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A New Multiple Inequality Similar to Hardy-Hilbert's Integral Inequality *

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Abstract

In this paper, by introducing some parameters we establish a new multiple inequality similar to Hardy-Hilbert's integral inequality with the best constant factor which involves the Γ function. Some particular results are obtained.

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1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0$, satisfy $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}}, \quad (1.1)$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy < pq \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}} \quad (1.2)$$

where the constant factors $\pi/(\sin \pi/p)$ and pq are the best possible. Inequalities (1.1) and (1.2) are called Hardy-Hilbert's inequalities (see [1]) and are important in analysis and their applications (cf. Mitrinović et al. [2]). At present, because of the requirement of higher-dimensional harmonic analysis and higher-dimensional operator theory, multiple Hardy-Hilbert's integral inequalities are researched (see [3, 4, 5]).

In 2005, Yang ([3]) gave the following inequality.

Theorem 1.1. *If* $n \in \mathbb{N}\backslash\{1\}, p_i > 1, \sum_{i=1}^n \frac{1}{p_i} = 1, \lambda > 0, f_i \geq 0$, *satisfy*

$$0 < \int_0^\infty x^{p_i-1-\lambda} f_i^{p_i}(x)dx < \infty \quad (i = 1, 2, \ldots, n),$$

then

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n f_i(x_i) dx_1 \ldots dx_n < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right) \left\{ \int_0^\infty x^{p_i-1-\lambda} f_i^{p_i}(x)dx \right\}^{\frac{1}{p_i}} \quad (1.3)$$

where the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{p_i}\right)$ *is the best possible.*

In the recent years, many new inequalities similar to (1.1) and (1.2) have been established; see ([6, 7, 9]). Recently Das and Sahoo ([6]) have given a new inequality similar to Hardy-Hilbert's inequality (1.1) as follows:

Theorem 1.2. *Let* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \lambda, s, r > 0, r + s = \lambda, f, g \geq 0$ *and* $F(x) = \int_0^x f(t)dt, G(x) = \int_0^x g(t)dt$. *If* $0 < \int_0^\infty f^p(x)dx < \infty$ *and* $0 < \int_0^\infty g^q(x)dx < \infty$, *then*

$$\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}-1}}{(x+y)^\lambda} F(x)G(y)dxdy < pqB(r,s) \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x)dx \right\}^{\frac{1}{q}} \quad (1.4)$$

where the constant factor $pqB(r,s)$ *is the best possible.*

Sulaiman ([8, 9]) derived three new integral inequalities similar to (1.1) as follows:

Theorem 1.3 ([8])**.** *Let* $f, g, h \geq 0,\ p, q, r > 2,\ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1,$

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt, \quad H(z) = \int_0^z h(t)dt.$$

Then

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{1/p} y^{1/q} z^{1/r} \sqrt{xyzF(x)G(y)H(z)}}{(x+y+z)^6} dxdydz < K_p K_q K_r \left(\int_0^\infty f^{p/2}(x)dx \right)^{1/p} \left(\int_0^\infty g^{q/2}(y)dy \right)^{1/q} \left(\int_0^\infty h^{r/2}(z)dz \right)^{1/r} \quad (1.5)$$

where

$$K_p = \sqrt{\frac{p/2}{p/2 - 1}} B^{1/p} \left(\frac{1}{p}, \frac{1}{p} \right) B^{1/p}(p,p).$$

Theorem 1.4 ([9])**.** *Let* $f, g \geq 0,\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ \lambda > 1, \alpha_p = p(\lambda - 1) + 1.$ *Define*

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt.$$

Then

$$\int_0^\infty \int_0^\infty \frac{F^{\frac{\alpha_p}{p}}(x) G^{\frac{\alpha_q}{q}}(y)}{(x+y)^{2\lambda}} dxdy < B(\lambda,\lambda) \left(\frac{\alpha_p}{\alpha_p - 1}\right)^{\frac{\alpha_p}{p}} \left(\frac{\alpha_q}{\alpha_q - 1}\right)^{\frac{\alpha_q}{q}} \times \left(\int_0^\infty f^{\alpha_p}(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^{\alpha_q}(y)dy\right)^{\frac{1}{q}}. \tag{1.6}$$

