regular, completely $\omega$-regular and $\omega$-normal space and investigate their fundamental properties, relationships and characterizations. The well-known Urysohn's Lemma and Tietze Extension Theorem are generalized to $\omega$-normal spaces. We improve some known results. Also, some other concepts are generalized and studied via $\omega$-open sets.

---

*2000 *Mathematics Subject Classification*. Primary 54D10, 54D15.

†E-mail: khalaf59@yahoo.com
‡E-mail: darweshyMath@yahoo.com
§Corresponding author. E-mail: anbukkannan@rediffmail.com

**Keywords and Phrases:** *$\omega$-open set, $\omega$-regular, Completely $\omega$-regular, $\omega$-normal.*

# 1. Introduction

Throughout this work, a space will always mean a topological space, $(X, \Im)$ and $(Y, \sigma)$ will denote spaces on which no separation axioms are assumed unless explicitly stated. The notations $T_{dis}$, $T_{ind}$ denote the discrete and indiscrete topologies and $\wp$ denotes the usual topology for the set of all real numbers $R$. For a subset $A$ of a space $(X, \Im)$, the closure and the interior of $A$ will be denoted by $Cl_X A$ and $Int_X A$ (or simply $ClA$ and $IntA$), respectively. A point $x \in X$ is called a condensation point of $A$ [13, pp. 90] if for each $G \in \Im$ with $x \in G$, the set $G \cap A$ is uncountable. A is called $\omega$-closed [8] if it contains all its condensation points. The complement of an $\omega$-closed set is called $\omega$-open. It is well known that a subset $U$ of a space $(X, \Im)$ is $\omega$-open if and only if for each $x \in U$, there exists $G \in \Im$ such that $x \in G$ and $G - U$ is countable. The family of all $\omega$-open subsets of a space $(X, \Im)$ is denoted by $\Im^\omega$, forms a topology on $X$ finer than $\Im$. The $\omega$-closure and $\omega$-interior, which are defined in the same way as $ClA$ and $IntA$, and they are denoted by $\omega ClA$ and $\omega IntA$, respectively. Several characterizations of $\omega$-closed subsets were provided in [3, 4, and 5]. A subset $A$ of a space $X$ is called $\omega$-dense [2] if $\omega ClA = X$. Authors in General topology used the notation of $\omega$-open sets to define some other types of sets, mappings and spaces, till Al-Hawary [1] and Rao and et al [6] used the nation of $\omega$-open sets in fuzzy and bitopological spaces, respectively. So we recall the following results and notions:

**Theorem 1.1.** [4] *If $U$ is $\omega$-open subset of $X$, then $U - C$ is $\omega$-open for every countable subset $C$ of $X$.*

**Theorem 1.2.** [4 and 5] *For any space $(X, \Im)$ and any subset $A$ of $X$,*

1. $\Im^{\omega\omega} = (\Im^\omega)^\omega = \Im^\omega$.

2. $(\Im_A)^\omega = (\Im^\omega)_A$.

**Definition 1.3.** [5] A space $(X, \Im)$ is said to be locally-countable if each point of $X$ has a countable open neighborhood.

It is easy to see that

**Theorem 1.4.** *Let* $(X, \Im)$ *be a space. Then* $\Im^{\omega} = T_{dis}$ *if and only if the space* $(X, \Im)$ *is locally countable.*

**Definition 1.5.** [5] A space $X$ is said to be anti-locally countable if each non-empty open subset of $X$ is uncountable.

Note that a space $(X, \Im)$ is anti-locally-countable if and only if $(X, \Im^{\omega})$ is so.

**Lemma 1.6.** [5] *If a space* $(X, \Im)$ *is anti-locally-countable, then*

1. $\omega ClA = ClA$*, for every* $\omega$*-open subset* $A$ *of* $X$*.*
2. $\omega IntA = IntA$*, for every* $\omega$*-closed subset* $A$ *of* $X$*.*

Al-Zoubi in [4] has improved part (1) of the above result, by proving the following lemma:

**Lemma 1.7.** [4] *If* $(A, \Im_A)$ *is an anti-locally countable subspace of a space* $(X, \Im)$*, then* $\omega ClA = ClA$*.*

**Definition 1.8.** [2] A space $X$ is said to be an $\omega$-space if every $\omega$-open set is open.

**Definition 1.9.** A function : $(X, \Im) \rightarrow (Y, \rho)$ is called

1. $\omega$-continuous [8] if $f^{-1}(U)$ is $\omega$-open in $X$, for each open subset $U$ of $Y$,
2. $\omega$-irresolute [2] if $f^{-1}(U)$ is $\omega$-open in X, for each $\omega$-open subset $U$ of $Y$,
3. almost $\omega$-continuous [10] if for each $x \in X$, and each open subset $V$ of $Y$ containing $f(x)$, there exists an $\omega$-open subset $U$ of $X$ that containing $x$ such that $f(U) \subseteq Int_Y Cl_Y V$,
4. almost $\omega$-continuous [2] if for each $x \in X$, and each open subset $V$ of $Y$ containing $f(x)$, there exists an $\omega$-open subset $U$ of $X$ containing $x$ such that $f(U) \subseteq \omega Int_Y Cl_Y V$,
5. almost weakly-$\omega$-continuous [2] if for each $x \in X$, and each open subset $V$ of $Y$ containing $f(x)$, there exists an $\omega$-open subset $U$ of $X$ that containing $x$ such that $f(U) \subseteq Cl_Y V$,

6. pre-$\omega$-open [2] if image of every $\omega$-open set is $\omega$-open.

We use the almost $\omega$-continuous mapping in the sense of Nour for a mapping that satisfies part (3) and almost-$\omega$-continuous in the sense of Omari and Noorani for mappings that satisfy part (4) of Definition 1.9. Simply, they are same if $Y$ is an anti-locally countable space and it is clear from the fact that $IntA \subseteq \omega IntA$, the almost $\omega$-continuity in sense of Nour implies the almost-$\omega$-continuity in the sense of Omari and Noorani. But the converse is not true in general. For example, the mapping $f : (R, \wp^{\omega}) \rightarrow (Y, \rho)$ defined by $f(x) = 1$ if $x \in Q$ and $f(x) = 3$ if $x \in Irr$ is almost-$\omega$-continuity in the sense of Omari and Noorani, but not in the sense of Nour, where $\rho = \{\phi, \{1\}, \{2\}, \{1, 2\}, X\}$ and $Q$ and $Irr$ denote the set of all rational and irrational numbers.

**Theorem 1.10.** [4] *Let $f : (X, \Im) \rightarrow (Y, \rho)$ be a mapping from an anti-locally countable space $(X, \Im)$ onto a regular space $(Y, \rho)$. Then the following are equivalent:*

1. *$f$ is continuous,*
2. *$f$ is $\omega$-continuous,*
3. *$f$ is almost $\omega$-continuous mapping in the sense of Nour,*
4. *$f$ is almost-$\omega$-continuous in the sense of Omari and Noorani,*
5. *$f$ is almost weakly $\omega$-continuous.*

**Theorem 1.11.** [9] *Let $A \subseteq X$ and $f : (X, \Im) \rightarrow (Y, \rho)$ be an $\omega$-continuous mapping. Then $f_A : (A, \Im_A) \rightarrow (Y, \rho)$ is $\omega$-continuous.*

**Lemma 1.12.** [4] *The open image of an $\omega$-open set is $\omega$-open.*

## 2. More Properties of $\omega$-open Sets and Some Other Results

It is easy to see that:

**Theorem 2.1.** *Let $(A, \Im_A)$ be any subspace of a space $(X, \Im)$. Then for any $B \subseteq A$, we have:*

1. $\omega Cl_A B = (\omega Cl_X B) \cap A$,

2. $\omega Int_X B = \omega Int_A B \cap \omega Int_X A$,

3. $\omega b_A(B) \subseteq (\omega b_X(B)) \cap A$,

4. $\omega b_A(B) = \omega Cl_X B \cap \omega Cl_X(A - B) \cap A$.

