Tamsui Oxford Journal of Information and Mathematical Sciences **28(3)** (2012) 293-301
Aletheia University

On a Class of Analytic Functions Defined by Generalized Al-Oboudi Differential Operator *

Serap BULUT†
Kocaeli University, Civil Aviation College, Arslanbey Campus,
41285 İzmit-Kocaeli, Turkey

and

Adriana CĂTAŞ‡
Department of Mathematics, University of Oradea,
str. Universitatii nr. 1, 410087, Oradea, Romania

Received February 16, 2011, Accepted July 31, 2012.

Abstract

In this paper, we introduce a new class of analytic functions by using generalized Al-Oboudi differential operator, and obtain some subordination results.

Keywords and Phrases: *Analytic functions, Differential operator, Differential subordination.*

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

---

*2000 *Mathematics Subject Classification.* Primary 30C80.

†Corresponding author. E-mail: serap.bulut@kocaeli.edu.tr
‡E-mail: acatas@uoradea.ro

and for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of the functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots .$$

Let $\mathcal{A}_n$ be the class of all functions of the form

$$f(z) = z + a_{n+1} z^{n+1} + \cdots \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}$ with

$$\mathcal{A}_1 = \mathcal{A}.$$

Also let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions f which are univalent in $\mathbb{U}$.

A function f analytic in $\mathbb{U}$ is said to be convex if it is univalent and $f(\mathbb{U})$ is convex.

Let

$$\mathcal{K} = \left\{ f \in \mathcal{A} : \operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0,\ z \in \mathbb{U} \right\}$$

denote the class of normalized convex functions in $\mathbb{U}$.

If f and g are analytic in $\mathbb{U}$, then we say that f is subordinate to g, written symbolically as

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in \mathbb{U})$$

if there exists a Schwarz function w which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$, $z \in \mathbb{U}$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

Let $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ be a function and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the (second-order) differential subordination

$$\psi\left(p(z),\ zp'(z),\ z^2 p''(z);\ z\right) \prec h(z) \quad (z \in \mathbb{U}), \tag{1.2}$$

then p is called a solution of the differential subordination.

The univalent function q is called a dominant of the solution of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.2).

A dominant $\tilde{q}$, which satisfies $\tilde{q} \prec q$ for all dominants q of (1.2) is said to be the best dominant of (1.2).

2. Definitions

The following definition of fractional derivative by Owa [5] (also by Srivastava and Owa [8]) will be required in our investigation.

The fractional derivative of order α is defined, for a function f, by

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\alpha} dt \quad (0 \leq \alpha < 1), \tag{2.1}$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin, and the multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $\log(z-t)$ to be real when $z - t > 0$.

It readily follows from (2.1) that

$$D_z^\alpha z^k = \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} z^{k-\alpha} \quad (0 \leq \alpha < 1, k \in \mathbb{N} = \{1, 2, \ldots\}).$$

Using $D_z^\alpha f$, Owa and Srivastava [6] introduced the operator $\Omega^\alpha : \mathcal{A} \to \mathcal{A}$, which is known as an extension of fractional derivative and fractional integral, as follows:

$$\Omega^\alpha f(z) = \Gamma(2-\alpha)\, z^\alpha D_z^\alpha f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k. \tag{2.2}$$

Note that

$$\Omega^0 f(z) = f(z).$$

In [2], Al-Oboudi and Al-Amoudi defined the linear multiplier fractional differential operator $D_\lambda^{n,\alpha}$ as follows:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D_\lambda^{1,\alpha} f(z) &= (1-\lambda)\, \Omega^\alpha f(z) + \lambda z\, (\Omega^\alpha f(z))' \\ &= D_\lambda^\alpha (f(z)), \quad \lambda \geq 0,\ 0 \leq \alpha < 1, \qquad (2.3) \\ D_\lambda^{2,\alpha} f(z) &= D_\lambda^\alpha \left(D_\lambda^{1,\alpha} f(z) \right), \\ &\vdots \\ D_\lambda^{m,\alpha} f(z) &= D_\lambda^\alpha \left(D_\lambda^{m-1,\alpha} f(z) \right), \quad m \in \mathbb{N}. \qquad (2.4) \end{aligned}$$