Theorem 1.5 ([9])**.** *Let* $f, g, h \geq 0, p, q, r > 1, \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, s, t \in \{p, q, r\}$,

$$\alpha_{s,t} = \lambda(1 + s/t) + (\lambda - 1)(s - 1), \quad \lambda > 1.$$

Define

$$F(x) = \int_0^x f(t)dt, \quad G(y) = \int_0^y g(t)dt, \quad H(z) = \int_0^z h(t)dt.$$

Suppose also that

$$0 < \int_0^\infty f^{\alpha_{p,q}}(t)dt < \infty, \quad 0 < \int_0^\infty g^{\alpha_{q,r}}(t)dt < \infty, \quad 0 < \int_0^\infty h^{\alpha_{r,p}}(t)dt < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{F^{\alpha_{p,q}/p}(x) G^{\alpha_{q,r}/q}(y) H^{\alpha_{r,p}/r}(z)}{(x+y+z)^{4\lambda}} dxdydz < K \left(\int_0^\infty f^{\alpha_{p,q}}(x)dx\right)^{1/p} \left(\int_0^\infty g^{\alpha_{q,r}}(y)dy\right)^{1/q} \left(\int_0^\infty h^{\alpha_{r,p}}(z)dz\right)^{1/r} \tag{1.7}$$

where

$$K = B(\lambda,\lambda)B(2\lambda,2\lambda) \left(\frac{\alpha_{p,q}}{\alpha_{p,q} - 1}\right)^{\alpha_{p,q}/p} \left(\frac{\alpha_{q,r}}{\alpha_{q,r} - 1}\right)^{\alpha_{q,r}/q} \left(\frac{\alpha_{r,p}}{\alpha_{r,p} - 1}\right)^{\alpha_{r,p}/r}.$$

But he cannot show that the constant factors in the new inequalities are the best possible.

In this paper, by introducing some parameters we establish a new multiple generalization of the above four inequalities similar to Hardy-Hilbert's integral inequality (1.3) with the best constant factor. Some particular results are obtained.

2. Preliminary Lemmas

In order to prove main result, we need the following lemmas.

Lemma 2.1. *If* $k \in \mathbb{N}, r_i > 1 (i = 1, 2, \ldots, k+1)$, *and* $\sum_{i=1}^{k+1} r_i = \lambda(k)$, *then*

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(1 + \sum_{j=1}^k u_j\right)^{\lambda(k)}} \prod_{j=1}^k u_j^{r_j - 1} du_1 \ldots du_k = \frac{\prod_{i=1}^{k+1} \Gamma(r_i)}{\Gamma(\lambda(k))}. \tag{2.1}$$

For the proof of Lemma 2.1 the reader is referred to ([3, 4]).

Lemma 2.2 (Hardy's inequality, cf. [1])**.** *If* $p > 1, f \geq 0$ *and* $F(x) = \int_0^x f(t)dt$, *then*

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx \tag{2.2}$$

unless $f \equiv 0$. *The constant is the best possible.*

Lemma 2.3. *Let* $p > \frac{1}{\beta}, 0 < \beta \leq 1, 0 < \varepsilon < \beta p - 1$ *for* $x \geq 1$, *then*

$$(x^{\frac{\beta p - (1+\varepsilon)}{\beta p}} - 1)^\beta \geq x^{\frac{\beta p - (1+\varepsilon)}{p}} - 1. \tag{2.3}$$

Proof. For $x \geq 1$, we set

$$f(x) = (x^{\frac{\beta p - (1+\varepsilon)}{\beta p}} - 1)^\beta - x^{\frac{\beta p - (1+\varepsilon)}{p}} + 1$$

Simple computations yield for $x > 1$

$$f'(x) = \frac{\beta p - (1+\varepsilon)}{p} x^{\frac{(\beta - 1)p - (1+\varepsilon)}{p}} \left((1 - x^{\frac{1+\varepsilon-\beta p}{\beta p}})^{\beta - 1} - 1\right) > 0.$$

f is increasing function on $(1, \infty)$ and continuous on $[1, \infty)$. In particular, we have $f(x) \geq f(1) = 0$, which gives the desired inequality.