Note that the following example shows that the particular case of part (iii) of [8, Theorem 3.1] is not true. It also shows that the general case of [8, Corollary 3.2] is not true:

**Example 2.2.** Let $X$ be an uncountable set equipped with the topology $\Im = \{\phi, A, B, X\}$, where $A$ and $B$ are uncountable disjoint subsets of $X$ such that $X = A \cup B$. Then $X$ is a hereditary lindelöf space and it is easy to see that a subset $G$ of $X$ is $\omega$-open if and only if $G = X - C$, $G = A - C$ or $G = B - C$, for some countable subset $C$ of $X$. Hence a subset $F$ of $X$ is $\omega$-closed if and only if $G = C$, $G = A \cup C$ or $G = B \cup C$, for some countable subset $C$ of $X$. But there is no $\omega$-open subset of $X$ which is a $G_\delta$-set, except for the open sets $\phi, A, B$ and $X$. Al-Zoubi [4] proved that the conditions that $X$ is anti-locally-countable and $Y$ is regular are essential in Theorem 1.2.16. But we can improve his result by dropping the condition that $f$ is surjection. For this, we need to prove the following lemma:

**Lemma 2.3.** *Let $(X, \Im)$ be an anti-locally countable space and let $A$ be a subset of $X$. If for a point $x \in A$, there exists an open subset $G$ of $X$ which contains $x$ and $G - A$ countable, then $ClG \subseteq ClA$.*

**Proof.** Let $x \in A$ and $G$ be an open set in $X$ such that $x \in G$ and $G - A$ is countable. Suppose that $y \in ClG - ClA$, then there exists an open set $V$ containing $y$ such that $V \cap A = \phi$. Since $y \in ClG$, $\phi \neq V \cap G \subseteq G - A$. This is a contradiction.

**Remark 2.4.** *The converse inclusion of Lemma 2.3 is not true in general. As a simple example in the usual space $(R, \wp^\omega)$, taking $A = Irr$, since $\sqrt{2} \in A$, $\sqrt{2} \in (1,2) \in \wp$ and $(1,2) - A$ is countable. But $R = ClA \nsubseteq [1,2] = Cl(1,2)$.*

As an immediate consequence of Lemma 2.3, we have the following corollary:

**Corollary 2.5.** *Let $(X, \Im)$ be an anti-locally countable space and $A$ be an $\omega$-open subset of $X$. Then for each point $x \in A$, there exists an open subset $G$ of $X$ containing $x$ such that $ClG \subseteq ClA$.*

The following theorem is an improvement version of Theorem 1.10.

**Theorem 2.6.** *Let $f$ be a mapping from an anti-locally countable space $(X, \Im)$ into a regular space $(Y, \rho)$. Then the following statements are equivalent:*

1. *$f$ is continuous,*
2. *$f$ is $\omega$-continuous,*
3. *$f$ is almost $\omega$-continuous in the sense of Nour,*
4. *$f$ is almost $\omega$-continuous in the sense of Omari and Noorani,*
5. *$f$ is almost weakly $\omega$-continuous.*

**Proof.** In general the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) follows from their definitions and the facts that $\Im \subseteq \Im^\omega$ and $\rho \subseteq \rho^\omega$, see [4]. To show the implication (5) $\Rightarrow$ (1), let $x \in X$ and $V$ be any open subset of $Y$ with $f(x) \in V$. By regularity of $Y$, we can choose two open sets $V_1$ and $V_2$ in $Y$ such that $f(x) \in V_1 \subseteq ClV_1 \subseteq V_2 \subseteq ClV_2 \subseteq V$. Since $f$ is almost weakly $\omega$-continuous, there exists an $\omega$-open subset $U$ in $X$ containing $x$ such that $f(U) \subseteq Cl_Y V_1$. Consequently, $U \subseteq f^{-1}(Cl_Y V_1)$. Since $x \in U \in \Im^\omega$, there exists an open set $G$ in $X$ with $x \in G$ and $G - U$ is countable. So by Lemma 1.6 and Lemma 2.3, we have $Cl_X G \subseteq Cl_X U = \omega Cl_X U$. Hence $G \subseteq \omega Cl_X U \subseteq \omega Cl_X(f^{-1}(Cl_Y V_1)) \subseteq (\omega Cl_X f^{-1}(V_2))$. Now, we have to show that $\omega Cl_X f^{-1}(V_2) \subseteq f^{-1}(Cl_Y V_2)$. Let $u \in \omega Cl_X f^{-1}(V_2)$. Suppose that $u \notin f^{-1}(Cl_Y V_2)$. Then $f(u) \notin Cl_Y V_2$. This implies that there exists an open set $W$ in $Y$ containing $f(u)$ such that $W \cap V_2 = \phi$. Hence $(Cl_Y W) \cap V_2 = \phi$. Since $f(u) \in W \in \rho$, by hypothesis there exists an $\omega$-open subset $H$ in $X$ containing $u$ such that $f(H) \subseteq Cl_Y W$. Since $u \in \omega Cl_X f^{-1}(V_2)$, $H \cap f^{-1}(V_2) \neq \phi$, and hence $f(H) \cap V_2 \neq \phi$. This implies that $Cl_Y W \cap V_2 \neq \phi$ which is impossible. Thus, $\omega Cl_X f^{-1}(V_2) \subseteq f^{-1}(Cl_Y V_2)$. Hence $G \subseteq f^{-1}(Cl_Y V_2)$. Therefore, $f(G) \subseteq Cl_Y V_2$. Hence $f$ is continuous.

In a similar way as continuity, it is easy to prove the following results:

**Theorem 2.7.** *Every constant mapping from $(X, \Im)$ into $(R, \wp)$ is $\omega$-continuous. Moreover, if $f$ and $g$ from $(X, \Im)$ into $(R, \wp)$ are $\omega$-continuous mappings, then the following statements are true:*

1. *$f \pm g$, $fg$, $| f |$, $min\{f, g\}$ and $max\{f, g\}$ are $\omega$-continuous mappings,*
2. *If $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is $\omega$-continuous.*

**Theorem 2.8.** *Let $f_n : (X, \Im) \to (R, \wp)$ be $\omega$-continuous mappings for all $n \in N$. If $f : (X, \Im) \to (R, \wp)$ is a mapping such that the series $\sum_{n=0}^{\infty} f_n(x)$ is uniformly convergent to $f(x)$, then $f$ is an $\omega$-continuous mapping.*

**Definition 2.9.** A subset $A$ of a space $(X, \Im)$ is said to be an $\omega$-zero-set of $X$ if there exists an $\omega$-continuous mapping $f : (X, \Im) \to (R, \wp)$ such that $A = \{x \in X; f(x) = 0\}$ and a subset is called co$\omega$-zero-set if it is the complement of an $\omega$-zero-set. Furthermore, if $f : (X, \Im) \to (R, \wp)$ is an $\omega$-continuous mapping, then the set $\omega Z(f) = \{x \in X; f(x) = 0\}$ is called the $\omega$-zero-set of $f$.

**Remark 2.10.**
1. *Every $\omega$-zero-set of a space is $\omega$-closed and hence every co$\omega$-zero-set is an $\omega$-open set,*
2. *Every zero-set of any space is an $\omega$-zero-set.*

The following examples show that the converse of neither parts of Remark 2.10 is true:

**Example 2.11.** Consider an $\omega$-closed subset $Q$ of the space $(R, \wp)$. We have to show the set $Q$ is not $\omega$-zero-set. Suppose that $Q$ is an $\omega$-zero-set. Then there exists an $\omega$-continuous mapping $f : R \to R$ such that $\{x \in R; f(x) = 0\}$ $= Q$ . Therefore, $f(x) = 0$ if and only if $x \in Q$. Since $f$ is $\omega$-continuous, $f$ is a continuous mapping {by Theorem 2.6}. Hence $Q$ is a zero-set. Consequently, $Q$ is a closed subset of $R$, which is a contradiction.

**Example 2.12.** Let $f : (X, \Im) \to (R, \wp)$ be a mapping defined by $f(a) = 0$ and $f(b) = 1 = f(c)$, where the $X = \{a, b, c\}$ and $\Im = \{\phi, X, \{a\}\}$. Then $f$ is $\omega$-continuous, but not a continuous function. Hence the set $\{a\}$ is an $\omega$-zero-set which is not zero-set.

**Lemma 2.13.** *If $A$ is an $\omega$-zero-set of a space $X$, then there exists an $\omega$-continuous mapping $f : X \to R$ such that $f \geq 0$ and $A = \omega Z(f)$.*

**Proof.** Since $A = \omega Z(g)$ for some $\omega$-continuous mapping $g : X \to R$, by Theorem 2.7, the mapping $f = | g | \geq 0$ is $\omega$-continuous and $A = \omega Z(f)$.

**Lemma 2.14.** *The intersection and union of any finite number of $\omega$-zero-sets is also an $\omega$-zero-set. If $\omega Z(f)$ and $\omega Z(g)$ are $\omega$-zero-sets of $f$ and $g$, then $\omega Z(f) \cup \omega Z(g) = \omega Z(fg)$, $\omega Z(f) \cap \omega Z(g) = \omega Z(h)$, where $h = f + g$.*

**Proof.** By Theorem 2.7, it follows that both $fg$ and $h = f + g$ are $\omega$-continuous. Therefore, $\omega Z(f) \cup \omega Z(g) = \omega Z(fg)$, $\omega Z(f) \cap \omega Z(g) = \omega Z(h)$ are $\omega$-zero-sets.