If f is given by (1.1), then by (2.2), (2.3) and (2.4), we see that

$$D_{\lambda}^{m,\alpha} f(z) = z + \sum_{k=2}^{\infty} \Psi_{k,m}(\alpha,\lambda) a_k z^k, \quad m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{2.5}$$

where

$$\Psi_{k,m}(\alpha,\lambda) = \left[\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} (1+(k-1)\lambda) \right]^m. \tag{2.6}$$

Remark 1. (**i**) When $\alpha = 0$, we get Al-Oboudi differential operator [1].

(**ii**) When $\alpha = 0$ and $\lambda = 1$, we get Sălăgean differential operator [7].

(**iii**) When $m = 1$ and $\lambda = 0$, we get Owa-Srivastava fractional differential operator [6].

Definition 1. Let $\mathcal{S}_{\alpha,\lambda}^{m}(\beta)$ be the class of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left\{ \left(D_{\lambda}^{m,\alpha} f(z) \right)' \right\} > \beta,$$

where $z \in \mathbb{U}$, $0 \leq \beta < 1$ and $D_{\lambda}^{m,\alpha}$ is the generalized Al-Oboudi differential operator.

Remark 2. In Definition 2, if we set

(**i**) $\alpha = 0$, then we get $\mathcal{S}_{0,\lambda}^{m}(\beta) \equiv \mathcal{R}^m(\lambda,\beta)$ defined by Al-Oboudi [1].

(**ii**) $\alpha = 0$ and $\lambda = 1$, then we get $\mathcal{S}_{0,1}^{m}(\beta) \equiv \mathcal{S}_m(\beta)$ defined by Taut et al. [9].

3. Preliminary Lemmas

In order to prove our main results, we will make use of the following lemmas.

Lemma 3.1. *[3] Let h be a convex function with $h(0) = a$ and let $\gamma \in \mathbb{C}^* := \mathbb{C} - \{0\}$ be a complex number with $\operatorname{Re}\gamma \geq 0$. If $p \in \mathcal{H}[a,n]$ and*

$$p(z) + \frac{1}{\gamma} z p'(z) \prec h(z) \quad (z \in \mathbb{U}),$$

then

$$p(z) \prec q(z) \prec h(z) \quad (z \in \mathbb{U}),$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z t^{\frac{\gamma}{n}-1} h(t) dt \quad (z \in \mathbb{U}).$$

The function q is convex and is the best dominant.

Lemma 3.2. *[4] Let $\operatorname{Re} r > 0$, $n \in \mathbb{N}$ and let*

$$w = \frac{n^2 + |r|^2 - |n^2 - r^2|}{4n \operatorname{Re} r}.$$

Let h be an analytic function in $\mathbb{U}$ with $h(0) = 1$ and suppose that

$$\operatorname{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -w.$$

If

$$p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$$

is analytic in $\mathbb{U}$ and

$$p(z) + \frac{1}{r} zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z),$$

where q is a solution of the differential equation

$$q(z) + \frac{n}{r} zq'(z) = h(z), \quad q(0) = 1,$$

given by

$$q(z) = \frac{r}{nz^{\frac{r}{n}}} \int_0^z t^{\frac{r}{n}-1} h(t) dt.$$

Moreover q is the best dominant.