3. Main Results

Theorem 3.1. *Let $n \in \mathbb{N}\backslash\{1\}, p_i > \frac{1}{\alpha_i}, 0 < \alpha_i \le 1, \sum_{i=1}^n \frac{1}{p_i} = 1, \lambda, s_i > 1,$ and $\sum_{i=1}^n s_i = \lambda$. Assume $F_i(x) = \int_0^x f_i(t)dt$ $(i = 1, 2, \dots, n)$. If $f_i \ge 0$ satisfy $0 < \int_0^\infty f_i^{\alpha_i p_i}(x)dx < \infty$ $(i = 1, 2, \dots, n)$ then the following inequality holds:*

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n x_i^{s_i + \frac{1}{p_i} - 1 - \alpha_i} F_i^{\alpha_i}(x_i) dx_1 \dots dx_n < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma(s_i) \left(\frac{\alpha_i p_i}{\alpha_i p_i - 1}\right)^{\alpha_i} \left\{\int_0^\infty f_i^{\alpha_i p_i}(x)dx\right\}^{\frac{1}{p_i}} \quad (3.1)$$

where the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma(s_i) \left(\frac{\alpha_i p_i}{\alpha_i p_i - 1}\right)^{\alpha_i}$ is the best possible.

Proof. By Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} J &= \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n x_i^{s_i + \frac{1}{p_i} - 1 - \alpha_i} F_i^{\alpha_i}(x_i) dx_1 \dots dx_n \\ &= \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n \left(x_i^{\frac{s_i}{p_i} - \alpha_i} \prod_{\substack{j=1 \\ (j \ne i)}}^n x_j^{\frac{s_j - 1}{p_i}} F_i^{\alpha_i}(x_i) \right) dx_1 \dots dx_n \\ &\le \prod_{i=1}^n \left\{ \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} x_i^{s_i} \prod_{\substack{j=1 \\ (j \ne i)}}^n x_j^{s_j - 1} \left(\frac{F_i(x_i)}{x_i}\right)^{\alpha_i p_i} dx_1 \dots dx_n \right\}^{\frac{1}{p_i}} \\ &= \frac{\prod_{j=1}^n \Gamma(s_j)}{\Gamma(\lambda)} \prod_{i=1}^n \left\{ \int_0^\infty \left(\frac{F_i(x_i)}{x_i}\right)^{\alpha_i p_i} dx_i \right\}^{\frac{1}{p_i}}. \qquad (3.2) \end{aligned}$$

Then by Hardy inequality (2.2), (3.1) is valid. For the best constant factor, let for sufficiently small $\varepsilon > 0$,

$$\widetilde{f}_i(x) = \begin{cases} 0, & \text{for } x \in (0, 1) \\ x^{-\frac{1+\varepsilon}{\alpha_i p_i}}, & \text{for } x \in [1, \infty) \end{cases} \quad (i = 1, 2, \dots, n)$$

then we find

$$\prod_{i=1}^{n}\left\{\int_0^\infty \widetilde{f}_i^{\alpha_i p_i}(x)dx\right\}^{\frac{1}{p_i}} = \frac{1}{\varepsilon} \tag{3.3}$$

and

$$\widetilde{F}_i(x_i) = \begin{cases} 0, & \text{for } x_i \in (0,1) \\ \frac{\alpha_i p_i}{\alpha_i p_i-(1+\varepsilon)}(x_i^{\frac{\alpha_i p_i-(1+\varepsilon)}{\alpha_i p_i}}-1), & \text{for } x_i \in [1,\infty) \end{cases} \quad (i=1,2,\ldots,n).$$