**Lemma 2.15.** *If $\alpha \in R$ and $f : X \to R$ is an $\omega$-continuous mapping, then the set $A = \{x \in X; f(x) \geq \alpha\}$ as well as $B = \{x \in X; f(x) \leq \alpha\}$ are $\omega$-zero-sets, and hence the sets $\{x \in X; f(x) < \alpha\}$ and $\{x \in X; f(x) > \alpha\}$ are co$\omega$-zero-sets.*

**Proof.** By using Theorem 2.7, it is easy to see that $A = \omega Z(min\{f(x) - \alpha, 0\})$ and $B = \omega Z(max\{f(x) - \alpha, 0\})$ are $\omega$-zero-sets.

**Lemma 2.16.** *If $A$ and $B$ are disjoint $\omega$-zero-sets the space $X$, then there exist disjoint co$\omega$-zero-sets $U$ and $V$ containing $A$ and $B$, respectively.*

**Proof.** Let $A = \omega Z(f)$ and $B = \omega Z(g)$. Then the mapping $h : X \to R$ given by $h(x) = \frac{f(x)}{f(x)+g(x)}$ is well-defined and in view of Theorem 2.7 it is $\omega$-continuous, $h(A) = \{0\}$ and $h(B) = \{1\}$. Then by Lemma 2.15, the sets $\{x \in X; h(x) > \frac{1}{2}\}$ and $\{x \in X; h(x) < \frac{1}{4}\}$ are the required co$\omega$-zero (hence $\omega$-open) sets.

**Corollary 2.17.** *If $X$ is anti-locally countable, then every $\omega$-zero-set of $X$ is a zero-set.*

**Proof.** It follows from Theorem 2.6.

Now, we recall the following known definition.

**Definition 2.18.** [8] A space $(X, \Im)$ is said to be $\omega$-$T_1$ (resp. $\omega$-$T_2$) if for each pair of distinct points $x$ and $y$ of $X$, there exist $\omega$-open sets $U$ and $V$ containing $x$ and $y$, respectively such that $y \notin U$ and $x \notin V$ (resp. $U \cap V = \phi$ ).

Since each countable subset of any space is $\omega$-closed, it is easy to see that each space is an $\omega$-$T_1$ space. Therefore, the results ( Theorem 3.12 and Corollary 3.13 of [2]) are trivial and they do not need to $Y$ satisfy any separating axiom. Note that every $T_2$-space is an $\omega$-$T_2$-space but not conversely.

**Theorem 2.19.** *Let $X$ be an anti-locally countable space. Then $X$ is an $\omega$-$T_2$ space if and only if $X$ is a $T_2$-space.*

**Proof.** Let $X$ be an anti-locally countable space. It is enough to prove that $X$ is a $T_2$-space if $X$ is an $\omega$-$T_2$-space. For this, let $x \neq y$ in an $\omega$-$T_2$-space $X$. Then it is easy to see that, there is an $\omega$-open set $U$ containing $x$ such that $y \notin \omega ClU$. Since $U$ is an $\omega$-open set, there exists an open set $G$ containing $x$ such that $G - U$ is countable. In virtue of Lemma 1.6 and Corollary 2.5, we have $\omega ClU = ClU$ and $ClG \subseteq ClU$. Thus $G$, $X - ClG$ are disjoint open sets in $X$ containing $x$ and $y$, respectively. Hence $X$ is a $T_2$-space.

## 3. $\omega$-Regular and Completely $\omega$-Regular Space

**Definition 3.1.** A space $(X, \Im)$ is called an $\omega$-regular space, if for each $\omega$-closed subset $H$ of $X$ and a point $x$ in $X$ such that $x \notin H$, there exist disjoint $\omega$-open sets $U$ and $V$ containing $x$ and $H$, respectively.

That is, a space $(X, \Im)$ is $\omega$-regular if and only if the space $(X, \Im^\omega)$ is regular. Now, we have the following results:

**Theorem 3.2.** *A space $X$ is $\omega$-regular if and only if for each point $x$ in $X$ and each $\omega$-open set $G$ containing $x$, there exists an $\omega$-open set $U$ such that $x \in U \subseteq \omega ClU \subseteq G$.*

**Proposition 3.3.** *Every locally-countable space is an $\omega$-regular space.*

The following example shows that the converse of Proposition 3.3 is not true in general.

**Example 3.4.** Consider the closed ordinal space $X = [0, \Omega]$, where $\Omega$ is the first uncountable ordinal and the subspace $[0, \Omega)$ of $X$ (see [12, Example 43, p. 68]). Since $X$ is an $\omega$-space and regular space, it is $\omega$-regular. Since any $\omega$-open set which contains $\Omega$ is uncountable, $X$ is not locally-countable.

**Theorem 3.5.** *If each point of a space* $(X, \Im)$ *contained in some* $\omega$*-open subset* $G$ *such that* $\omega ClG$ *is an* $\omega$*-regular subspace of* $X$*, then* $(X, \Im)$ *is* $\omega$*-regular.*

**Proof.** Let $x \in G \in \Im^{\omega}$. Then by hypothesis, there exists an $\omega$-open set $V$ containing $x$ such that $(H, \Im_H)$ is an $\omega$-regular subspace of $X$, where $H = \omega ClV$. Since $x \in G \cap H \in \Im_H^{\omega}$, by $\omega$-regularity of $(H, \Im_H)$, there exists an $\omega$-open subset $U$ of $H$ such that $x \in U \subseteq \omega Cl_H U \subseteq G \cap H \subseteq G$. Since $H$ is $\omega$-closed and $x \in V \subseteq H$, $x \in \omega Int_X H$. Thus by Theorem 2.1, we have $x \in \omega Int_X U \subseteq \omega Cl_X(U) = \omega Cl_X(U) \cap H = \omega Cl_H U \subseteq G$. Hence $X$ is an $\omega$-regular space.

**Theorem 3.6.** *If a space* $X$ *is an anti-locally countable* $\omega$*-regular space, then* $X$ *is a regular and* $\omega$*-space.*

**Proof.** Let $G$ be any open set in $X$ and let $x$ be a point in $X$ such that $x \in G$. Then by Theorem 3.2, there exists an $\omega$-open set $U$ in $X$ such that $x \in U \subseteq \omega ClU \subseteq G$. Since $x \in U$, there exists an open set $V$ such that $x \in V$ and $V - U$ is countable. Hence by Lemma 1.6 and Corollary 2.5, we have $ClU = \omega ClU$ and $ClV \subseteq ClU$. Thus, $x \in V \subseteq ClV \subseteq G$. Therefore, $X$ is regular.

If $G$ is an arbitrary $\omega$-open set in $X$ and $x$ is any point of $G$, then by the above argument, we can prove that $G$ is an open set. This implies that $X$ is an $\omega$-space.

The following result gives the relationship between $\omega$-regular and an $\omega$-$T_2$-space:

**Proposition 3.7.** *Every* $\omega$*-regular space is an* $\omega$*-*$T_2$ *space.*

**Proof.** Obvious.

The following examples show that the converse of Proposition 3.7 is not true in general.

**Example 3.8.** Consider the Smirnov's Deleted Sequence Topology [12, Example 64, pp. 88] $\eta$ on the set of all real number $R$, which is defined as: if $A = \{\frac{1}{n}; n \in N\}$ , then $\eta = \{U \subseteq R; U = G - B, G \in \wp$ and $B \subseteq A\}$. Since this topology is finer than the usual topology $\wp$, $(R, \eta)$ is a $T_2$-space. Hence, $(R, \eta)$ is an $\omega$-$T_2$ space. Since $(R, \eta)$ is a non regular anti locally-countable space, $(R, \eta)$ is not $\omega$-regular {by Theorem 3.6}.

The following proposition gives a partial converse of Proposition 3.7 and another relationship between regularity and $\omega$-regularity:

**Proposition 3.9.** *1. Every $\omega$-compact $\omega$-$T_2$ space is an $\omega$-regular space,*

*2. Every $\omega$-compact $T_2$ space is both regular and $\omega$-regular space.*

**Proof.** Straightforward.

The following theorem shows that the property of $\omega$-regularity is a hereditary property:

**Theorem 3.10.** *Every subspace of an $\omega$-regular space is also $\omega$-regular.*

**Proof.** Obvious.

**Definition 3.11.** A space $(X, \Im)$ is said to be a completely $\omega$-regular space if for every $\omega$-closed subset $F$ of $X$ and every point $x \in X - F$, there exists an $\omega$-continuous mapping $f : (X, \Im) \to (I, \wp_I)$ (simply, $f : X \to I$), such that $f(x) = \{0\}$ and $f(F) = \{1\}$.

That is, a space $(X, \Im)$ is completely $\omega$-regular if and only if $(X, \Im^\omega)$ is completely regular.