4. Main Results

Theorem 4.1. *The set $\mathcal{S}^m_{\alpha,\lambda}(\beta)$ is convex.*

Proof. Let

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \quad (z \in \mathbb{U};\ j = 1, 2, \ldots, l) \tag{4.1}$$

be in the class $\mathcal{S}_{\alpha,\lambda}^{m}(\beta)$. Then, by Definition 1, we have

$$\operatorname{Re}\left\{(D_\lambda^{m,\alpha} f_j(z))'\right\} = \operatorname{Re}\left\{1 + \sum_{k=2}^{\infty} k\Psi_{k,m}(\alpha,\lambda)\, a_{k,j} z^{k-1}\right\} > \beta. \tag{4.2}$$

For any nonnegative numbers $\mu_1, \mu_2, \ldots, \mu_l$ such that $\mu_1 + \mu_2 + \ldots + \mu_l = 1$, we must show that the function

$$h(z) = \sum_{j=1}^{l} \mu_j f_j(z) \tag{4.3}$$

is in $\mathcal{S}_{\alpha,\lambda}^{m}(\beta)$, that is

$$\operatorname{Re}\left\{(D_\lambda^{m,\alpha} h(z))'\right\} > \beta.$$

By (4.1) and (4.3), we have

$$h(z) = z + \sum_{k=2}^{\infty} \left(\sum_{j=1}^{l} \mu_j a_{k,j}\right) z^k.$$

Therefore we get

$$D_\lambda^{m,\alpha} h(z) = z + \sum_{k=2}^{\infty} \Psi_{k,m}(\alpha,\lambda) \left(\sum_{j=1}^{l} \mu_j a_{k,j}\right) z^k, \tag{4.4}$$

where $\Psi_{k,m}(\alpha,\lambda)$ defined as in (2.6). Differentiating (4.4) with respect to z, we obtain

$$(D_\lambda^{m,\alpha} h(z))' = 1 + \sum_{k=2}^{\infty} k\Psi_{k,m}(\alpha,\lambda) \left(\sum_{j=1}^{l} \mu_j a_{k,j}\right) z^{k-1}.$$

So we get

$$\begin{aligned}
\operatorname{Re}\left\{(D_\lambda^{m,\alpha} h(z))'\right\} &= 1 + \sum_{j=1}^{l} \mu_j \operatorname{Re}\left\{\sum_{k=2}^{\infty} k\Psi_{k,m}(\alpha,\lambda)\, a_{k,j} z^{k-1}\right\} \\
&> 1 + \sum_{j=1}^{l} \mu_j (\beta - 1) \qquad \text{(by (4.2))} \\
&= \beta
\end{aligned}$$

since $\mu_1 + \mu_2 + \ldots + \mu_l = 1$. Therefore we get desired result.

Theorem 4.2. *Let q be convex function in $\mathbb{U}$ with $q(0) = 1$ and let*

$$h(z) = q(z) + \frac{1}{c+2} zq'(z) \quad (z \in \mathbb{U}),$$

where c is a complex number with $\operatorname{Re} c > -2$.

If $f \in \mathcal{S}^m_{\alpha,\lambda}(\beta)$ and $F = I^\alpha_c f$, where

$$F(z) = I^\alpha_c f(z) = \frac{c+2}{z^{c+1}} \int_0^z t^c \Omega^\alpha f(t) dt \quad (0 \leq \alpha < 1), \tag{4.5}$$

then

$$\left(D^{m,\alpha}_\lambda \left(\Omega^\alpha f(z)\right)\right)' \prec h(z) \tag{4.6}$$

implies

$$\left(D^{m,\alpha}_\lambda F(z)\right)' \prec q(z),$$

and this result is sharp.

Proof. From the equality (4.5), we get

$$z^{c+1} F(z) = (c+2) \int_0^z t^c \Omega^\alpha f(t) dt. \tag{4.7}$$

Differentiating (4.7) with respect to z, we have

$$(c+1) F(z) + zF'(z) = (c+2) \Omega^\alpha f(z)$$

and

$$(c+1) D^{m,\alpha}_\lambda F(z) + z \left(D^{m,\alpha}_\lambda F(z)\right)' = (c+2) D^{m,\alpha}_\lambda \left(\Omega^\alpha f(z)\right). \tag{4.8}$$

Differentiating (4.8) with respect to z, we obtain

$$\left(D^{m,\alpha}_\lambda F(z)\right)' + \frac{1}{c+2} z \left(D^{m,\alpha}_\lambda F(z)\right)'' = \left(D^{m,\alpha}_\lambda \left(\Omega^\alpha f(z)\right)\right)'. \tag{4.9}$$