Denote $\phi(\varepsilon) = \prod_{i=1}^n \left(\frac{\alpha_i p_i}{\alpha_i p_i-(1+\varepsilon)}\right)^{\alpha_i}$. Then $\phi(\varepsilon) \to \prod_{i=1}^n \left(\frac{\alpha_i p_i}{\alpha_i p_i-1}\right)^{\alpha_i}$, as $\varepsilon \to 0^+$ and for $x_i \geq 1$, by Lemma 2.3, we have

$$\prod_{i=1}^{n}\widetilde{F}_i^{\alpha_i}(x_i) = \phi(\varepsilon)\prod_{i=1}^{n}(x_i^{\frac{\alpha_i p_i-(1+\varepsilon)}{\alpha_i p_i}}-1)^{\alpha_i} > \phi(\varepsilon)\prod_{i=1}^{n}(x_i^{\frac{\alpha_i p_i-(1+\varepsilon)}{p_i}}-1).$$

Hence, we have

$$\begin{aligned}
J(\varepsilon) =& \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda}\prod_{i=1}^{n} x_i^{s_i+\frac{1}{p_i}-1-\alpha_i}\widetilde{F}_i^{\alpha_i}(x_i)dx_1\ldots dx_n \\
>&\phi(\varepsilon)\int_1^\infty \cdots \int_1^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda}\prod_{i=1}^{n} x_i^{s_i+\frac{1}{p_i}-1-\alpha_i}(x_i^{\frac{\alpha_i p_i-(1+\varepsilon)}{p_i}}-1)dx_1\ldots dx_n \\
=&\phi(\varepsilon)\int_1^\infty \cdots \int_1^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda}\Bigg(\prod_{i=1}^{n} x_i^{s_i-\frac{\varepsilon}{p_i}-1} \\
&+\sum_{k=1}^{n}(-1)^k\sum_{cyc}\prod_{i=1}^{k} x_i^{s_i+\frac{1}{p_i}-1-\alpha_i}\prod_{j=k+1}^{n} x_j^{s_j-\frac{\varepsilon}{p_j}-1}\Bigg)dx_1\ldots dx_n \\
>&\phi(\varepsilon)\int_1^\infty \cdots \int_1^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda}\Bigg(\prod_{i=1}^{n} x_i^{s_i-\frac{\varepsilon}{p_i}-1} \\
&-\sum_{\substack{1\leq k\leq n\\ k-odd}}\sum_{cyc}\prod_{i=1}^{k} x_i^{s_i+\frac{1}{p_i}-1-\alpha_i}\prod_{j=k+1}^{n} x_j^{s_j-\frac{\varepsilon}{p_j}-1}\Bigg)dx_1\ldots dx_n \\
=&\phi(\varepsilon)\Bigg(M-\sum_{\substack{1\leq k\leq n\\ k-odd}}\sum_{cyc}N(k)\Bigg).
\end{aligned}$$

Taking $u_i = \frac{x_{i+1}}{x_1}(i = 1, 2, \ldots, n-1)$ and using (2.1), we have