Now, it is easy to show the following results:

**Theorem 3.12.** *A space $(X, \Im)$ is a completely $\omega$-regular space if and only if for every $\omega$-open subset $G$ of $X$ and every point $x \in G$, there exists an $\omega$-continuous mapping $f : X \to I$ such that $f(x) = 0$ and $f(y) = 1$ for all $y \notin G$.*

**Proof.** Obvious.

**Proposition 3.13.** *Every completely $\omega$-regular space is an $\omega$-regular space.*

**Proof.** Obvious.

**Proposition 3.14.** *Every locally-countable space is completely $\omega$-regular.*

**Proof.** Obvious.

The converse of the Proposition 3.14 is not true; see Example 3.4.

**Question:** Is the converse of Proposition 3.13 true?

**Theorem 3.15.** *A space $(X, \Im)$ is completely $\omega$-regular if and only if the collection of all co$\omega$-zero-sets of $X$ form a base for $\Im^\omega$.*

**Proof.** Let $V$ be any $\omega$-open set in a completely $\omega$-regular space $X$ and let $v \in V$. Then by Theorem 3.12, there exists an $\omega$-continuous mapping $g : X \to I$ such that $g(v) = 1$ and $g(X - V) = \{0\}$. Set $U = \{x \in X; g(x) \geq \frac{2}{3}\}$ and $G = \{x \in X; g(x) > \frac{2}{3}\}$. By Lemma 2.15, $U$ is an $\omega$-zero set and $G$ is a $co\omega$-zero set such that $x \in G \subseteq U \subseteq V$.

Conversely, suppose that the condition of theorem holds. Let $a \in X$ and $H$ be an $\omega$-closed set in $X$ such that $a \notin H$. Then by hypothesis, there exists an $\omega$-zero set, say $\omega Z(h)$ such that $a \in X - \omega Z(h) \subseteq X - H$, where $h : X \to I$ is an $\omega$-continuous mapping. Hence we have $h(a) = t > 0$. We define $f : X \to I$ by putting $f(x) = \min \{1, \frac{|h(x)|}{t}\}$. Then by Theorem 2.7, $f$ is an $\omega$-continuous mapping. Consequently, we have $f(a) = 1$ and $x \in \omega Z(h)$ for each $x \in H$. Therefore, $f(x) = 0$ for each $x \in H$. Hence $X$ is completely $\omega$-regular.

The following examples show that completely regularity and completely $\omega$-regularity are independent topological concepts:

**Example 3.16.** Let $X = \{a, b, c\}$ and $\Im = \{\phi, X, \{a\}\}$. Then by Proposition 3.14, $(X, \Im)$ is a completely $\omega$-regular, but not completely regular because it is not regular.

**Example 3.17.** The usual space $(R, \wp)$ is completely regular but not a completely $\omega$-regular space because it is not $\omega$-regular.

The following theorem shows that the property of $\omega$-regularity is a hereditary property:

**Theorem 3.18.** *Every subspace of a completely $\omega$-regular space is also a completely $\omega$-regular space.*

**Proof.** Let $(X, \Im)$ be a completely $\omega$-regular space and let $(Y, \Im_Y)$ be a subspace of $(X, \Im)$. Suppose that $A$ is any $\omega$-closed set in $Y$ and $y$ is a point of $Y$ such that $y \notin A$. Since $A$ is an $\omega$-closed subset of $Y$, by Theorem 1.2, there exists an $\omega$-closed subset $H$ of $X$ such that $A = H \cap Y$. Since $y \in Y$ and $y \notin A$, $y \notin H$. By completely $\omega$-regularity of $X$, there exists an $\omega$-continuous mapping $f : X \to I$ such that $f(y) = \{0\}$ and $f(A) = \{1\}$. Hence by Theorem 1.11, the restriction mapping $f_Y : Y \to I$ is an $\omega$-continuous mapping and $f_Y(y) = 0 = f(y)$. Since $A \subseteq H$ and $A \subseteq Y$, $f_Y(A) \subseteq f(H) = \{1\}$. Thus $f_Y(A) = \{1\}$.

The following theorem gives a relationship between completely $\omega$-regularity and completely regularity:

**Theorem 3.19.** *If $X$ is an anti-locally countable and completely $\omega$-regular space, then $X$ is completely regular and $X$ is an $\omega$-space.*

**Proof.** Let $X$ be an anti-locally-countable completely $\omega$-regular space. Let $H$ be a closed set in $X$ and suppose that $x$ is an arbitrary point in $X$ such that $x \notin H$. Since every closed set is $\omega$-closed and by completely $\omega$-regularity of a space $X$, there exists an $\omega$-continuous mapping $f : X \to I$ such that $f(y) = \{0\}$ and $f(H) = \{1\}$. Since $X$ is anti locally-countable and $I$ is a regular space, by Theorem 2.6, $f : X \to I$ is a continuous mapping. Hence $X$ is completely regular. Since $X$ is completely $\omega$-regular, by Theorem 2.6, it is $\omega$-regular and then by Theorem 3.6, it is $\omega$-space.

# 4. $\omega$-Normal Space

**Definition 4.1.** A space $X$ is called an $\omega$-normal space if for each pair of disjoint $\omega$-closed sets $A$ and $B$ in $X$, there exist disjoint $\omega$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

That is, a space $(X, \Im)$ is an $\omega$-normal space if and only if $(X, \Im^{\omega})$ is a normal space

It is not difficult to prove the following characterizations of an $\omega$-normal space:

**Theorem 4.2.** *A space $X$ is an $\omega$-normal space if for each pair of $\omega$-open sets $U$ and $V$ in $X$ such that $X = U \cup V$, there exist $\omega$-closed sets $A$ and $B$ which are contained in $U$ and $V$, respectively and $X = A \cup B$.*

**Theorem 4.3.** *If $X$ is any space, then the following statements are equivalent:*

1. *The space $X$ is $\omega$-normal,*
2. *For each $\omega$-closed set $A$ in $X$ and each $\omega$-open set $G$ in $X$ containing $A$, there is an $\omega$-open set $U$ such that $A \subseteq U \subseteq \omega ClU \subseteq G$.*
3. *For each $\omega$-closed set $A$ and each $\omega$-open set $G$ containing $A$, there exist $\omega$-open sets $\{U_n, n \in N\}$ such that $A \subseteq \cup\{U_n; n \in N\}$ and $\omega ClU_n \subseteq G$ for each $n \in N$.*

Now, we can establish the following Urysohn's type lemma of $\omega$-normality which is an important characterization of the $\omega$-normal space:

**Theorem 4.4.** *Let $X$ be any space. Then the following statements are equivalent:*

1. *$X$ is an $\omega$-normal space,*
2. *For each $\omega$-closed subset $A$ and $\omega$-open subset $B$ of $X$ such that $A \subseteq B$, there exists an $\omega$-continuous mapping $f : X \to I$ such that $f(A) = \{0\}$ and $f(X - B) = \{1\}$,*
3. *For each pair of disjoint $\omega$-closed subsets $F$ and $H$ of $X$, there exists an $\omega$-continuous mapping $f : X \to I$ such that $f(F) = \{0\}$ and $f(H) = \{1\}$.*

**Proof.** $(1) \Rightarrow (2)$ : Suppose that $B$ is an $\omega$-open subset of an $\omega$-normal space $X$ containing an $\omega$-closed subset $A$ of $X$. Then by Theorem 4.3, there exists an $\omega$-open set which we denote by $U_{\frac{1}{2}}$ such that $A \subseteq U_{\frac{1}{2}} \subseteq \omega ClU_{\frac{1}{2}} \subseteq B$. Then $U_{\frac{1}{2}}$ and $B$ are $\omega$-open subsets of $X$ containing the $\omega$-closed sets $A$ and $\omega ClU_{\frac{1}{2}}$, respectively. In the same way, there exist $\omega$-open sets, say $U_{\frac{1}{4}}$ and $U_{\frac{3}{4}}$ such that $A \subseteq U_{\frac{1}{4}} \subseteq \omega ClU_{\frac{1}{4}} \subseteq U_{\frac{1}{2}}$ and $\omega ClU_{\frac{1}{2}} \subseteq U_{\frac{3}{4}} \subseteq \omega ClU_{\frac{3}{4}} \subseteq B$. Continuing in this process, for each rational number in the open interval (0, 1) of the form $t = \frac{m}{2^n}$, where $n$ = 1, 2, ... and $m$ = 1, 2, ... , $2^{n-1}$, we obtain $\omega$-open sets of the form $U_t$ such that for each $s < t$ then $A \subseteq U_s \subseteq \omega ClU_s \subseteq U_t \subseteq \omega ClU_t$. We denote the set of all such rational numbers by $\Psi$, and define $f : X \to I$ as follows:

$$f(x) = \begin{cases} 1 \text{ if } x \in X - B \\ \inf \{t; t \in \Psi \text{ and } x \in U_t \ \} \end{cases}$$