Using (4.9), the differential subordination (4.6) becomes

$$\left(D^{m,\alpha}_\lambda F(z)\right)' + \frac{1}{c+2} z \left(D^{m,\alpha}_\lambda F(z)\right)'' \prec h(z). \tag{4.10}$$

Let us define

$$p(z) = \left(D_{\lambda}^{m,\alpha} F(z)\right)'. \tag{4.11}$$

Then a simple computation yields

$$\begin{aligned} p(z) &= \left[z + \sum_{k=2}^{\infty} \Psi_{k,m}(\alpha, \lambda) \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} \frac{c+2}{k+c+1} a_k z^k \right]' \\ &= 1 + p_1 z + p_2 z^2 + \ldots, \qquad p \in \mathcal{H}[1,1]. \end{aligned}$$

Using (4.11) in the subordination (4.10), we have

$$p(z) + \frac{1}{c+2} z p'(z) \prec h(z) = q(z) + \frac{1}{c+2} z q'(z) \quad (z \in \mathbb{U}).$$

Using Lemma 3.1, we obtain

$$p(z) \prec q(z)$$

which is desired result. Moreover q is the best dominant.

Theorem 4.3. *Let* $\operatorname{Re} c > -2$ *and let*

$$w = \frac{1 + |c+2|^2 - |c^2 + 4c + 3|}{4 \operatorname{Re}(c+2)}.$$

Let h be an analytic function in $\mathbb{U}$ with $h(0) = 1$ and suppose that

$$\operatorname{Re}\left\{1 + \frac{z h''(z)}{h'(z)}\right\} > -w.$$

If $f \in \mathcal{S}_{\alpha,\lambda}^{m}(\beta)$ and $F = I_c^{\alpha} f$, where F is defined by (4.5), then

$$\left(D_{\lambda}^{m,\alpha}\left(\Omega^{\alpha} f(z)\right)\right)' \prec h(z)$$

implies

$$\left(D_{\lambda}^{m,\alpha} F(z)\right)' \prec q(z),$$

where q is the solution of the differential equation

$$h(z) = q(z) + \frac{1}{c+2} z q'(z), \quad q(0) = 1,$$

given by

$$q(z) = \frac{c+2}{z^{c+2}} \int_0^z t^{c+1} h(t) dt.$$

Moreover q is the best dominant.

Proof. We consider $n = 1$ and $r = c + 2$ in Lemma 3.2. Then the proof is easily seen by means of the proof of Theorem 4.2.

References

[1] F. M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, *Int. J. Math. Math. Sci.*, no. 25-28 (2004), 1429-1436.

[2] F. M. Al-Oboudi and K. A. Al-Amoudi, On classes of analytic functions related to conic domains, *J. Math. Anal. Appl.*, **339** no. 1 (2008), 655-667.

[3] D. J. Hallenbeck and S. Ruscheweyh,Subordination by convex functions, *Proc. Amer. Math. Soc.*, **52** (1975), 191-195.

[4] G. Oros and G. I. Oros, A class of holomorphic functions II, *Libertas Math.*, **23** (2003), 65-68.

[5] S. Owa,On the distortion theorems. I, *Kyungpook Math. J.*, **18** no. 1 (1978), 53-59.

[6] S. Owa and H. M. Srivastava,Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.*, **39** no. 5 (1987), 1057-1077.

[7] G. Ş. Sălăgean, Subclasses of univalent functions, *Complex Analysis-Fifth Romanian-Finnish seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math.*, **vol. 1013**, Springer, Berlin, (1983), 362-372.

[8] H. M. Srivastava and S. Owa, (Eds.), *Univalent Functions, Fractional Calculus, and Their Applications*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, UK; JohnWiley & Sons, New York, NY, USA, 1989.

[9] A. O. Tăut, G. I. Oros and R. Şendruţiu, On a class of univalent functions defined by Sălăgean differential operator, *Banach J. Math. Anal.*, **3** no. 1 (2009), 61-67.