$$\begin{aligned}
M &= \int_1^\infty \cdots \int_1^\infty \frac{\prod_{i=1}^n x_i^{s_i - \frac{\varepsilon}{p_i} - 1}}{\left(\sum_{j=1}^n x_j\right)^\lambda} dx_1 \ldots dx_n \\
&= \int_1^\infty x_1^{-\varepsilon-1} dx_1 \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i=1}^{n-1} u_i^{s_{i+1} - \frac{\varepsilon}{p_{i+1}} - 1}}{\left(1 + \sum_{i=1}^{n-1} u_i\right)^\lambda} du_1 \ldots du_{n-1} \\
&\quad - \int_1^\infty x_1^{-\varepsilon-1} dx_1 \int_0^{1/x_1} \cdots \int_0^{1/x_1} \frac{\prod_{i=1}^{n-1} u_i^{s_{i+1} - \frac{\varepsilon}{p_{i+1}} - 1}}{\left(1 + \sum_{i=1}^{n-1} u_i\right)^\lambda} du_1 \ldots du_{n-1} \\
&> \frac{1}{\varepsilon} \frac{\Gamma(s_1 - \sum_{i=1}^{n-1} \frac{\varepsilon}{p_{i+1}}) \prod_{i=1}^{n-1} \Gamma(s_{i+1} - \frac{\varepsilon}{p_{i+1}})}{\Gamma(\lambda)} \\
&\quad - \int_1^\infty x_1^{-\varepsilon-1} dx_1 \int_0^{1/x_1} \cdots \int_0^{1/x_1} \prod_{i=1}^{n-1} u_i^{s_{i+1} - \frac{\varepsilon}{p_{i+1}} - 1} du_1 \ldots du_{n-1} \\
&= \frac{1}{\varepsilon} \frac{\Gamma(s_1 - \sum_{i=1}^{n-1} \frac{\varepsilon}{p_{i+1}}) \prod_{i=1}^{n-1} \Gamma(s_{i+1} - \frac{\varepsilon}{p_{i+1}})}{\Gamma(\lambda)} \\
&\quad - \frac{1}{\left(\frac{\varepsilon}{p_1} + \sum_{i=1}^{n-1} s_{i+1}\right) \prod_{i=1}^{n-1} (s_{i+1} - \frac{\varepsilon}{p_{i+1}})}.
\end{aligned}$$

Again taking $u_i = \frac{x_{i+1}}{x_1}(i = 1, 2, \ldots, n-1)$ and using (2.1), we have

$$\begin{aligned}
N(k) &= \int_1^\infty \cdots \int_1^\infty \frac{\prod_{i=1}^k x_i^{s_i + \frac{1}{p_i} - 1 - \alpha_i} \prod_{j=k+1}^n x_j^{s_j - \frac{\varepsilon}{p_j} - 1}}{\left(\sum_{j=1}^n x_j\right)^\lambda} dx_1 \cdots dx_n \\
&= \int_1^\infty x_1^{-1-\gamma} dx_1 \int_0^\infty \cdots \int_0^\infty g(u_1, \ldots, u_{n-1}) du_1 \ldots du_{n-1} \\
&\quad - \int_1^\infty x_1^{-1-\gamma} dx_1 \int_0^{1/x_1} \cdots \int_0^{1/x_1} g(u_1, \ldots, u_{n-1}) du_1 \ldots du_{n-1} \\
&< \int_1^\infty x_1^{-1-\gamma} dx_1 \int_0^\infty \cdots \int_0^\infty g(u_1, \ldots, u_{n-1}) du_1 \ldots du_{n-1} \\
&= \frac{1}{\gamma} \cdot \frac{\Gamma(s_1 + \gamma - \alpha_1 - \frac{1}{p_1}) \prod_{i=1}^{k-1} \Gamma(s_{i+1} + \frac{1}{p_{i+1}} - \alpha_{i+1}) \prod_{i=k}^{n-1} \Gamma(s_{i+1} - \frac{\varepsilon}{p_{i+1}})}{\Gamma(\lambda)}
\end{aligned}$$

where $\gamma = \sum_{i=1}^{k} \alpha_i - \sum_{i=1}^{k} \frac{1}{p_i} + \sum_{i=k+1}^{n} \frac{\varepsilon}{p_i}$, and

$$g(u_1, \ldots, u_{n-1}) = \frac{\prod_{i=1}^{k-1} u_i^{s_{i+1}+\frac{1}{p_{i+1}}-1-\alpha_{i+1}} \prod_{i=k}^{n-1} u_i^{s_{i+1}-\frac{\varepsilon}{p_{i+1}}-1}}{\left(1+\sum_{i=1}^{n-1} u_i\right)^{\lambda}}.$$