$f(X - B) = \{1\}$ and if $x \in A$, then $x \in U_t$ for all $t \in \Psi$. Therefore, by the definition of $f$, we have $f(x) = inf\Psi = 0$. Hence $f(B) = \{0\}$ and $f(x) \in I$ for all $x \in X$. It remains only to show that $f$ is an $\omega$-continuous mapping since the intervals of the form [0,a) and (b,1], where $a, b \in (0, 1)$ form an open subbase of the space $I$. If $x \in U_t$ for some $t < a$, then $f(x) = \inf \{s; s \in \psi \text{ and } x \in U_s\}$ $= r \leq t < a$. Thus $0 \leq f(x) < a$. If $f(x) = 0$, then $x \in U_t$ for all $t \in \Psi$. Hence $x \in U_t$ for some $t < a$. If $0 < f(x) < a$, by definition of $f$, we have $f(x) = \{s; s \in \Psi \text{ and } x \in U_s\} < a$ {Since $a < 1$}. Thus $f(x) = t$ for some $t < a$, and hence $x \in U_t$ for some $t < a$. Therefore, we conclude that $0 \leq f(x) < a$ if and only if $x \in U_t$ for some $t < a$. Hence $f^{-1}([0, a)) = \cup\{U_t; t \in \Psi \text{ and } x \in U_t\}$

which is an $\omega$-open subset of $X$. Also it is easy to assert that: $0 \leq f(x) \leq b$ if and only if $x \in U_t$ for all $t > b$. Let $x \in X$ such that $0 \leq f(x) \leq b$. It is evident that $f(x) < t$ for all $t > b$ which implies that $x \in U_t$ for all $t > b$. For the converse, let $x \in U_t$ for all $t > b$. Then $f(x) \leq t$ for all $t > b$. Thus $f(x) \leq b$ and it is clear from the definition of $f$, that $f(x) \geq 0$. This proves our assertion. Since for all $t > b$, there is $r \in \Psi$ such that $t > r > b$. Then $\omega ClU_r \subseteq U_t$. Consequently we have $\cap\{U_t; t \in \Psi \text{ and } t > b\} = \cap\{\omega ClU_r; r \in \Psi$ and $r > b\}$. Therefore, $f^{-1}([0,b]) = \{x; 0 \leq f(x) \leq b\} = \cap\{U_t; t \in \Psi$ and $t > b\} = \cap\{\omega ClU_r; r \in \Psi$ and $r > b\}$. Since $f^{-1}((0,1]) = f^{-1}(I - [0,b]) = X - f^{-1}([0,b]) = \cup\{X - \omega ClU_r; r \in Psi$ and $r > b\}$ which is $\omega$-open, and this completes the proof of this part.

(2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (1) : Let $A$ and $B$ be two disjoint $\omega$-closed subsets of $X$. Then by hypothesis, there exists an $\omega$-continuous mapping $f : X \to I$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Then the disjoint open sets $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ in $I$ containing $f(A)$ and $f(B)$, respectively. The $\omega$-continuity of $f$ gives that $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ are disjoint $\omega$-open sets in $X$ containing $A$ and $B$, respectively. It completes the proof.

**Corollary 4.5.** *Every $\omega$-normal space is completely $\omega$-regular and hence it is $\omega$-regular.*

**Question:** Is the converse of corollary 4.5 true?

Recalling that the space $(X, \Im^{\omega})$ is lindelöf if and only if $(X, \Im)$ is lindelöf [9].

**Theorem 4.6.** *Every $\omega$-regular lindelöf space $X$ is $\omega$-normal.*

**Proof.** Let $F$ be any $\omega$-closed and $U$ be any $\omega$-open subset of an $\omega$-regular lindelöf space $X$ such that $F \subseteq U$. Then by Theorem 3.2, for each $x \in F$, there exists an $\omega$-open set $V_x$ such that $x \in V_x \subseteq \omega ClV_x \subseteq U$. Since $F$ is $\omega$-closed, by part (i) of [8, Theorem 3.3], $F$ is also a lindelöf subspace. Therefore, the cover $\{V_x; x \in F\}$ of $F$ has a countable subcover, say $\{V_n; n \in N\}$. Thus $F \subseteq \{V_n; n \in N\}$ and $\omega ClV_n \subseteq U$ for each $n \in N$. Hence by Theorem 4.3, $X$ is $\omega$-normal.

In virtue of Theorem 2.8, Theorem 4.4 and the fact that every bounded closed intervals of $R$ are homeomorphic, we can generalize the Tietze Extension Theorem to $\omega$-normality which is also an important characterization of $\omega$-normal space.

**Theorem 4.7.** *A space $X$ is $\omega$-normal if and only if every $\omega$-continuous mapping $g$ on an $\omega$-closed subset of $X$ into any closed interval $[a, b]$ has an $\omega$-continuous extension $f$ over $X$ into $[a, b]$.*

The following result contains the relationship between $\omega$-normal and an $\omega$-zero-set:

**Proposition 4.8.** *Let $X$ be a space. Then*

1. *An $\omega$-zero-set of $X$ is $\omega$-closed and it is the intersection of many countable $\omega$-open sets,*
2. *Let $H$ be an $\omega$-closed subset of $X$ which is the intersection of many countable $\omega$-open sets. If $X$ is $\omega$-normal, then $H$ is an $\omega$-zero-set.*

**Proof.** (1) Let $F$ be an $\omega$-zero-set of a space $X$. Then by Remark 2.10, $F$ is an $\omega$-closed subset of $X$. Then by Lemma 2.13, there exists an $\omega$-continuous mapping $f : X \to R$ such that $f \geq 0$ and $F = \omega Z(f)$. Hence $F = \cap\{U_n; n \in Z^+\}$, where $U_n = \{x \in X; f(x) < \frac{1}{n}\}$.

(2) Let $H$ be an $\omega$-closed subset of an $\omega$-normal space $X$ such that $H = \cap\{U_n; n \in Z^+\}$, where $U_n$ is an $\omega$-open set for each $n \in Z^+$. Since $H \subseteq U_n$ for each $n \in Z^+$ and $X$ is an $\omega$-normal space, for each $n \in Z^+$, there exists an $\omega$-continuous mapping $f_n : X \to \left[0, \frac{1}{3^n}\right]$ such that $f_n(H) = \{0\}$ and $f_n(X - U_n) = \left\{\frac{1}{3^n}\right\}$ {by Theorem 4.4}. Since $\sum\limits_{n=0}^{\infty} f_n(x) \leq \sum\limits_{n=0}^{\infty} \frac{1}{3^n}$ and the series $\sum\limits_{n=0}^{\infty} \frac{1}{3^n}$ is absolutely convergent, the mapping $f : X \to R$ given by $f(x) = \sum\limits_{n=0}^{\infty} f_n(x)$ for each $x \in X$ is an $\omega$-continuous mapping and $H = \omega Z(f)$ {by Theorem 2.8}.

**Corollary 4.9.** *Every locally-countable space is $\omega$-normal.*

The converse of Corollary 4.9 is not true; see Example 3.4 and [12, Example 43, p. 68]. Now, since the usual topological space $(R, \wp)$ is not $\omega$-regular and by Corollary 4.5, $(R, \wp)$ is not $\omega$-normal. However, it is a normal space. This means that the $\omega$-normality is not implied by normality. The following example shows that the $\omega$-normality does not imply normality and hence this means that the $\omega$-normality and normality are independent topological concepts.

**Example 4.10.** Let $X = \{a, b, c\}$ and $\Im = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then by Corollary 4.9, the space $(X, \Im)$ is $\omega$-normal, but not a normal space because there are no disjoint open sets containing the disjoint closed sets $\{b\}$ and $\{c\}$, respectively.

The following theorem shows a relationship between $\omega$-normality and normality:

**Theorem 4.11.** *Let $X$ be an anti-locally countable $\omega$-normal space. Then $X$ is a normal space and it is an $\omega$-space.*

**Proof.** Let $F$ and $H$ be two disjoint closed subsets of an anti-locally countable $\omega$-normal space $X$. Then there exist $\omega$-open sets $U$ and $V$ such that $F \subseteq U$, $H \subseteq V$ and $U \cap V = \phi$. This implies that $\omega ClU \cap V = \phi$ and $U \cap \omega ClV = \phi$. Since $X$ is anti-locally-countable, we get $ClU \cap V = \phi$ and $U \cap ClV = \phi$ {by Lemma 1.6}. Since $IntClU \subseteq ClU$ and $IntClV \subseteq ClV$, $IntClU \cap V = \phi$, $U \cap IntClV = \phi$. This implies that $IntClU \cap ClV = \phi$ and $ClU \cap IntClV = \phi$. Thus $IntClU \cap IntClV = \phi$. Hence $IntClU$ and $IntClV$ are disjoint open sets in $X$ containing $F$ and $H$, respectively. This implies that $X$ is a normal space. By Corollary 4.5 and Theorem 3.6, it follows that $X$ is an $\omega$-space.

Recalling that a space $X$ is said to be a separable space [7, Definition 8.7.1, p. 175], if it contains a countable dense subset. Then it is easy to obtain the following results:

**Proposition 4.12.** *There is no anti-locally countable separable space which is $\omega$-regular (and hence $\omega$-normal).*

**Proof.** Obvious.

Note that Proposition 4.12, gives another way to proving that $(R, \wp)$ is neither $\omega$-regular nor $\omega$-normal. However, the space $(R, \wp)$ is an anti-locally-countable regular separable space.

**Proposition 4.13.** *There is no uncountable space $(X, \Im)$ for which $(X, \Im^{\omega})$ is separable.*

**Proof.** Obvious.

**Theorem 4.14.** *Every $\omega$-closed subspace of an $\omega$-normal space is also an $\omega$-normal space.*

**Proof.** Obvious.

We recall that a topological space is said to be completely normal [11, Definition 1.4.1, p. 27] if every subspace of the space is normal. The following example shows that the property that being an $\omega$-normal of a space is not hereditary.

**Example 4.15.** Consider the Tychonoff Plank space $X = [0, \Omega] \times [0, \Omega_0]$ [12, Example 86, p. 106] and [7, Example 3.4, p. 145], where $\Omega$ and $\Omega_0$ denoted the first uncountable and first infinite countable ordinals. Since both ordinal spaces $[0, \Omega]$ and $[0, \Omega_0]$, are $\omega$-spaces, $X = [0, \Omega] \times [0, \Omega_0]$ is also an $\omega$-space. Since this space is normal, it is $\omega$-normal. Since this space is not a completely normal space, it is not hereditary normal. Hence it is not hereditary $\omega$-normal.

# 5. Some Covering and Characterizations of $\omega$-Normal Space

We begin this section with the following definition:

**Definition 5.1.** The family $\{A_\lambda; \lambda \in \Lambda\}$ of subsets of a space $(X, \Im)$ is called:

1. $\omega$-locally-finite if for each $x \in X$, there exists an $\omega$-open set $G$ containing $x$ such that the set $\{\lambda \in \Lambda; G \cap A_\lambda \neq \phi\}$ is finite,

2. $\omega$-discrete if for each point $x \in X$, there is an $\omega$-open set $G$ containing $x$ such that the set $\{\lambda \in \Lambda; G \cap A_\lambda \neq \phi\}$ has at most one member.

**Proposition 5.2.** *Every locally-finite family of subsets of any space $(X, \Im)$ is $\omega$-locally-finite.*

**Proof.** It follows from the fact that $\Im \subseteq \Im^\omega$.

The following example shows that the converse implication of Proposition 5.2 is not true in general.

**Example 5.3.** Consider the set $X = N$ equipped with the indiscrete topology $T_ind$. Then the family $\{\{n\}; n \in X\}$ is an $\omega$-discrete (and hence $\omega$-locally-finite) but not locally-finite (and hence not discrete).

The following example shows that the arbitrary union of $\omega$-closed sets need not be $\omega$-closed. That is, the union of $\omega$-closure of sets does not equal to the $\omega$-closure of their union as well as it show that the unions of closed sets need not be closed.

**Example 5.4.** Consider the usual topological space $(R, \wp)$. Then $\{\{x\}; x \in (0,1)\}$ is a family of $\omega$-closed subsets of $R$. Thus $\cup\{\omega Cl\{x\}; x \in (0,1)\} = \cup\{\{x\}; x \in (0,1)\} =$ (0,1) which is not $\omega$-closed. But $\omega Cl(0,1) =$ [0,1].

**Proposition 5.5.** *If $\{A_\lambda; \lambda \in \Lambda\}$ is an $\omega$-locally-finite family of subsets of a space $X$, then $\{\omega Cl A_\lambda; \lambda \in \Lambda\}$ is also $\omega$-locally-finite and $\omega Cl(\cup\{A_\lambda; \lambda \in \Lambda\})$ $= \cup\{\omega Cl A_\lambda; \lambda \in \Lambda\}$.*

**Proof.** Let $x \in X$. Since $\{A_\lambda; \lambda \in \Lambda\}$ is $\omega$-locally-finite, there exists an $\omega$-open set $G$ containing $x$ such that the set $\{\lambda \in \Lambda; G \cap A_\lambda \neq \phi\}$ is finite. Since $G \cap A_\lambda = \phi$ if and only if $G \cap \omega Cl A_\lambda = \phi$, $\{\lambda \in \Lambda; G \cap \omega Cl A_\lambda \neq \phi\}$ is finite. $\{\omega Cl A_\lambda; \lambda \in \Lambda\}$ is also $\omega$-locally-finite. Since $\cup\{\omega Cl A_\lambda; \lambda \in \Lambda\} \subseteq \omega Cl(\cup\{A_\lambda; \lambda \in \Lambda\})$, we have only to prove that $\omega Cl(\cup\{A_\lambda; \lambda \in \Lambda\}) \subseteq \cup\{\omega Cl A_\lambda; \lambda \in \Lambda\}$. Let $x \notin \cup\{\omega Cl A_\lambda; \lambda \in \Lambda\}$. Since $\{\omega Cl A_\lambda; \lambda \in \Lambda\}$ is $\omega$-locally-finite, there exists an $\omega$-open set $U$ containing $x$ such that $\Lambda_0 = \{\lambda \in \Lambda; U \cap \omega Cl A_\lambda \neq \phi\}$ is finite. Set $V = U \cap (\cup\{X - \omega Cl A_\lambda; \lambda \in \Lambda_0\}$ is an $\omega$-open subsets of $X$ containing $x$ such that $V \cap (\cup\{A_\lambda; \lambda \in \Lambda\}) = \cup\{V \cap A_\lambda; \lambda \in \Lambda\} = \phi$. Thus $x \notin \omega Cl(\cup\{A_\lambda; \lambda \in \Lambda\})$. This completes the proof.

**Corollary 5.6.** *If $\{A_\lambda; \lambda \in \Lambda\}$ is a locally-finite family of subsets of a space $X$, then $\{\omega Cl A_\lambda; \lambda \in \Lambda\}$ is also locally-finite and $\omega Cl(\cup\{A_\lambda; \lambda \in \Lambda\}) = \cup\{\omega Cl A_\lambda; \lambda \in \Lambda\}$.*

**Proof.** It follows from Proposition 5.2 and Proposition 5.5.

It is easy to see that for any subset $A$ and any $\omega$-open subset $G$ of any space $X$, $G \cap A = \phi$ if and only if $G \cap \omega Cl A = \phi$. But the following example shows that with this fact, the $\omega$-locally-finiteness of the $\omega$-closure of a family does not imply the $\omega$-locally-finiteness of the family. Also, it shows that the locally-finiteness of the closure of a family does not imply the locally-finiteness of the family.

**Example 5.7.** Consider the set $X = R$ equipped with the topology $\rho = \{\phi, X, Irr\}$. Then the family $\{(p, p+1); p \in Z\}$ is neither $\omega$-locally-finite nor locally-finite. It is easy to see that each non-empty $\omega$-open set $U$ of $X$ contains a set of the form $X - C$ or $Irr - C$, where $C$ is a countable subset of $X$. Hence $U \cap (p, p+1) \neq \phi$, for each $p \in Z$. Thus $\{\omega Cl(p, p+1); p \in Z\} = \{X\}$ which is locally-finite (and hence $\omega$- locally-finite).

**Proposition 5.8.** *Let $\{A_\lambda; \lambda \in \Lambda\}$ be a family of subsets of a space $X$ and $\{B_\gamma; \gamma \in \Gamma\}$ be an $\omega$-locally-finite $\omega$-closed cover of $X$ such that for each $\gamma \in \Gamma$, the set $\{\lambda \in \Lambda; B_\gamma \cap A_\lambda \neq \phi\}$ is finite. Then there exists an $\omega$-locally-finite family $\{G_\lambda; \lambda \in \Lambda\}$ of $\omega$-open sets of $X$ such that $A_\lambda \subseteq G_\lambda$ for each $\lambda \in \Lambda$.*

**Proof.** For each $\lambda$, let $G_\lambda = X - (\cup\{B_\gamma; B_\gamma \cap A_\lambda = \phi\})$. So it is easy to see that $A_\lambda \subseteq G_\lambda$ and by Proposition 5.6, $G_\lambda$ is $\omega$-open for each $\lambda$. Let $x \in X$. Since $\{B_\gamma; \gamma \in \Gamma\}$ is $\omega$-locally-finite, there is an $\omega$-open set $U$ containing $x$ such that the set $\Gamma_0 = \{\gamma \in \Gamma; U \cap B_\gamma \neq \phi\}$ is finite. Thus $U \cap B_\gamma = \phi$ for each $\gamma \notin \Gamma_0$. Therefore, $U \subseteq \cup\{B_\gamma; \gamma \in \Gamma_0\}$. Also, since for each $\gamma \in \Gamma_0$, $G_\lambda \cap B_\gamma = \phi$ if and only if $A_\lambda \cap B_\gamma = \phi$, the finiteness of $\{\lambda \in \Lambda; B_\gamma \cap A_\lambda \neq \phi\}$ implies the finiteness of $\{\lambda \in \Lambda; U \cap G_\lambda \neq \phi\}$ and this completes the proof.