Hence, we have

$$J(\varepsilon) > \phi(\varepsilon) \left\{ \frac{1}{\varepsilon} \frac{\Gamma(s_1 - \sum_{i=1}^{n-1} \frac{\varepsilon}{p_{i+1}}) \prod_{i=1}^{n-1} \Gamma(s_{i+1} - \frac{\varepsilon}{p_{i+1}})}{\Gamma(\lambda)} - \bigcirc(1) \right\}. \tag{3.4}$$

If the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma(s_i) \left(\frac{\alpha_i p_i}{\alpha_i p_i - 1}\right)^{\alpha_i}$ in (3.1) is not the best possible, then there exists a positive constant K such that $K < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma(s_i) \left(\frac{\alpha_i p_i}{\alpha_i p_i - 1}\right)^{\alpha_i}$ and (3.1) still remains valid if $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma(s_i) \left(\frac{\alpha_i p_i}{\alpha_i p_i - 1}\right)^{\alpha_i}$ is replaced by K. In particular by (3.3) and (3.4), we have

$$\begin{aligned}
&\phi(\varepsilon) \left\{ \frac{\Gamma(s_1 - \sum_{i=1}^{n-1} \frac{\varepsilon}{p_{i+1}}) \prod_{i=1}^{n-1} \Gamma(s_{i+1} - \frac{\varepsilon}{p_{i+1}})}{\Gamma(\lambda)} - \varepsilon \bigcirc(1) \right\} \\
&\quad < \varepsilon \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^{n} x_j\right)^{\lambda}} \prod_{i=1}^{n} x_i^{s_i + \frac{1}{p_i} - 1 - \alpha_i} \widetilde{F}_i^{\alpha_i}(x_i) dx_1 \ldots dx_n \\
&\quad < \varepsilon K \prod_{i=1}^{n} \left\{ \int_0^\infty \widetilde{f}_i^{\alpha_i}(x) dx \right\}^{\frac{1}{p_i}} = K.
\end{aligned}$$

Then $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma(s_i) \left(\frac{\alpha_i p_i}{\alpha_i p_i - 1}\right)^{\alpha_i} \le K$ as $\varepsilon \to 0^+$. This contradicts the fact $K < \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma(s_i) \left(\frac{\alpha_i p_i}{\alpha_i p_i - 1}\right)^{\alpha_i}$. Hence the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma(s_i) \left(\frac{\alpha_i p_i}{\alpha_i p_i - 1}\right)^{\alpha_i}$ in (3.1) is the best possible. The theorem is proved.

Now we discuss some particular cases of (3.1). Taking $\alpha_i = 1$ in Theorem 3.1, we get the following multiple extension of (1.4), with the constant factor is the best possible.

Corollary 3.2. *Let* $n \in \mathbb{N}\backslash\{1\}, p_i > 1, \sum_{i=1}^{n} \frac{1}{p_i} = 1, \lambda, s_i > 1,$ *and* $\sum_{i=1}^{n} s_i = \lambda$. *Assume* $F_i(x) = \int_0^x f_i(t)dt$ $(i = 1, 2, \ldots, n)$. *If* $f_i \ge 0$ *satisfy* $0 <$

$\int_0^\infty f_i^{p_i}(x)dx < \infty$ $(i = 1, 2, \ldots, n)$ *then the following inequality holds:*

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n x_j\right)^\lambda} \prod_{i=1}^n x_i^{s_i + \frac{1}{p_i} - 2} F_i(x_i) dx_1 \ldots dx_n$$

$$< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma(s_i) \left(\frac{p_i}{p_i - 1}\right) \left\{\int_0^\infty f_i^{p_i}(x)dx\right\}^{\frac{1}{p_i}} \quad (3.5)$$

where the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma(s_i) \left(\frac{p_i}{p_i-1}\right)$ *is the best possible.*

Taking $n = 3, \alpha_i = 1/2$ in Theorem 3.1, we get the following generalization of (1.5), with the constant factor is the best possible.

Corollary 3.3. *Let* $p, q, r > 2, \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \lambda, s_1, s_2, s_3 > 1$, *and* $s_1 + s_2 + s_3 = \lambda$. *Assume* $F(x) = \int_0^x f(t)dt, G(x) = \int_0^x g(t)dt$, *and* $H(x) = \int_0^x h(t)dt$. *If* $f, g, h \geq 0$ *satisfy* $0 < \int_0^\infty f^{\frac{p}{2}}(x)dx < \infty, 0 < \int_0^\infty g^{\frac{q}{2}}(x)dx < \infty, 0 < \int_0^\infty h^{\frac{r}{2}}(x)dx < \infty$ *then the following inequality holds:*

$$\int_0^\infty \int_0^\infty \int_0^\infty \frac{x^{s_1 + \frac{1}{p} - \frac{3}{2}} y^{s_2 + \frac{1}{q} - \frac{3}{2}} z^{s_3 + \frac{1}{r} - \frac{3}{2}}}{(x + y + z)^\lambda} \sqrt{F(x)G(y)H(z)} dxdydz$$

$$< C_\lambda \left(\int_0^\infty f^{p/2}(x)dx\right)^{1/p} \left(\int_0^\infty g^{q/2}(y)dy\right)^{1/q} \left(\int_0^\infty h^{r/2}(z)dz\right)^{1/r} \quad (3.6)$$

where the constant factor $C_\lambda = \frac{1}{\Gamma(\lambda)} K_{p,s_1} K_{q,s_2} K_{r,s_3}$ *is the best possible and* $K_{p,s_1} = \Gamma(s_1)\sqrt{\frac{p/2}{p/2-1}}$.

Taking $n = 2, \alpha_1 = \alpha_p/p, \alpha_2 = \alpha_q/q, s_1 = s_2 = \lambda$ in Theorem 3.1, we get the following result.

Corollary 3.4. *Let* $p > \frac{1}{\alpha_1}, q > \frac{1}{\alpha_2}, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 1, \alpha_p = p(\lambda - 1) + 1$, *and* $1 - \min\{\frac{1}{p}, \frac{1}{q}\} < \lambda \leq 2 - \max\{\frac{1}{p}, \frac{1}{q}\}$. *Assume* $F(x) = \int_0^x f(t)dt$, *and* $G(y) = \int_0^y g(t)dt$. *If* $f, g \geq 0$ *satisfy* $0 < \int_0^\infty f^{\alpha_p}(x)dx < \infty, 0 < \int_0^\infty g^{\alpha_q}(x)dx < \infty$ *then the following inequality holds:*

$$\int_0^\infty \int_0^\infty \frac{F^{\frac{\alpha_p}{p}}(x) G^{\frac{\alpha_q}{q}}(y)}{(x + y)^{2\lambda}} dxdy < B(\lambda, \lambda) \left(\frac{\alpha_p}{\alpha_p - 1}\right)^{\frac{\alpha_p}{p}} \left(\frac{\alpha_q}{\alpha_q - 1}\right)^{\frac{\alpha_q}{q}}$$

$$\times \left(\int_0^\infty f^{\alpha_p}(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^{\alpha_q}(y)dy\right)^{\frac{1}{q}} \quad (3.7)$$

where the constant factor $B(\lambda,\lambda)\left(\frac{\alpha_p}{\alpha_p-1}\right)^{\frac{\alpha_p}{p}}\left(\frac{\alpha_q}{\alpha_q-1}\right)^{\frac{\alpha_q}{q}}$ *is the best possible.*

Taking $n=3, \alpha_1=\alpha_{p,q}/p, \alpha_2=\alpha_{q,r}/q, \alpha_3=\alpha_{r,p}/r, s_1=\lambda\left(\frac{1}{q}+1\right), s_2=\lambda\left(\frac{1}{r}+1\right), s_3=\lambda\left(\frac{1}{p}+1\right)$ in Theorem 3.1, we get the following result.