**Definition 5.9.** An $\omega$-open covering $\{U_\lambda; \lambda \in \Lambda\}$ of a space $X$ is said to be $\omega$-shrinkable if there exists an $\omega$-open covering $\{V_\lambda; \lambda \in \Lambda\}$ of $X$ such that $\omega ClV_\lambda \subseteq U_\lambda$ for each $\lambda \in \Lambda$.

**Theorem 5.10.** *Let $X$ be a space. Then the following statements are equivalent:*

1. *$X$ is $\omega$-normal,*
2. *Each point-finite $\omega$-open covering of $X$ is $\omega$-shrinkable,*
3. *Each finite $\omega$-open covering of $X$ has a locally-finite $\omega$-closed refinement,*
4. *Each finite $\omega$-open covering of $X$ has an $\omega$-locally-finite $\omega$-closed refinement.*

**Proof.** (1) $\Rightarrow$ (2) : Let $\{U_\lambda; \lambda \in \Lambda\}$ be a point-finite $\omega$-open covering of an $\omega$-normal space $X$. Assume that $\Lambda$ is well-ordered. We shall construct the $\omega$-shrinking to $\{U_\lambda; \lambda \in \Lambda\}$ by the transfinite induction. For this; let $\mu$ be an element of $\Lambda$ and suppose that for each $\lambda < \mu$, we have an $\omega$-open set $V_\lambda$ such that $\omega ClV_\lambda \subseteq U_\lambda$ and for each $\upsilon < \mu$, $[\cup\{V_\lambda; \lambda < \upsilon\}] \cup [\cup\{U_\lambda; \lambda \geq \upsilon\} = X$. Let $x \in X$. Since $\{U_\lambda; \lambda \in \Lambda\}$ is point-finite, there is the largest element, say

$\xi \in \Lambda$ such that $x \in U_\xi$. If $\xi \geq \mu$, then $x \in \cup\{U_\lambda; \lambda \geq \mu\}$. However, if $\xi < \mu$, then $x \in [\cup\{V_\lambda; \lambda < \mu\}]$. Hence $[\cup\{V_\lambda; \lambda < \mu\}] \cup [\cup\{U_\lambda; \lambda \geq \mu\}] = X$. Thus $U_\mu$ contains the complement of an $\omega$-open set $[\cup\{V_\lambda; \lambda < \mu\}] \cup [\cup\{U_\lambda; \lambda > \mu\}]$. Since $X$ is an $\omega$-normal space, there exits an $\omega$-open set, say $V_\mu$ such that $(X - [\cup\{V_\lambda; \lambda < \mu\}] \cup [\cup\{U_\lambda; \lambda > \mu\}]) \subseteq V_\mu \subseteq \omega ClV_\mu \subseteq U_\mu$ {by Theorem 4.3}. Hence $[\cup\{V_\lambda; \lambda \leq \mu\}] \cup [\cup\{U_\lambda; \lambda \geq \mu\}] = X$. Hence the construction of the $\omega$-shrinking of $\{U_\lambda; \lambda \in \Lambda\}$ is completed by transfinite induction.

(2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (4) : Follows from Proposition 5.2.

(4) $\Rightarrow$ (1) : Let $X$ be a space which satisfies condition (4) and let $U$ and $V$ be two $\omega$-open subsets of $X$ such that $U \cup V = X$. Then $\{U, V\}$ is a finite $\omega$-open covering of $X$. Then by hypothesis, this covering has an $\omega$-locally-finite $\omega$-closed refinement, say $\Psi$. Let $F$ and $H$ be the union of these members of $\Psi$ which is contained in $U$ and $V$, respectively. Then by Proposition 5.5, $F$ and $H$ are $\omega$-closed subsets of $X$. Since $\Psi$ is a cover of $X$, in view of Theorem 4.2, $X$ is $\omega$-normal.

**Theorem 5.11.** *Let $\{U_\lambda; \lambda \in \Lambda\}$ be an $\omega$-locally-finite family of an $\omega$-open set of an $\omega$-normal space $X$, and let $\{E_\lambda; \lambda \in \Lambda\}$ be a family of $\omega$-closed sets such that $E_\lambda \subseteq G_\lambda$ for each $\lambda \in \Lambda$. Then there exists a family $\{V_\lambda; \lambda \in \Lambda\}$ of $\omega$-open sets such that $E_\lambda \subseteq V_\lambda \subseteq \omega ClV_\lambda \subseteq G_\lambda$ for each $\lambda \in \Lambda$ and the families $\{E_\lambda; \lambda \in \Lambda\}$ and $\{\omega ClV_\lambda; \lambda \in \Lambda\}$ are similar.*

**Proof.** Assume that $\Lambda$ is well-ordered. We shall construct a family $\{V_\lambda; \lambda \in \Lambda\}$ of $\omega$-open sets such that $E_\lambda \subseteq V_\lambda \subseteq \omega ClV_\lambda \subseteq G_\lambda$ for each $\lambda \in \Lambda$ by using the transfinite induction. First, we define the family $\{A^{\upsilon}_\lambda; \lambda \in \Lambda\}$ by

$$A^{\upsilon}_\lambda = \begin{cases} \omega ClV_\lambda \text{ if } \lambda \leq \mu \\ E_\lambda \text{ if } \lambda > \mu \end{cases}$$

Suppose that $\mu \in \Lambda$ and $V_\lambda$ are defined for each $\upsilon < \mu$ such that the family $\{A^{\upsilon}_\lambda; \lambda \in \Lambda\}$ is similar to $\{E_\lambda; \lambda \in \Lambda\}$. Let $\{B_\lambda; \lambda \in \Lambda\}$ be the family given by

$$B_\lambda = \begin{cases} \omega ClV_\lambda \text{ if } \lambda \leq \mu \\ E_\lambda \text{ if } \lambda > \mu \end{cases}.$$

To show $\{B_\lambda; \lambda \in \Lambda\}$ is similar to $\{E_\lambda; \lambda \in \Lambda\}$. Suppose that $\lambda_1, \lambda_2, \lambda_3, ...,$ $\lambda_k \in \Lambda$ such that $\lambda_1 < \lambda_2 < ... < \lambda_j < \mu < \lambda_{j+1} < ... < \lambda_k$. Since $\lambda_j < \lambda_\mu$, $\{A^{\lambda_j}_\lambda; \lambda \in \Lambda\}$ and $\{E_\lambda; \lambda \in \Lambda\}$ are similar. Since $\cap\{B_{\lambda_i}; i = 1, 2, 3, ..., k\} =$