Corollary 3.5. *Let* $p>\frac{1}{\alpha_1}, q>\frac{1}{\alpha_2}, r>\frac{1}{\alpha_3}, \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1, s,t\in\{p,q,r\}, 1<\lambda\leq\frac{2-\max\{\frac{1}{p},\frac{1}{q},\frac{1}{r}\}}{1+\max\{\frac{1}{p},\frac{1}{q},\frac{1}{r}\}}$, *and* $\alpha_{s,t}=\lambda(1+s/t)+(\lambda-1)(s-1)$. *Assume* $F(x)=\int_0^x f(t)dt, G(y)=\int_0^y g(t)dt$, *and* $H(z)=\int_0^z h(t)dt$. *If* $f,g,h\geq 0$ *satisfy* $0<\int_0^\infty f^{\alpha_{p,q}}(t)dt<\infty, 0<\int_0^\infty g^{\alpha_{q,r}}(t)dt<\infty, 0<\int_0^\infty h^{\alpha_{r,p}}(t)dt<\infty$ *then the following inequality holds:*

$$\int_0^\infty\int_0^\infty\int_0^\infty\frac{F^{\alpha_{p,q}/p}(x)G^{\alpha_{q,r}/q}(y)H^{\alpha_{r,p}/r}(z)}{(x+y+z)^{4\lambda}}dxdydz$$
$$<K\left(\int_0^\infty f^{\alpha_{p,q}}(x)dx\right)^{1/p}\left(\int_0^\infty g^{\alpha_{q,r}}(y)dy\right)^{1/q}\left(\int_0^\infty h^{\alpha_{r,p}}(z)dz\right)^{1/r} \tag{3.8}$$

where the constant factor

$$K=\frac{\Gamma(s_1)\Gamma(s_2)\Gamma(s_3)}{\Gamma(4\lambda)}\left(\frac{\alpha_{p,q}}{\alpha_{p,q}-1}\right)^{\alpha_{p,q}/p}\left(\frac{\alpha_{q,r}}{\alpha_{q,r}-1}\right)^{\alpha_{q,r}/q}\left(\frac{\alpha_{r,p}}{\alpha_{r,p}-1}\right)^{\alpha_{r,p}/r}$$

is the best possible.

References

[1] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press Cambridge, 1952.

[2] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities Involving Function and Their Integrals and Derivatives*, Kluwer Acadamic Publishers, Boston, 1991.

[3] B. Yang, On a New Multiple Extension of Hilbert's integral inequality, *J. Math. Pure and Appl. Math.*, **6** no. 2 (2005), Art. 39.

[4] B. Yang and L. Debnath, On An Extended Multiple Hardy-Hilbert's Integral Inequality, *Tamsui Oxford J. Math. Sci*, **21** no. 2 (2005), 157-169.

[5] Q. Huang and B. Yang, On a Multiple Hilbert-Type Integral Operator and Application, *J. Inequal. Appl*, (2009), Article ID 192197, 13 pages.

[6] N. Das and S. Sahoo, New Inequalities Similar to Hardy-Hilbert's inequality, *Turk. J. Math*, **34** (2010), 153-165.

[7] N. Das and S. Sahoo, On a generalization of Hardy-Hilbert's integral inequality, *Bul. Acad. Sţiinţe Repub. Mold. Mat*, **2** (2010), 91-110.

[8] W. T. Sulaiman, On Three Inequalities Similar to Hardy-Hilbert's Integral Inequality, *Acta Math. Univ. Comenianae*, **LXXVI** no. 2 (2007), 273-278.

[9] W. T. Sulaiman, On Two New Inequalities Similar to Hardy-Hilbert's Integral Inequality, *Soochow J. Math*, **33** (2007), 497-501.