$\cap\{A^{\lambda_j}_{\lambda_i}; i = 1,2,3,...,k\}$, $\cap\{B_{\lambda_i}; i = 1,2,3,...,k\} = \phi$ if and only if $\cap\{E_{\lambda_i}; i = 1,2,3,...,k\} = \phi$. Thus the families $\{B_\lambda; \lambda \in \Lambda\}$ and $\{E_\lambda; \lambda \in \Lambda\}$ are similar. Also, since $B_\lambda \subseteq G_\lambda$ for each $\lambda \in \Lambda$, the family $\{B_\lambda; \lambda \in \Lambda\}$ is $\omega$-locally-finite. Thus, if $\Gamma$ is the family of finite subsets of $\Lambda$ and for each $\gamma \in \Gamma$, we set $F_\gamma = \cap\{B_\lambda; \lambda \in \gamma\}$. Then the family $\{F_\gamma; \gamma \in \Gamma\}$ is an $\omega$-locally-finite family of $\omega$-closed sets. Hence by Proposition 5.5, we obtain that $F = \cup\{F_\gamma; F_\gamma \cap E_\mu\}$ is $\omega$-closed and it is disjoint from $E_\mu$. Therefore, by Theorem 4.3, there exists an $\omega$-open set $V_\mu$ such that $E_\mu \subseteq V_\mu \subseteq \omega ClV_\mu \subseteq G_\mu$ and $\omega ClV_\mu \cap F = \phi$. Hence the $\omega$-open sets $V_\lambda$ are defined for each $\lambda \leq \mu$. It remains only to show that the family $\{A^\mu_\lambda; \lambda \in \Lambda\}$ is similar to $\{E_\lambda; \lambda \in \Lambda\}$. For this, it is sufficient to show that it is similar to $\{B_\lambda; \lambda \in \Lambda\}$. Suppose that $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_t \in \Lambda$ and $\cap\{B_{\lambda_i}; i = 1,2,3,...,t\} = \phi$, we have to show $\cap\{A^\mu_{\lambda_i}; i = 1,2,3,...,t\} = \phi$. Consider $\lambda_1 < \lambda_2 < ... < \lambda_j \leq \mu < \lambda_{j+1} < ... < \lambda_t$. If $\lambda_j \neq \mu$, then the proof is completed. If $\lambda_j = \mu$, then $\cap\{B_{\lambda_i}; i = 1,2,3,...,t\} \cap E_\mu = \phi$. Hence by our construction $\cap\{B_{\lambda_i}; i = 1,2,3,...,t\} \cap \omega ClV_\mu = \phi$. Thus $\cap\{A^\mu_{\lambda_i}; i = 1,2,3,...,t\} = \phi$. This completes the proof.

**Corollary 5.12.** *Let $X$ be a topological space. Then the followings statements are equivalent:*

1. *$X$ is $\omega$-normal,*
2. *For each finite family $\{E_i; i = 1,2,3,...,k\}$ of $\omega$-closed sets of $X$, there is a family $\{V_i; i = 1,2,3,...,k\}$ of $\omega$-open sets such that $E_i \subseteq V_i$ for each $i$ = 1, 2, ..., $k$, and the families $\{E_i; i = 1,2,3,...,k\}$ and $\{\omega ClV_i; i = 1,2,3,...,k\}$ are similar,*
3. *For each pair $E_1$ and $E_2$ of disjoint $\omega$-closed sets of $X$, there is a pair $V_1$ and $V_2$ of $\omega$-open sets of $X$ such that $\omega ClV_1$ and $\omega ClV_2$ are disjoint.*

**Proof.** (1) ⇒ (2) : It follows by putting $G_i = X$, for each $i$ = 1, 2, ..., $k$ in Theorem 5.11.

(2) ⇒ (3) and (3) ⇒ (1) are obvious.

**Corollary 5.13.** *Let $(X, \Im)$ be a space and $B \subseteq A \subseteq X$. Then the following statements are true:*

1. *Let $A \in \Im^\omega$. Then $B \in \Im^\omega_A$ if and only if $B \in \Im^\omega$,*

2. *Let $A$ be $\omega$-closed. Then $B$ is $\omega$-closed in $A$ if and only if it is $\omega$-closed in $X$.*

**Proof.** It follows from Theorem 1.2.

**Theorem 5.14.** *Let $E$ be an $\omega$-closed subset of an $\omega$-normal $X$ and let $\{G_\lambda; \lambda \in \Lambda\}$ be an $\omega$-locally-finite family of an $\omega$-open set of $X$ such that $E \subseteq \cup\{G_\lambda; \lambda \in \Lambda\}$. Then there exists a family $\{V_\lambda; \lambda \in \Lambda\}$ of $\omega$-open sets of $X$ such that $\omega ClV_\lambda \subseteq G_\lambda$ for each $\lambda \in \Lambda$, $E \subseteq \cup\{G_\lambda; \lambda \in \Lambda\}$ and $\{\omega ClV_\lambda; \lambda \in \Lambda\}$ are similar to $\{E \cap G_\lambda; \lambda \in \Lambda\}$.*

**Proof.** Let $\Gamma$ be the family of finite subsets of $\Lambda$ such that $E \cap (\cap\{G_\lambda; \lambda \in \gamma\} \neq \phi$ for each $\gamma \in \Gamma$. The family $\{E \cap (\cap\{G_\lambda; \lambda \in \gamma\}); \gamma \in \Gamma\}$ of non-empty $\omega$-open subsets of $E$ is $\omega$-locally-finite and from Theorem 4.14, $E$ is $\omega$-normal. Hence by Corollary 4.5, $E$ is $\omega$-regular. Therefore, for each $\gamma \in \Gamma$, there exists a non-empty $\omega$-closed set $D_\gamma$ of $E$ such that $D_\gamma \subseteq E \cap (\cap\{G_\lambda; \lambda \in \gamma\})$. Since $\{G_\lambda; \lambda \in \Lambda\}$ is $\omega$-locally-finite and $E$ is an $\omega$-closed subset of $X$, the family $\{D_\gamma; \gamma \in \Gamma\}$ consists of $\omega$-closed subsets of $X$ and it is $\omega$-locally-finite in $X$ {by Corollary 5.13}. Since $E$ is $\omega$-normal and each $\omega$-locally-finite family is point-finite, it follows that there is an $\omega$-locally-finite $\omega$-closed covering $\{H_\lambda; \lambda \in \Lambda\}$ of $E$ such that $H_\lambda \subseteq E \cap G_\lambda$ for each $\lambda$ {by Theorem 5.10}. Let $F_\lambda = E_\lambda \cup \{D_\gamma; \gamma \in \Gamma\}$ for each $\lambda$. Then by Proposition 5.5, $F_\lambda = E_\lambda \cup \{D_\gamma; \gamma \in \Gamma\}$ is $\omega$-closed in both $E$ and $X$ and $F_\lambda = E_\lambda \cap G_\lambda$ for each $\lambda$. Also $\{F_\lambda; \lambda \in \Lambda\}$ is an $\omega$-closed covering of $E$. Furthermore, the families $\{F_\lambda; \lambda \in \Lambda\}$ and $\{E \cap G_\lambda; \lambda \in \Lambda\}$ are similar. For if $\gamma \in \Gamma$, then $D_\gamma \subseteq \cap\{F_\lambda; \lambda \in \gamma\}$. Hence $\cap\{F_\lambda; \lambda \in \Lambda\} = \phi$. Since $X$ is an $\omega$-normal space, there exists a family $\cup\{V_\lambda; \lambda \in \Lambda\}$ of $\omega$-open subsets of $X$ such that $F_\lambda \subseteq V_\lambda \subseteq \omega ClV_\lambda \subseteq G_\lambda$ for each $\lambda$ {by Theorem 5.11}. Therefore, $\{\omega ClV_\lambda; \lambda \in \Lambda\}$ and $\{E \cap G_\lambda; \lambda \in \Lambda\}$ are similar. This completes the proof.

## References


[1] T. A. Al-Hawary, Fuzzy $\omega^o$-Open Sets, *Bull. Korean Math. Soc.*, **Vol. 45** (2009), 749-755.

[2] A. Al-Omari and M. S. Noorani, Contra-$\omega$-continuous and Almost Contra $\omega$-continuous, *Int. J. of Math. and Math. Sci.*, **Vol. 2007** Article ID 40469 (2007), 13 pages.

[3] A. Al-Omari and M. S. Noorani, Regular Generalized $\omega$-closed sets, *Int. J. of Math. and Math. Sci.*, **Vol. 2007** Article ID16292 (2007), 11 pages.

[4] K. Y. Al-Zoubi, On Generalized $\omega$-closed sets, *Int. J. of Math. and Math. Sci.*, **Vol. 2005** no. 13 (2005), 2011-2021.

[5] K. Y. Al-Zoubi and B. Al-Nashief, The Topology of $\omega$-open subsets, *Al-Manarah*, **Vol 9(2)** (2003), 169-179.

[6] K. Chandrasekhara Rao and D. Narasimhan, Semi Star Generalized $\omega$-Closed Sets in Bitopological Spaces, *Int. J. Contermp. Math. Sci.*, **Vol. 4** (12) (2009), 587-595.

[7] J. Dugundji, *Topology, Allyn and Bacon Inc. Boston*, 1966.

[8] H. Z. Hdeib, $\omega$-closed maping, *Revista colombiana de mathematics*, **Vol. 16** (1982), 65-78.

[9] H. Z. Hdeib, $\omega$-continuous Functions, *Dirasat J.*, **Vol. 16(2)** (1989), 136-153.

[10] T. M. Nour, Almost $\omega$-Continuous Functions, *Eur. J. of Sci. Res.*, **Vol. 18(1)** (2005), 43-47.

[11] A. R. Pears, *Dimension Theory of General Spaces*, Cambridge University press, Cambridge, 1975.

[12] I. L. Steen and J. A. Jr. Seebanch, *Counterexamples in Topology*, Springer-Verlag, New York, 1978.

[13] S. Willard, *General Topology*, Addision Weasly, London, 1970.